Pensieve header: The main program and demo, updated and corrected.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\USC-240205"];
```

tex

{\bf\red Implementation} (sources: \url{http://drorbn.net/icerm23/ap}). I like it most when the implementation matches the math perfectly. We failed here.

pdf

In[∘]:=
```
Once[<< KnotTheory`];
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

tex

\par{\bf\red Utilities.} The step function, algebraic numbers, canonical forms.

pdf

In[∘]:=
```
θ[x_] /; NumericQ[x] := UnitStep[x]
```

pdf

In[∘]:=
```
ω2[v_][p_] := Module[{q = Expand[p], n, c},
    If[q === 0, 0, c = Coefficient[q, ω, n = Exponent[q, ω]];
     c v^n + ω2[v][q - c (ω + ω^-1)^n]]];
```

pdf

In[∘]:=
```
sign[ℰ_] := Module[{n, d, v, p, rs, e, k},
   {n, d} = NumeratorDenominator[ℰ];
   {n, d} /= ω^(Exponent[n,ω]/2+Exponent[n,ω,Min]/2);
   p = Factor[ω2[v]@n * ω2[v]@d /. v → 4 u^2 - 2];
   rs = Solve[p == 0, u, Reals];
   If[rs === {}, Sign[p /. u → 0],
    rs = Union@(u /. rs);
    Sign[(-1)^(e=Exponent[p,u]) Coefficient[p, u, e]] + Sum[
      k = 0; While[(d = RootReduce[∂_{u,++k} p /. u → r]) == 0];
      If[EvenQ[k], 0, 2 Sign[d]] * θ[u - r],
      {r, rs}]
   ]
  ]
```

pdf

In[∘]:=
```
SetAttributes[B, Orderless];
CF[b_B] := RotateLeft[#, First@Ordering[#] - 1] & /@ DeleteCases[b, {}]
```

pdf

In[∘]:=
```
CF[ℰ_] := Module[{γs = Union@Cases[ℰ, γ_ | γ̄_, ∞]},
   Total[CoefficientRules[ℰ, γs] /. (ps_ → c_) :→ Factor[c] Times @@ γs^ps] ]
```

pdf

```
In[∘]:= CF[{}] = {};
CF[𝒞_List] := Module[{γs = Union@Cases[𝒞, γ_, ∞], γ},
  CF /@ DeleteCases[0][
    RowReduce[Table[∂_γ r, {r, 𝒞}, {γ, γs}]].γs] ]
```

pdf

```
In[∘]:= (ℰ_)* := ℰ /. {γ̄ → γ, γ → γ̄, ω → ω^-1, c_Complex ⧴ c*};
r_Rule⁺ := {r, r*}
```

pdf

```
In[∘]:= RulesOf[γ_i_ + rest_.] := (γ_i → -rest)⁺;
CF[PQ[𝒞_, q_]] := Module[{n𝒞 = CF[𝒞]},
  PQ[n𝒞, CF[q /. Union @@ RulesOf /@ n𝒞]] ]
```

pdf

```
In[∘]:= CF[Σ_b_[σ_, pq_]] := Σ_CF[b][σ, CF[pq]]
```

tex

\needspace{32mm}
\par{\bf\red Pretty-Printing.}

pdf

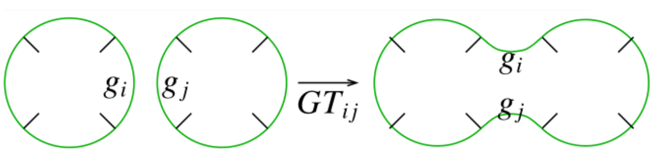
```
In[∘]:= Format[Σ_b_B[σ_, PQ[𝒞_, q_]]] := Module[{γs},
   γs = γ_# & /@ Join @@ b;
   Column[{TraditionalForm@σ,
     TableForm[Join[
       Prepend[""] /@ Table[TraditionalForm[∂_c r], {r, 𝒞}, {c, γs}],
       {Prepend[""][
         Join @@ (b /. {l_, m___, r_} ⧴ {DisplayForm@RowBox[{"(", l}],
               m, DisplayForm@RowBox[{r, ")"}]}) /. i_Integer ⧴ γ_i ]},
       MapThread[Prepend, {Table[TraditionalForm[∂_r,c q], {r, γs*}, {c, γs}], γs*}]
      ], TableAlignments → Center]
    }, Center] ];
```

tex

\par{\bf\red The Face-Centric Core.}

pdf

```
In[∘]:= Σ_b1_[σ1_, PQ[𝒞1_, q1_]] ⊕ Σ_b2_[σ2_, PQ[𝒞2_, q2_]] ^:=
  CF@Σ_Join[b1,b2][σ1 + σ2, PQ[𝒞1 ⋃ 𝒞2, q1 + q2]];
```



tex

\par GT for Gap Touch: \hfill \input{figs/GT.pdf_t}

pdf

In[∘]:=
```
GT_{i_,j_}@Σ_B[{li___,i_,ri___},{lj___,j_,rj___},bs___][σ_, PQ[C_, q_]] :=
 CF@Σ_B[{ri,li,j,rj,lj,i},bs][σ, PQ[C ⋃ {γ_i - γ_j}, q]]
```

cor·don (kôr′dn)

n.

1. A line of people, military posts, or ships stationed around an area to enclose or guard it: *a police cordon.*

2. A rope, line, tape, or similar border stretched around an area, usually by the police, indicating that access is restricted.

tex

\par\vskip 1mm\par\Cordon

pdf

In[∘]:=
```
Cordon_{i_}@Σ_B[{li___,i_,ri___},bs___][σ_, PQ[C_, q_]] :=
 Module[{ϕ = ∂_{γ_i} C, λ = ∂_{γ̄_i,γ_i} q, nσ = σ, nC, nq, p},
  {p} = FirstPosition[(# =!= 0) & /@ ϕ, True, {0}];
  {nC, nq} = Which[
    p > 0,  {C, q} /. (γ_i → -C⟦p⟧ / ϕ⟦p⟧)^+ /. (γ_i → 0)^+,
    λ =!= 0, (nσ += sign[λ]; {C, q /. (γ_i → -(∂_{γ̄_i} q) / λ)^+ /. (γ_i → 0)^+}),
    λ === 0, {C ⋃ {∂_{γ̄_i} q}, q /. (γ_i → 0)^+}];
  CF@Σ_B[Most@{ri,li},bs][nσ, PQ[nC, nq] /. (γ_{Last@{ri,li}} → γ_{First@{ri,li}})^+] ]
```

tex

\par\needspace{20mm}

{\bf\red Strand Operations.} c for contract, mc for magnetic contract:

pdf

In[∘]:=
```
c_{i_,j_}@t : Σ_B[{li___,i_,ri___},{___,j_,___},___][__] := t // GT_{j,First@{ri,li}} // Cordon_j
```

pdf

In[∘]:=
```
c_{i_,j_}@t : Σ_B[{___,i_,j_,___},___][__] := Cordon_j@t
c_{i_,j_}@t : Σ_B[{j_,___,i_},___][__] := Cordon_j@t
c_{i_,j_}@t : Σ_B[{___,j_,i_,___},___][__] := Cordon_i@t
c_{i_,j_}@t : Σ_B[{i_,___,j_},___][__] := Cordon_i@t
```

pdf

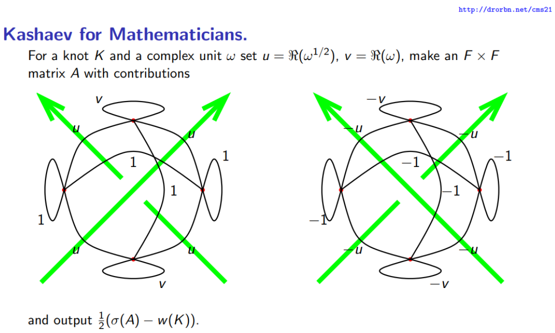
In[∘]:=
```
mc[ℰ_] := ℰ //.
  t : Σ_B[{___,i_,___},{___,j_,___},___][__] | Σ_B[{___,i_,j_,___},___][__] | Σ_B[{j_,___,i_},___][__] /;
   i + j == 0 :→ c_{i,j}@t
```

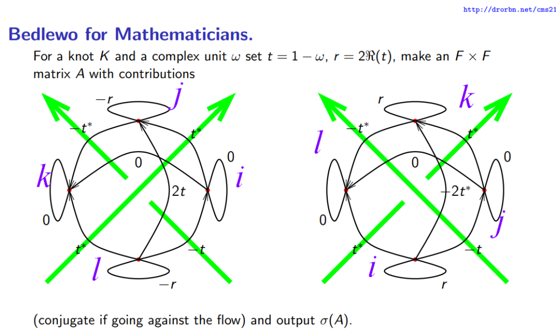
tex

\par{\bf\red The Crossings} (and empty strands).

pdf

In[∘]:=
```
Kas@P_{i_,j_} := CF@Σ_{B[{i,j}]}[0, PQ[{}, 0]];
TL@P_{i_,j_} := CF@Σ_{B[{i,j}]}[0, PQ[{}, 0]]
```





pdf

In[∘]:=
```
Kas[x : X[i_, j_, k_, l_]] := Kas@If[PositiveQ[x], X_{-i,j,k,-l}, X̄_{-j,k,l,-i}];
Kas[(x : X | X̄)_{fs__}] := Module[{v = 2 u^2 - 1, p, γs, m},
  γs = γ_# & /@ {fs}; p = (x === X);
  m = If[p, ( v u 1 u
              u 1 u 1
              1 u v u
              u 1 u 1 ), - ( v u 1 u
                             u 1 u 1
                             1 u v u
                             u 1 u 1 )];
  CF@Σ_{B[{fs}]}[If[p, -1, 1], PQ[{}, γs*.m.γs]]]
```

pdf

In[∘]:=
```
TL[x : X[i_, j_, k_, l_]] := TL@If[PositiveQ[x], X_{-i,j,k,-l}, X̄_{-j,k,l,-i}];
TL[(x : X | X̄)_{fs__}] := Module[{t = 1 - ω, r, γs, m},
  r = t + t*; γs = γ_# & /@ {fs};
  m = If[x === X,
     ( -r   -t  2 t   t*
       -t*   0   t*   0
       2 t*  t  -r   -t*
       t     0  -t    0 ), ( r    -t  -2 t*  t*
                             -t*   0   t*    0
                             -2 t  t   r    -t*
                             t     0  -t     0 )];
  CF@Σ_{B[{fs}]}[0, PQ[{}, γs*.m.γs]]]
```

tex

\par{\bf\red Evaluation on Tangles and Knots.}

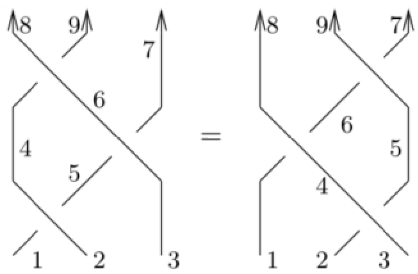
pdf

In[∘]:=
```
Kas[K_] := Fold[mc[#1 ⊕ #2] &, Σ_{B[]}[0, PQ[{}, 0]], List @@ (Kas /@ PD@K)];
KasSig[K_] := Expand[Kas[K][[1]] / 2]
```

pdf

In[∘]:=
```
TL[K_] := Fold[mc[#1⊕#2] &, Σ_B[][0, PQ[{}, 0]], List @@ (TL /@ PD@K)] /.
    θ[c_ + u] /; Abs[c] ≥ 1 :→ θ[c];
TLSig[K_] := TL[K][[1]]
```

Reidemeister 3



tex

\par\needspace{20mm}
\parpic[r]{\input{figs/R3.pdf_t}}
{\bf\red Reidemeister 3.}

pdf

In[∘]:=
```
R3L = PD[X_{-2,5,4,-1}, X_{-3,7,6,-5},
    X_{-6,9,8,-4}];
R3R = PD[X_{-3,5,4,-2}, X_{-4,6,8,-1},
    X_{-5,7,9,-6}];
{TL@R3L == TL@R3R, Kas@R3L == Kas@R3R}
```

Out[∘]=
pdf

{True, True}

In[∘]:=
```
TL@R3L
```

Out[∘]=

$$-1$$

$$\begin{array}{c|cccccc}
 & (\gamma_{-3} & \gamma_7 & \gamma_9 & \gamma_8 & \gamma_{-1} & \gamma_{-2}) \\
\overline{\gamma}_{-3} & \frac{\omega^2+1}{\omega} & \omega-1 & -2\omega & 2 & 0 & -\frac{\omega+1}{\omega} \\
\overline{\gamma}_7 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 & 0 & 0 \\
\overline{\gamma}_9 & -\frac{2}{\omega} & 1-\omega & \frac{\omega^2+1}{\omega} & -\frac{\omega+1}{\omega} & 0 & \frac{2}{\omega} \\
\overline{\gamma}_8 & 2 & 0 & -\omega-1 & \frac{\omega^2+1}{\omega} & -\frac{\omega-1}{\omega} & -\frac{2}{\omega} \\
\overline{\gamma}_{-1} & 0 & 0 & 0 & \omega-1 & 0 & 1-\omega \\
\overline{\gamma}_{-2} & -\omega-1 & 0 & 2\omega & -2\omega & \frac{\omega-1}{\omega} & \frac{\omega^2+1}{\omega}
\end{array}$$

tex

\needspace{15mm}

pdf

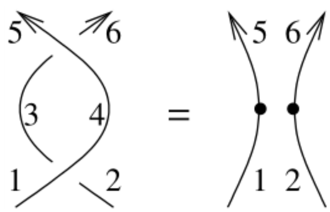
In[∘]:= **Kas@R3L**

Out[∘]=
pdf

$$2\,\theta\left(u-\tfrac{1}{2}\right)-2\,\theta\left(u+\tfrac{1}{2}\right)-2$$

	$(\gamma_{-3}$	γ_7	γ_9	γ_8	γ_{-1}	$\gamma_{-2})$
$\bar\gamma_{-3}$	$\frac{2u^2\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(2u+1)}$	$-\frac{2u}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-}{(2u-1)\,(2}$
$\bar\gamma_7$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$\frac{2\,(2u^2-1)}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(2u+1)}$	$-\frac{2u}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(}$
$\bar\gamma_9$	$-\frac{1}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$\frac{2u^2\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(2u+1)}$	$-\frac{2u}{(2u-1)\,(}$
$\bar\gamma_8$	$-\frac{2u}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$\frac{2u^2\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(}$
$\bar\gamma_{-1}$	$-\frac{1}{(2u-1)\,(2u+1)}$	$-\frac{2u}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$\frac{2\,(2u^2-1)}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-}{(2u-1)\,(2}$
$\bar\gamma_{-2}$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(2u+1)}$	$-\frac{2u}{(2u-1)\,(2u+1)}$	$-\frac{1}{(2u-1)\,(2u+1)}$	$\frac{u\,(4u^2-3)}{(2u-1)\,(2u+1)}$	$\frac{2u^2\,(4u^2}{(2u-1)\,(2}$

Reidemeister 2b



tex

\par
\parpic[r]{\input{figs/R2.pdf_t}}
{\bf\red Reidemeister 2.}

pdf

In[∘]:= $\mathbf{TL@PD}\left[\mathbf{X}_{-2,4,3,-1},\ \bar{\mathbf{X}}_{-4,6,5,-3}\right]$

Out[∘]=
pdf

$$\begin{array}{ccccc}
 & & 0 & & \\
 & 1 & 0 & -1 & 0 \\
 & (\gamma_{-2} & \gamma_6 & \gamma_5 & \gamma_{-1}) \\
\bar\gamma_{-2} & 0 & 0 & 0 & 0 \\
\bar\gamma_6 & 0 & 0 & 0 & 0 \\
\bar\gamma_5 & 0 & 0 & 0 & 0 \\
\bar\gamma_{-1} & 0 & 0 & 0 & 0
\end{array}$$

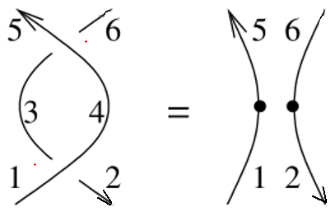
pdf

In[∘]:= $\{\mathbf{TL@PD}[\mathbf{X}_{-2,4,3,-1},\ \bar{\mathbf{X}}_{-4,6,5,-3}] == \mathbf{GT}_{5,-2}\mathbf{@TL@PD}[\mathbf{P}_{-1,5},\ \mathbf{P}_{-2,6}],$
$\mathbf{Kas@PD}[\mathbf{X}_{-2,4,3,-1},\ \bar{\mathbf{X}}_{-4,6,5,-3}] == \mathbf{GT}_{5,-2}\mathbf{@Kas@PD}[\mathbf{P}_{-1,5},\ \mathbf{P}_{-2,6}]\}$

Out[∘]=
pdf

{True, True}

Reidemeister 2c



In[∘]:= $\mathrm{TL@PD}\left[\overline{X}_{-1,2,4,-3},\ X_{-6,5,3,-4}\right]$

Out[∘]=

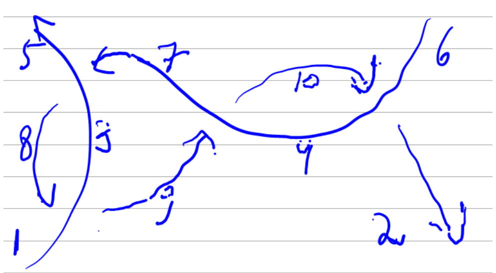
$$\begin{array}{ccccc} & & 0 & & \\ & (\Upsilon_{-6} & \Upsilon_{5} & \Upsilon_{-1} & \Upsilon_{2}) \\ \bar{\Upsilon}_{-6} & 0 & \omega-1 & 0 & 1-\omega \\ \bar{\Upsilon}_{5} & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 \\ \bar{\Upsilon}_{-1} & 0 & 1-\omega & 0 & \omega-1 \\ \bar{\Upsilon}_{2} & \frac{\omega-1}{\omega} & 0 & -\frac{\omega-1}{\omega} & 0 \end{array}$$

In[∘]:= $\left\{\mathrm{TL@PD}\left[\overline{X}_{-1,2,4,-3},\ X_{-6,5,3,-4}\right] == \mathrm{GT}_{5,2}\mathrm{@TL@PD}\left[P_{-1,5},\ P_{-6,2}\right],\right.$
$\left.\mathrm{Kas@PD}\left[\overline{X}_{-1,2,4,-3},\ X_{-6,5,3,-4}\right] == \mathrm{GT}_{5,2}\mathrm{@Kas@PD}\left[P_{-1,5},\ P_{-6,2}\right]\right\}$

Out[∘]=

$$\left\{\begin{array}{ccccc} & & 0 & & \\ & (\Upsilon_{-6} & \Upsilon_{5} & \Upsilon_{-1} & \Upsilon_{2}) \\ \bar{\Upsilon}_{-6} & 0 & \omega-1 & 0 & 1-\omega \\ \bar{\Upsilon}_{5} & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 \\ \bar{\Upsilon}_{-1} & 0 & 1-\omega & 0 & \omega-1 \\ \bar{\Upsilon}_{2} & \frac{\omega-1}{\omega} & 0 & -\frac{\omega-1}{\omega} & 0 \end{array} == \begin{array}{ccccc} & & 0 & & \\ & 0 & -1 & 0 & 1 \\ & (\Upsilon_{-6} & \Upsilon_{5} & \Upsilon_{-1} & \Upsilon_{2}) \\ \bar{\Upsilon}_{-6} & 0 & 0 & 0 & 0 \\ \bar{\Upsilon}_{5} & 0 & 0 & 0 & 0 \\ \bar{\Upsilon}_{-1} & 0 & 0 & 0 & 0 \\ \bar{\Upsilon}_{2} & 0 & 0 & 0 & 0 \end{array},\ \mathrm{True}\right\}$$

That’s a fail for TL!

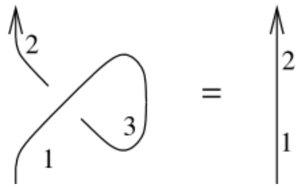


In[∘]:= $\mathrm{TL@PD}\left[\overline{X}_{-1,9,3,-8},\ X_{-7,5,8,-3},\ \overline{X}_{-4,10,7,-9},\ X_{-10,4,2,-6}\right]$

Out[∘]=

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & -1 & 0 & 1 \\ & (\Upsilon_{-6} & \Upsilon_{5} & \Upsilon_{-1} & \Upsilon_{2}) \\ \bar{\Upsilon}_{-6} & 0 & 0 & 0 & 0 \\ \bar{\Upsilon}_{5} & 0 & 0 & 0 & 0 \\ \bar{\Upsilon}_{-1} & 0 & 0 & 0 & 0 \\ \bar{\Upsilon}_{2} & 0 & 0 & 0 & 0 \end{array}$$

Reidemeister 1



tex

\par
\parpic[r]{\input{figs/R1.pdf_t}}
{\bf\red Reidemeister 1.}

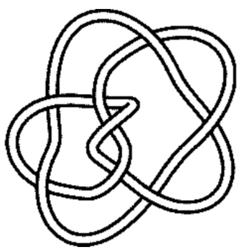
pdf

In[∘]:=
```
{TL@PD[X_{-3,3,2,-1}] == TL@P_{-1,2},
 Kas@PD[X_{-3,3,2,-1}] == Kas@P_{-1,2}}
```

Out[∘]=
pdf

{True, True}

A Knot



tex

\par
\parpic[r]{\includegraphics[width=1in]{8_5.png}}
{\bf\red A Knot.}

pdf

In[∘]:=
```
f = TLSig[Knot[8, 5]]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$$2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]-2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]-2\,\theta\left[u-\boxed{-0.630\ldots}\right]+2\,\theta\left[u-\boxed{0.630\ldots}\right]$$

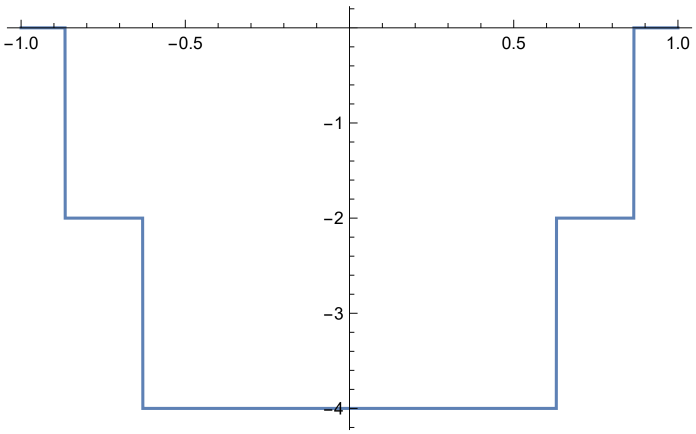
tex

\par\picskip{0}{
 \def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
 \def\nbpdfOutput#1{\hfill\includegraphics[width=0.4\linewidth,valign=t]{#1}}

pdf

In[ ]:= `Plot[f, {u, -1, 1}]`

Out[ ]=
pdf



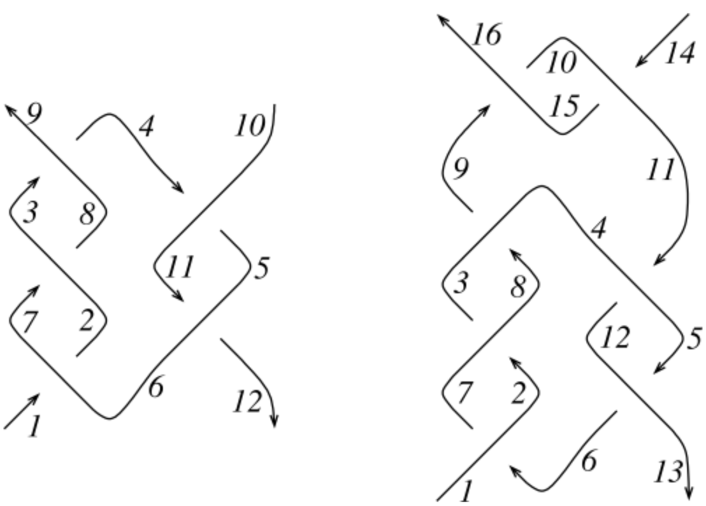
tex

}

Some Tangles

tex

\needspace{30mm}
\par\parpic[r]{\includegraphics[width=1.88in]{figs/CKT.pdf}}
{\bf\red The Conway-Kinoshita-Terasaka Tangles.}



pdf

In[ ]:= $\mathbf{T1} = \mathrm{PD}\left[\overline{X}_{-6,2,7,-1}, \overline{X}_{-2,8,3,-7}, \overline{X}_{-8,4,9,-3}, X_{-11,6,12,-5}, X_{-4,11,5,-10}\right];$

$\mathbf{T2} = \mathrm{PD}\left[X_{-6,2,7,-1}, X_{-2,8,3,-7}, X_{-8,4,9,-3}, \overline{X}_{-12,6,13,-5}, \overline{X}_{-4,12,5,-11}, \overline{X}_{-10,15,11,-14}, \overline{X}_{-15,10,16,-9}\right];$

tex

\par\needspace{10mm}

pdf

In[∘*]:=*
```
Column@{TL[T1], Kas[T1]}
```

Out[∘*]=*
pdf

$$-2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-1$$

$$\begin{array}{c|cccc}
 & (\Upsilon_{-10} & \Upsilon_{9} & \Upsilon_{-1} & \Upsilon_{12}) \\
\bar{\Upsilon}_{-10} & 0 & 1-\omega & 0 & \omega-1 \\
\bar{\Upsilon}_{9} & \frac{\omega-1}{\omega} & \frac{2\,\omega}{\omega^2-\omega+1} & -\frac{\omega-1}{\omega} & -\frac{2\,\omega}{\omega^2-\omega+1} \\
\bar{\Upsilon}_{-1} & 0 & \omega-1 & 0 & 1-\omega \\
\bar{\Upsilon}_{12} & -\frac{\omega-1}{\omega} & -\frac{2\,\omega}{\omega^2-\omega+1} & \frac{\omega-1}{\omega} & \frac{2\,\omega}{\omega^2-\omega+1}
\end{array}$$

$$-2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-1$$

$$\begin{array}{c|cccc}
 & (\Upsilon_{-10} & \Upsilon_{9} & \Upsilon_{-1} & \Upsilon_{12}) \\
\bar{\Upsilon}_{-10} & 2\,(u-1)\,(u+1)\,\left(4\,u^2-3\right) & 0 & -2\,(u-1)\,(u+1)\,\left(4\,u^2-3\right) & 0 \\
\bar{\Upsilon}_{9} & 0 & \frac{1}{2\,\left(4\,u^2-3\right)} & 0 & -\frac{1}{2\,\left(4\,u^2-3\right)} \\
\bar{\Upsilon}_{-1} & -2\,(u-1)\,(u+1)\,\left(4\,u^2-3\right) & 0 & 2\,(u-1)\,(u+1)\,\left(4\,u^2-3\right) & 0 \\
\bar{\Upsilon}_{12} & 0 & -\frac{1}{2\,\left(4\,u^2-3\right)} & 0 & \frac{1}{2\,\left(4\,u^2-3\right)}
\end{array}$$

pdf

In[∘*]:=*
```
Column@{TL[T2], Kas[T2]}
```

Out[∘*]=*
pdf

$$0$$

$$\begin{array}{c|cccc}
 & (\Upsilon_{-14} & \Upsilon_{16} & \Upsilon_{-1} & \Upsilon_{13}) \\
\bar{\Upsilon}_{-14} & 0 & 1-\omega & 0 & \omega-1 \\
\bar{\Upsilon}_{16} & \frac{\omega-1}{\omega} & -\frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1} & -\frac{\omega-1}{\omega} & \frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1} \\
\bar{\Upsilon}_{-1} & 0 & \omega-1 & 0 & 1-\omega \\
\bar{\Upsilon}_{13} & -\frac{\omega-1}{\omega} & \frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1} & \frac{\omega-1}{\omega} & -\frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1}
\end{array}$$

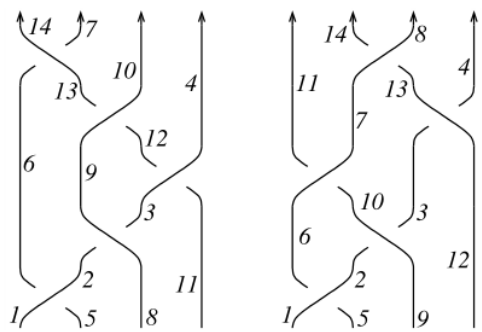
$$1$$

$$\begin{array}{c|cccc}
 & (\Upsilon_{-14} & \Upsilon_{16} & \Upsilon_{-1} & \Upsilon_{13}) \\
\bar{\Upsilon}_{-14} & \frac{1}{2}\,\left(-16\,u^4+28\,u^2-13\right) & 0 & \frac{1}{2}\,\left(16\,u^4-28\,u^2+13\right) & 0 \\
\bar{\Upsilon}_{16} & 0 & -\frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13} & 0 & \frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13} \\
\bar{\Upsilon}_{-1} & \frac{1}{2}\,\left(16\,u^4-28\,u^2+13\right) & 0 & \frac{1}{2}\,\left(-16\,u^4+28\,u^2-13\right) & 0 \\
\bar{\Upsilon}_{13} & 0 & \frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13} & 0 & -\frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13}
\end{array}$$

Some Braids

tex

\parpic[r]{\includegraphics[width=1.88in]{figs/B1B2.pdf}}
{\bf\red Examples with non-trivial codimension.}



In[∘]:= PD[X[5, 2, 6, 1], X[2, 9, 3, 10], X[10, 7, 11, 6], X[3, 12, 4, 13], X[13, 8, 14, 7]] /.
$x : X[i\_, j\_, k\_, l\_] \mapsto \text{If}\left[\text{PositiveQ@}x,\ X_{-i,j,k,-l},\ \overline{X}_{-j,k,l,-i}\right]$

Out[∘]=

$\text{PD}\left[X_{-5,2,6,-1},\ \overline{X}_{-9,3,10,-2},\ X_{-10,7,11,-6},\ \overline{X}_{-12,4,13,-3},\ X_{-13,8,14,-7}\right]$

pdf

In[∘]:= $\text{B1} = \text{PD}\left[X_{-5,2,6,-1},\ \overline{X}_{-8,3,9,-2},\ X_{-11,4,12,-3},\ X_{-12,10,13,-9},\ \overline{X}_{-13,7,14,-6}\right];$

$\text{B2} = \text{PD}\left[X_{-5,2,6,-1},\ \overline{X}_{-9,3,10,-2},\ X_{-10,7,11,-6},\ \overline{X}_{-12,4,13,-3},\ X_{-13,8,14,-7}\right];$

pdf

In[•]:=
```
Column@{TL[B1], Kas[B1]}
```

Out[•]=
pdf

0

	1	0	-1	0	$\frac{1}{\omega}$	0	$-\frac{1}{\omega}$	0
	0	0	0	-1	$\frac{1}{\omega}$	0	$-\frac{1}{\omega}$	1
	$(\Upsilon_{-11}$	Υ_{4}	Υ_{10}	Υ_{7}	Υ_{14}	Υ_{-1}	Υ_{-5}	$\Upsilon_{-8})$
$\bar{\Upsilon}_{-11}$	0	0	0	0	0	0	0	0
$\bar{\Upsilon}_{4}$	0	0	0	0	$\frac{\omega-1}{\omega^2}$	0	$-\frac{\omega-1}{\omega^2}$	0
$\bar{\Upsilon}_{10}$	0	0	0	0	$-\frac{\omega-1}{\omega}$	0	$\frac{\omega-1}{\omega}$	0
$\bar{\Upsilon}_{7}$	0	0	0	0	$\frac{(\omega-1)^2}{\omega^2}$	0	$-\frac{(\omega-1)^2}{\omega^2}$	0
$\bar{\Upsilon}_{14}$	0	$-((\omega-1)\,\omega)$	$\omega-1$	$(\omega-1)^2$	0	$-\frac{\omega-1}{\omega}$	$\frac{\omega-1}{\omega}$	0
$\bar{\Upsilon}_{-1}$	0	0	0	0	$\omega-1$	0	$1-\omega$	0
$\bar{\Upsilon}_{-5}$	0	$(\omega-1)\,\omega$	$1-\omega$	$-(\omega-1)^2$	$1-\omega$	$\frac{\omega-1}{\omega}$	$\frac{(\omega-1)^2}{\omega}$	0
$\bar{\Upsilon}_{-8}$	0	0	0	0	0	0	0	0

0

	1	0	-1	0	1	0	-1	0
	$(\Upsilon_{-11}$	Υ_{4}	Υ_{10}	Υ_{7}	Υ_{14}	Υ_{-1}	Υ_{-5}	$\Upsilon_{-8})$
$\bar{\Upsilon}_{-11}$	0	0	0	0	0	0	0	0
$\bar{\Upsilon}_{4}$	0	0	0	-1	$-u$	0	u	1
$\bar{\Upsilon}_{10}$	0	0	0	$-u$	$1-2u^2$	0	$2u^2-1$	u
$\bar{\Upsilon}_{7}$	0	-1	$-u$	$2u^2-3$	$-u$	-1	0	1
$\bar{\Upsilon}_{14}$	0	$-u$	$1-2u^2$	$-u$	-1	$-u$	$-2(u-1)(u+1)$	u
$\bar{\Upsilon}_{-1}$	0	0	0	-1	$-u$	0	u	1
$\bar{\Upsilon}_{-5}$	0	u	$2u^2-1$	0	$-2(u-1)(u+1)$	u	$4u^2-3$	0
$\bar{\Upsilon}_{-8}$	0	1	u	1	u	1	0	$1-2u$

tex

\par\needspace{10mm}

pdf

In[∘]:= **Column@{TL[B2], Kas[B2]}**

Out[∘]=
pdf

$$0$$

	$(\curlyvee_{-12}$	$\curlyvee_{4}$	$\curlyvee_{8}$	$\curlyvee_{14}$	$\curlyvee_{11}$	$\curlyvee_{-1}$	⋯
$\overline{\curlyvee}_{-12}$	$\frac{(\omega-1)^2}{\omega}$	$\omega-1$	$-2(\omega-1)$	$\frac{2(\omega-1)^2}{\omega}$	$\frac{2(\omega-1)}{\omega^2}$	0	$-\frac{2}{\cdots}$
$\overline{\curlyvee}_{4}$	$-\frac{\omega-1}{\omega}$	0	$\frac{\omega-1}{\omega}$	0	0	0	
$\overline{\curlyvee}_{8}$	$\frac{2(\omega-1)}{\omega}$	$1-\omega$	$\frac{(\omega-1)^2}{\omega}$	$-\frac{(\omega-1)(2\omega-3)}{\omega}$	$-\frac{2(\omega-1)}{\omega^2}$	0	$\frac{2(\cdots}{\cdots}$
$\overline{\curlyvee}_{14}$	$\frac{2(\omega-1)^2}{\omega}$	0	$-\frac{(\omega-1)(3\omega-2)}{\omega}$	$\frac{3(\omega-1)^2}{\omega}$	$-\frac{(\omega-2)(\omega-1)}{\omega^2}$	0	$-\frac{2}{\cdots}$
$\overline{\curlyvee}_{11}$	$-2(\omega-1)\omega$	0	$2(\omega-1)\omega$	$-((\omega-1)(2\omega-1))$	$\frac{(\omega-1)^2}{\omega}$	$-\frac{\omega-1}{\omega}$	$\frac{2(\cdots}{\cdots}$
$\overline{\curlyvee}_{-1}$	0	0	0	0	$\omega-1$	0	1
$\overline{\curlyvee}_{-5}$	$2(\omega-1)\omega$	0	$-2(\omega-1)\omega$	$2(\omega-1)\omega$	$-2(\omega-1)$	$\frac{\omega-1}{\omega}$	$\frac{(\omega\cdots}{\cdots}$
$\overline{\curlyvee}_{-9}$	$-\frac{(\omega-1)(3\omega-2)}{\omega}$	0	$\frac{2(\omega-1)(2\omega-1)}{\omega}$	$-\frac{2(\omega-1)(2\omega-1)}{\omega}$	$\frac{2(\omega-1)^2}{\omega^2}$	0	$-\frac{(\omega-2\cdots}{\cdots}$

$$2\,\Theta\left(u-\frac{\sqrt{3}}{2}\right)-2\,\Theta\left(u+\frac{\sqrt{3}}{2}\right)$$

	1	$\frac{1}{2u}$	0	$-\frac{1}{2u}$	-1	$-\frac{1}{2u}$
	$(\curlyvee_{-12}$	$\curlyvee_{4}$	$\curlyvee_{8}$	$\curlyvee_{14}$	$\curlyvee_{11}$	$\curlyvee_{-1}$
$\overline{\curlyvee}_{-12}$	0	0	0	0	0	0
$\overline{\curlyvee}_{4}$	0	$-\frac{(2u-1)(2u+1)(2u^2-1)}{4u^2(4u^2-3)}$	$-\frac{2u^2-1}{2u}$	$\frac{1}{4u^2(4u^2-3)}$	0	$-\frac{(2u-1)(2u\cdots}{4u^2(4u^2-\cdots}$
$\overline{\curlyvee}_{8}$	0	$-\frac{2u^2-1}{2u}$	$-2(u-1)(u+1)$	$\frac{2u^2-1}{2u}$	0	$-\frac{1}{2u}$
$\overline{\curlyvee}_{14}$	0	$\frac{1}{4u^2(4u^2-3)}$	$\frac{2u^2-1}{2u}$	$\frac{(2u^2-1)(16u^4-16u^2+1)}{4u^2(4u^2-3)}$	0	$-\frac{8u^4-10u^2+\cdots}{4u^2(4u^2-\cdots}$
$\overline{\curlyvee}_{11}$	0	0	0	0	0	0
$\overline{\curlyvee}_{-1}$	0	$-\frac{(2u-1)(2u+1)}{4u^2(4u^2-3)}$	$-\frac{1}{2u}$	$-\frac{8u^4-10u^2+1}{4u^2(4u^2-3)}$	0	$\frac{8u^4-10u^2-\cdots}{4u^2(4u^2-3\cdots}$
$\overline{\curlyvee}_{-5}$	0	$-\frac{1}{2u(4u^2-3)}$	0	$\frac{1}{2u(4u^2-3)}$	0	$\frac{8u^4-10u^2+\cdots}{2u(4u^2-3)}$
$\overline{\curlyvee}_{-9}$	0	$\frac{8u^4-6u^2-1}{4u^2(4u^2-3)}$	$\frac{1}{2u}$	$\frac{1}{4u^2(4u^2-3)}$	0	$\frac{16u^4-16u^2+\cdots}{4u^2(4u^2-3\cdots}$