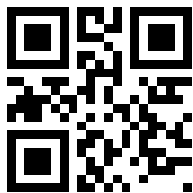


The Hardest Math I've Ever Really Used, 1

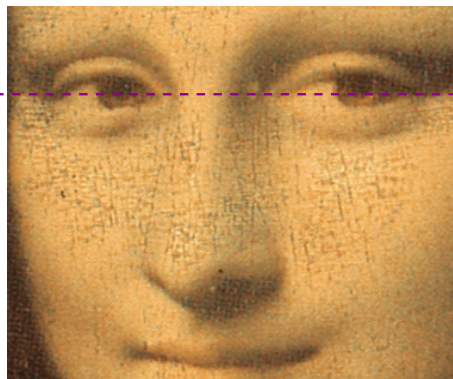
Abstract. We'll talk about How to Talk Mathematics, at least partially, by talking mathematics. What's the hardest math I've ever used in real life? Me, myself, directly - not by using a cellphone or a GPS device that somebody else designed? And in "real life" — not while studying or teaching mathematics? I use addition and subtraction daily, adding up bills or calculating change. I use percentages often, though mostly it is just "add 15 percent". I seldom use multiplication and division: when I buy in bulk, or when I need to know how many tiles I need to replace my kitchen floor. I've used powers twice in my life, doing calculations related to mortgages. I've used a tiny bit of geometry and algebra for a tiny bit of non-math-related computer graphics I've played with. And for a long time, that was all. In my talk I will tell you how recently a math topic discovered only in the 1800s made a brief and modest appearance in my non-mathematical life. There are many books devoted to that topic and a lot of active research. Yet for all I know, nobody ever needed the actual formulas for such a simple reason before.



<http://drorbn.net/to26>

I could be a mathematician ...

...or an art historian...



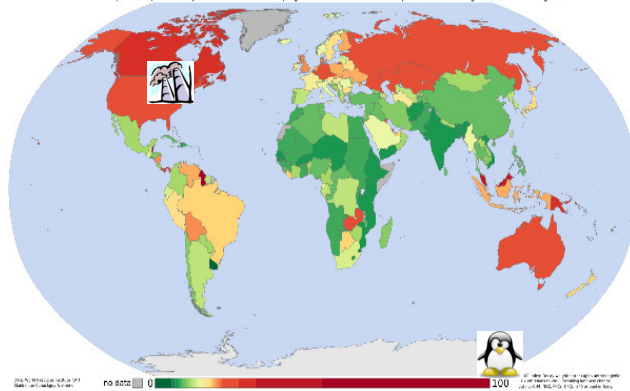
...or an

environmentalist.

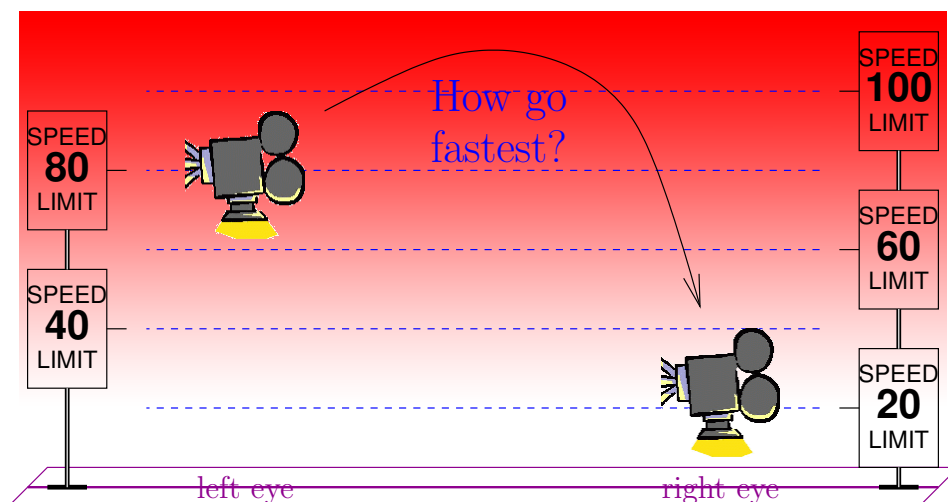


Al Gore in Futurama, circa 3000AD

Per capita responsibility for current anthropogenic CO2 in the atmosphere (including land-use change)



Goal. Find the least-blur path to go from Mona's left eye to Mona's right eye in fixed time. Alternatively, fix your blur-tolerance, and find the fastest path to do the same. For fixed blur, our camera moves at a speed proportional to its distance from the image plane:



Small print on giving talks. • Use lots of pictures! • Avoid traditional slides! • Make a hand-out! The handout *is* the talk, not just besides the talk. • You must know your subject in and out. • Prepare! I almost never "write my talk the night before", and often start weeks or months in advance. • Your talk must tell an interesting story. Choose your research so that it would. Then *tell a story*. • Don't be shy, tape yourself. • Classes? Classes are different.

Joint study with
Itai Bar-Natan

Non-Commutative Gaussian Elimination and Rubik's Cube

The Problem. Let $G = (g_1, \dots, g_n)$ be a subgroup of S_n , with $n = O(100)$. Before you die, understand G :

1. Compute $|G|$.
2. Given $\sigma \in S_n$, decide if $\sigma \in G$.
3. Write a $\sigma \in G$ in terms of g_1, \dots, g_n .
4. Produce random elements of G .

The Commutative Analog. Let $V = \text{span}(v_1, \dots, v_n)$ be a subspace of \mathbb{R}^n . Before you die, understand V .

Solution: Gaussian Elimination. Prepare an empty table.

1	2	3	4	...	n-1	n
---	---	---	---	-----	-----	---

Space for a vector $u_i \in V$, of the form $u_i = (0, 0, 0, 1, *, \dots, *)$; $1 = \text{"the pivot"}$

Feed v_1, \dots, v_n in order. To feed a non-zero v , find its pivotal position i .

1. If box i is empty, put v there.
2. If box i is occupied, find a combination v' of v and u_i that eliminates the pivot, and feed v' .

Non-Commutative Gaussian Elimination
Prepare a mostly-empty table.

1,1			
1,2	2,2		
1,3	2,3	3,3	
...
1,n	2,n	3,n	...
			n,n

Space for a $\sigma_{i,j} \in S_n$ of the form $(1, 2, \dots, i-2, i-1, j, *, *, \dots, *)$
So $\sigma_{i,j}$ fixes $1, \dots, i-1$,
sends "the pivot" i to j and goes wild afterwards, and $\sigma_{i,j}^{-1}$ "does sticker".

Feed g_1, \dots, g_n in order. To feed a non-identity σ , find its pivotal position i and let $j := \sigma(i)$.

1. If box (i, j) is empty, put σ there.
2. If box (i, j) contains $\sigma_{i,j}$, feed $\sigma' := \sigma_{i,j}^{-1}\sigma$.

The Twist. When done, for every occupied (i, j) and (k, l) , feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

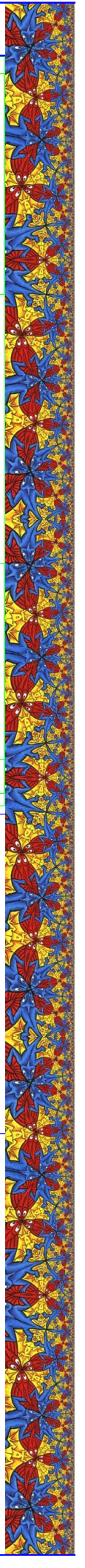
Claim. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table T .

Claim. Anything fed in T is a monotone product in T :
 f was fed $\Rightarrow f \in M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\dots\sigma_{n,j_n} : \forall i, j_i \geq i \ \& \ \sigma_{i,j_i} \in T\}$

Homework Problem 1. Can you do cosets?

Homework Problem 2. Can you do categories (groupoids)?

The Results
 $\text{In}[3] := (\text{Feed}[*]; \text{Product}[1 * \text{Length}[\text{Select}[\text{Range}[n], \text{Head}[s[i], \#]] == *], \{i, n\}]] \ \& \ / \ \& \ \text{ge}$
 $\text{Out}[3] := \{4, 16, 159993501696000, 21119142223872000, 43252003274498986000, 43252003274498986000\}$
<http://www.math.toronto.edu/~drorbn/Talks/Mathcamp-0907/> and links there



The Hardest Math I've Ever Really Used, 2

Picture credits: Mona: Leonrado; Al Gore: Futurama; Map 1: en.wikipedia.org/wiki/Greenhouse_gas; Smokestacks: gbuaped.org/complaint.htm; Penguin: brentpabst.com/bp/2007/12/15/BrentGoesPenguin.asp; Map 2: flightpedia.org; Segway: co2calculator.wordpress.com/2008/10; Lobachevsky: en.wikipedia.org/wiki/Nikolai_Lobachevsky; Eschiers: www.josleys.com/show_gallery.php?galid=325;

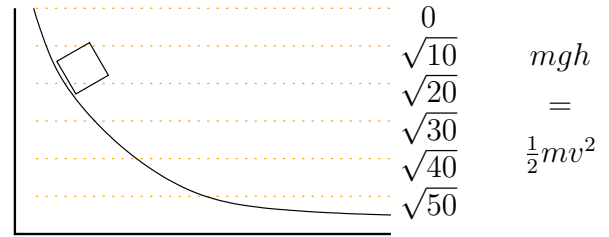


Fermat's Principle

$c \sim 300,000$
 $c \sim 250,000$



The Brachistochrone



Bernoulli on Newton. "I recognize the lion by his paw".

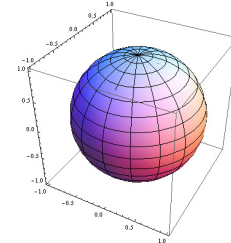
Flatlanders airline route map



The Least Action Principle. Everywhere in physics, a system goes from A to B along the path of least action.

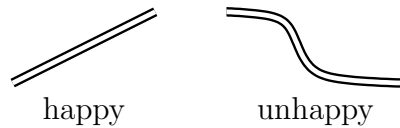
With small print for quantum mechanics.

```
ParametricPlot3D[
  Sin[u] Cos[v],
  Sin[u] Sin[v],
  Cos[u]
], {u, 0, pi}, {v, 0, 2pi}
```

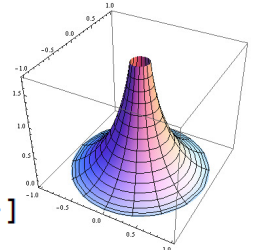


The Happy Segway Principle

A Segway is happy iff both its wheels are

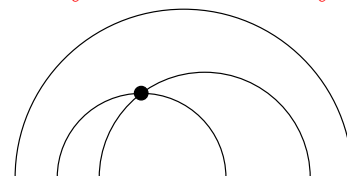


```
ParametricPlot3D[
  Sech[u] Cos[v],
  Sech[u] Sin[v],
  u - Tanh[u]
], {u, 0, e}, {v, 0, 2pi}
```



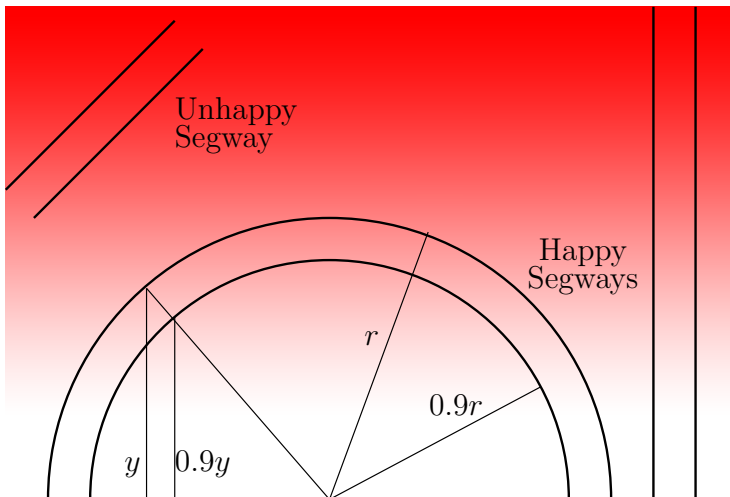
The Bolyai-Lobachevsky Plane

Two parallels through one point



Further Fun Facts. • In small scale, $\pi^H \rightarrow \pi^E$. In large scale, $\pi^H \rightarrow \infty$. • The sum of the angles of a triangle is always less than π . In fact, $\text{sum} + \text{area} = \pi$, so the largest possible area of a triangle is π . • If your friend walks away, she'll drop out of sight before you know it. • There are so many places just a stone throw away! But you'd better remember your way back well!

Happy camera-carrying Segways above the Mona Plane



The Mona Plane



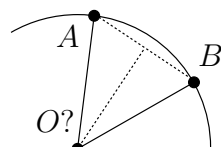
The Actual Code

```
p3.y = p2.y + b*x3p;
x = p1.x-p2.x; y = p1.y-p2.y;
d1 = p1.d; d2 = p2.d;
norm = sqrt(x*x + y*y);
a = x/norm; b = y/norm;
x1p = a*x + b*y;
x0 = (x1p + (d1*d1-d2*d2)/x1p)/2;
r = sqrt((x1p-x0)*(x1p-x0)+d1*d1);
x1pp = (x1p-x0)/r; x2pp = -x0/r;
theta1 = acos(x1pp);
theta2 = acos(x2pp);
t1 = log(tan(theta1/2));
t2 = log(tan(theta2/2));
t3 = t1 + s*(t2-t1);
theta3 = 2*atan(exp(t3));
x3pp = cos(theta3);
d3pp = sin(theta3);
x3p.d = x0 + r*x3pp;
p3.d = r*d3pp;
p3.x = p2.x + a*x3p;
```

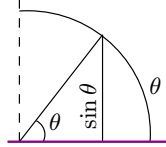
Ops used. +, -, x, /, sqrt, cos, sin, tan, arccos, arctan, log, exp.

Some further basic geometry also occurs:

Finding the Centre



Parametrization



$$\theta'(t) = \sin \theta(t)$$

$$\Downarrow$$

$$\theta = 2 \arctan e^t$$

