# The Hardest Math I've Ever Really Used, 1

Dror Bar–Natan at the How to Talk Mathematics Seminar

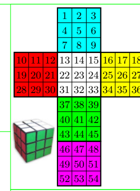
Toronto, February 2026

Dror Bar-Natan: Talks: Mathcamp-0907:

## Non-Commutative Gaussian Elimination and Rubik's Cube

Joint study with Itai Bar-Natan

**The Problem.** Let $G = \langle g_1, \ldots, g_\alpha \rangle$ be a subgroup of $S_n$, with $n = O(100)$. Before you die, understand $G$:
1. Compute $|G|$.
2. Given $\sigma \in S_n$, decide if $\sigma \in G$.
3. Write a $\sigma \in G$ in terms of $g_1, \ldots, g_\alpha$.
4. Produce *random* elements of $G$.




**The Commutative Analog.** Let $V = \mathrm{span}(v_1, \ldots, v_\alpha)$ be a subspace of $\mathbb{R}^n$. Before you die, understand $V$.

**Solution: Gaussian Elimination.** Prepare an empty table,

| 1 | 2 | 3 | 4 | $\cdots$ | $n-1$ | $n$ |
|---|---|---|---|---|---|---|
| | | | | | | |

Space for a vector $u_4 \in V$, of the form $u_4 = (0, 0, 0, 1, *, \ldots, *)$; $1 :=$ "the pivot".

Based on an algorithm by

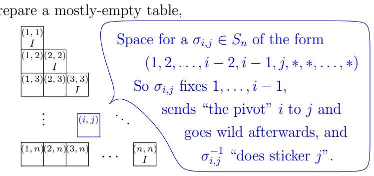
Otto Schreier Charles Sims

See also *Permutation Group Algorithms* by Ákos Seress.

**Feed $v_1, \ldots, v_\alpha$ in order.** To feed a non-zero $v$, find its pivotal position $i$.
1. If box $i$ is empty, put $v$ there.
2. If box $i$ is occupied, find a combination $v'$ of $v$ and $u_i$ that eliminates the pivot, and feed $v'$.

**Non-Commutative Gaussian Elimination**

Prepare a mostly-empty table,




**Feed $g_1, \ldots, g_\alpha$ in order.** To feed a non-identity $\sigma$, find its pivotal position $i$ and let $j := \sigma(i)$.
1. If box $(i, j)$ is empty, put $\sigma$ there.
2. If box $(i, j)$ contains $\sigma_{i,j}$, feed $\sigma' := \sigma_{i,j}^{-1}\sigma$.

**The Twist.** When done, for every occupied $(i, j)$ and $(k, l)$, feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

**Claim.** The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table $T$.

**Claim.** Anything fed in $T$ is a monotone product in $T$:

$$f \text{ was fed } \Rightarrow f \in M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \ \&\ \sigma_{i,j_i} \in T\}$$

**The Generators**

```
In[1]:= gs = {
  purple = P[18,27,36,4,5,6,7,8,9,3,11,12,13,14,15,16,17,
   45,2,20,21,22,23,24,25,26,44,1,29,30,31,32,33,34,35,43,
   37,38,39,40,41,42,10,19,28,52,49,46,53,50,47,54,51,48],
  white = P[1,2,3,4,5,6,16,25,34,10,11,9,15,24,33,39,17,
   18,19,20,8,14,23,32,38,26,27,28,29,7,13,22,31,37,35,36,
   12,21,30,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54],
  green = P[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,
   19,20,21,22,23,24,25,26,27,31,32,33,34,35,36,48,47,46,
   39,42,45,38,41,44,37,40,43,30,29,28,49,50,51,52,53,54],
  blue = P[3,6,9,2,5,8,1,4,7,54,53,52,10,11,12,13,14,15,
   19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,
   37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,18,17,16],
  red = P[13,2,3,22,5,6,31,8,9,12,21,30,37,14,15,16,17,
   18,11,20,29,40,23,24,25,26,27,10,19,28,43,32,33,34,35,
   36,46,38,39,49,41,42,52,44,45,1,47,48,4,50,51,7,53,54],
  yellow = P[1,2,48,4,5,51,7,8,54,10,11,12,13,14,3,18,27,
   36,19,20,21,22,23,6,17,26,35,28,29,30,31,32,9,16,25,34,
   37,38,15,40,41,24,43,44,33,46,47,39,49,50,42,52,53,45]
};        Enter
```

**Theorem.** $G = M_1$. *$G^{-1}$ is more fun!*

$$G = M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$$

**Proof.** The inclusions $M_1 \subset G$ and $\{g_1, \ldots, g_\alpha\} \subset M_1$ are obvious. The rest follows from the following

**Lemma.** $M_1$ is closed under multiplication.

**Proof.** By backwards induction. Let

$$M_k := \{\sigma_{k,j_k}\cdots\sigma_{n,j_n} : \forall i \geq k, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$$

Clearly $M_n M_n \subset M_n$. Now assume that $M_5 M_5 \subset M_5$ and show that $M_4 M_4 \subset M_4$. Start with $\sigma_{8,j} M_4 \subset M_4$:

$$\sigma_{8,j}(\sigma_{4,j_4} M_5) \stackrel{1}{=} (\sigma_{8,j}\sigma_{4,j_4}) M_5 \stackrel{2}{\subset} M_4 M_5$$

$$\stackrel{3}{=} \sigma_{4,j_4}(M_5 M_5) \stackrel{4}{\subset} \sigma_{4,j_4} M_5 \subset M_4$$

(1: associativity, 2: thank the twist, 3: associativity and tracing $i_4$, 4: induction). Now the general case

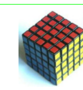
$$(\sigma_{4,j_4'}\sigma_{5,j_5'}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$

falls like a chain of dominos.

Problem Solved!

**A Demo Program**

```
In[2]:= ($RecursionLimit = 2^16;
 n = 54;
 P /: p_P ** P[a___] := p[[{a}]];
 Inv[p_P] := P @@ Ordering[p];
 Feed[P @@ Range[n]] := Null;
 Feed[p_P] := Module[{i, j},
   For[i = 1, p[[i]] == i, ++i];
   j = p[[i]];
   If[Head[s[i, j]] === P,
     Feed[Inv[s[i, j]] ** p],
   (* Else *) s[i, j] = p;
     Do[If[Head[s[k, l]] == P,
       Feed[s[i, j] ** s[k, l]];
       Feed[s[k, l] ** s[i, j]]
     ], {k, n}, {l, n}]
   ]]);   Enter
```



www.geocities.com/jaapsch/puzzles/

www.powerstrike.net/puzzles/

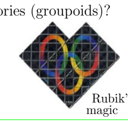
http://lodelnic.free.fr/

**Homework Problem 1.** Can you do cosets?


www.powerstrike.net/puzzles/

**Homework Problem 2.** Can you do categories (groupoids)?

| | | | |
|---|---|---|---|
| 7 | 9 | 2 | 5 |
| 1 | 4 | 8 | 3 |
| 6 | 10 | 11 | 12 |
| 13 | 14 | 15 | |



Rubik's magic

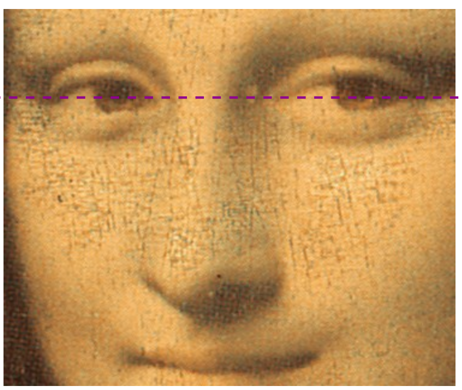
**The Results**

```
In[3]:= (Feed[#]; Product[1 + Length[Select[Range[n], Head[s[i, #]] === P &]], {i, n}]) & /@ gs   Enter
Out[3]= {4, 16, 159993501696000, 21119142223872000, 43252003274489856000, 43252003274489856000}
```

that's cool!

http://www.math.toronto.edu/~drorbn/Talks/Mathcamp-0907/ and links there

http://drorbn.net/to26



Al Gore in Futurama, circa 3000AD