Toronto-1609 handout on Sep 21

September 21, 2016 9:33 PM

Work in Progress! **A Poly-Time Knot Polynomial Via Solvable Approximation**

Abstract. I will construct the first poly-time-computable knot polynomial since Alexander's [Al, 1928] by using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.

Expected!

~~Why "expected"?~~ Finite-type invariants include all coefficients of all quantum knot polynomials (appropriately parametrized), and each is computable in poly-time. Yet

d	2	3	4	5	6	7	8	…
known f.t. invts in $O(n^d)$	1	1	∞	3	4	8	11	…

This is an unreasonable picture! So there ought to be further poly-time polynomial invariants.

Also. • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral. Rozansky

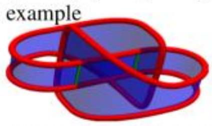
Paradise!

~~Why "paradise"?~~ Foremost ~~answer~~ reason: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Secondary ~~answer~~ reason: may disprove {ribbon} = {slice}: (see [BN2])

a ribbon singularity a clasp singularity

example

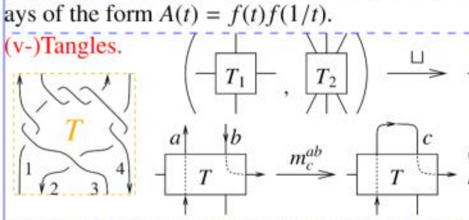


A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

(v-)Tangles.



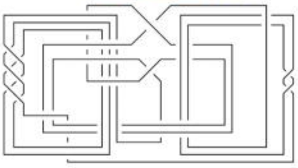
(meta-associativity: $m_x^{ab} /\!/ m_y^{xc} = m_x^{bc} /\!/ m_y^{ax}$)

Gompf, Scharlemann, Thompson [GST]

$U \in \mathcal{T}_n$ $\quad 1 \in \mathcal{A}_n$

$\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$ (with τ and κ arrows) with $\mathcal{R} := \kappa(\tau^{-1}(1))$

ribbon $K \in \mathcal{T}_1$ $\quad z(K) \in \mathcal{R} \subseteq \mathcal{A}_1$



Faster is better, leaner is meaner!

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a : a \in S\})$:

$$\left({}_a\nearrow_b,\ {}_b\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m_c^{ab}} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta}\end{array}\right)$$

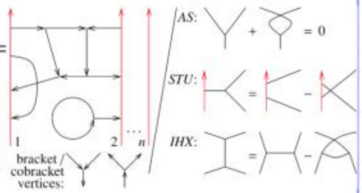
For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!

Dunfield: 1000-crossing fast.

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$Z\colon \left\{\begin{matrix} S\text{-component} \\ (v/b\text{-})\text{tangles}\end{matrix}\right\} \to \mathcal{A}_S^v :=$

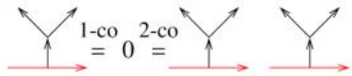
Etingof Kazhdan Haviv Enriquez Ševera

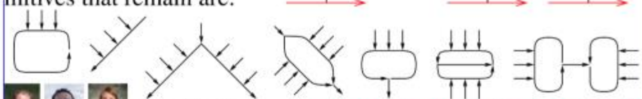


(it is enough to know Z on ⤫ and have disjoint union and stitching formulas) …exponential and too hard!

Idea. Look for "ideal" quotients of $\mathcal{A}_S^v$ that have poly-sized descriptions; …specifically, limit the co-brackets.

1-co and 2-co, aka TC and TC^2, on the right. The primitives that remain are:



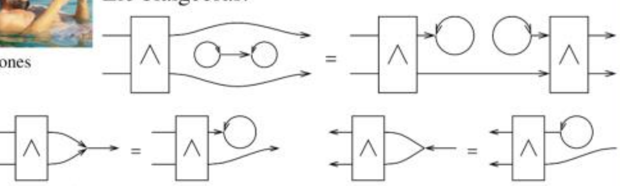


Alekseev, Torossian, Dancso

…manageable but still exponential!

Jones

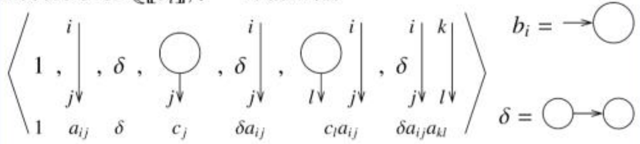
The 2D relations come from the relation with 2D Lie bialgebras:



We let $\mathcal{A}^{2,2}$ be $\mathcal{A}^v$ modulo 2-co and $2D$, and $z^{2,2}$ be the projection of $\log Z$ to $\mathcal{P}^{2,2} := \pi\mathcal{P}^v$, where $\mathcal{P}^v$ are the primitives of $\mathcal{A}^v$.

Main Claim. $z^{2,2}$ is poly-time computable.

Main Point. $\mathcal{P}^{2,2}$ is poly-size, so how hard can it be? Indeed, as a module over $\mathbb{Q}[\![b_i]\!]$, $\mathcal{P}^{2,2}$ is at most



Claim. $R_{jk} = e^{a_{jk}} e^{\rho_{jk}}$ is a solution of the Yang-Baxter / R3 equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in $\exp \mathcal{P}^{2,2}$, with $\rho_{jk} :=$

$$\psi(b_j)\left(-c_k + \frac{c_k a_{jk}}{b_j} - \frac{\delta a_{jk} a_{jk}}{b_j^2}\right) + \frac{\phi(b_j)\psi(b_k)}{b_k \phi(b_k)}\left(c_k a_{kk} - \frac{\delta a_{jk} a_{kk}}{b_j}\right),$$

and with $\phi(x) := e^{-x} - 1 = -x + x^2/2 - \ldots$, and $\psi(x) := ((x+2)e^{-x} - 2 + x)/(2x) = x^2/12 - x^3/24 + \ldots$.

But how do we multiply in $\exp(\mathcal{P}^{2,2})$? How do we stitch?

~~Help Needed! Disorganized~~ videos of talks in a private seminar ~~are~~ at ωεβ/PP.

Many thanks: Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

4-hour version of this talk at web/LD.

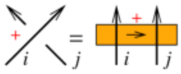

van der Veen

1-Smidgen sl_2 (with van der Veen). Let $\mathfrak{g}_1$ be the 4D Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over $\mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and $[w,c] = w$, $[c,u] = u$, and $[u,w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset 0$. In a certain sense, $\mathfrak{g}_1$ is more valuable than sl_2. (note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon = 0$, or $\mathbb{Q}\langle b,c,u,w\rangle/([b,\cdot] = 0, [c,u] = u, [c,w] = -w, [u,w] = b$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{a}^* \rtimes \mathfrak{a}$ where $\mathfrak{a}$ is the 2D Lie algebra $\mathbb{Q}\langle b, u\rangle$ and (c, w) is the dual basis of (b, u). It is even more valuable than $\mathfrak{g}_1$, but topology already got by other means almost everything $\mathfrak{g}_0$ has to give.

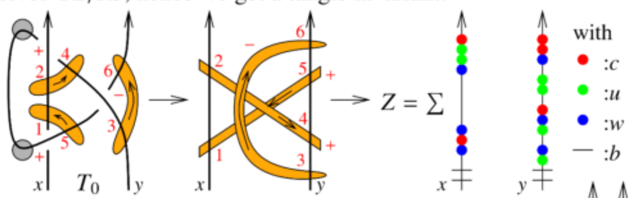
How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w\rangle/[w,c] = w$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (c, w) \mapsto (0, c\wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get $\mathfrak{g}_0$. At $k = 1$, get $[w,c] = w$, $[w,b'] = -\epsilon w$, $[c,u] = u$, $[b',u] = -\epsilon u$, $[b',c] = 0$, and $[u,w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is $\mathfrak{g}_1$.

0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,

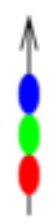


$$= R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$$

solves YB/R3, hence we get a tangle invariant:



Goal. Sort Z to be as on the right, with $f_k \in \mathbb{Q}[\![b_i]\!]$. $\sum_k f_k$



Better, with $\zeta \in \mathbb{Q}[\![b_x, c_x, u_x, w_x, b_y, c_y, u_y, w_y]\!]$, write

$$Z = \mathbb{O}\left(\zeta | x\colon c_x u_x w_x,\ y\colon c_y u_y w_y\right) \qquad \text{(cuw form)}$$

Here $\mathbb{O}\,(poly \mid specs)$ plants the variables of $poly$ in $\mathcal{S}(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,

$$\mathbb{O}\left(c_1^3 u_1 c_2 e^{u_3} w_3^9 | x\colon w_3 c_1,\ y\colon u_1 u_3 c_2\right) = w^9 c^3 \otimes u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$$

Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \mathbb{O}\left(\exp\left(b_i c_j + \frac{e^{b_i}-1}{b_i} u_i w_j\right) | i\colon u_i,\ j\colon c_j w_j\right)$ $= \sum_{m,n} \frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!} u^n \otimes c^m w^m$.

Example. $Z(T_0) =$

$$\mathbb{O}\left(1 \exp\left(b_5 c_1 + \frac{e^{b_5}-1}{b_5} u_5 w_1 + b_2 c_4 + \frac{e^{b_2}-1}{b_2} u_2 w_4 - b_3 c_6 + \frac{e^{-b_3}-1}{b_3} u_3 w_6\right) \middle| x\colon c_1 w_1 u_2,\ y\colon u_3 c_4 w_4 u_5 c_6 w_6\right)$$

$$= \mathbb{O}\left(? | x\colon c_x u_x w_x,\ y\colon c_y u_y w_y\right)$$

$\mathbb{O}(\omega e^{L+Q})$: L bilinear in b_i and c_i, and Q a balanced quadratic in u_i and w_i with coefficients in $\mathbb{Q}(b_i, e^{b_i}) \ni \omega$. "Admissible"

The Big $\mathfrak{g}_0$ Lemma. Under $[c,u] = u$, $[c,w] = -w$, and $[u,w] = b$:

1. $N_k^{c_i c_j} := \mathbb{O}(\zeta | c_i c_j) \stackrel{\rightarrow}{=} \mathbb{O}(\zeta/(c_i, c_j \to c_k) | c_k)$ (Meaning, $N_k^{c_i c_k}\colon \zeta \mapsto (\zeta/(c_i, c_j \to c_k))$ and the diagram commutes. Trivial, also for b, u, w.)

$$\begin{tikzcd} \mathcal{S} \arrow[r, "N_k^{c_i c_j}"] \arrow[dr, "\mathbb{O}(-|c_i c_j)"'] & \mathcal{S} \arrow[d, "\mathbb{O}(-|c_k)"] \\ & \mathcal{U} \end{tikzcd}$$

2a. $N^{uc} := \mathbb{O}(e^{\gamma c + \beta u} | uc) \stackrel{\rightarrow}{=} \mathbb{O}(e^{\gamma c + e^{-\gamma}\beta u} | cu)$ (means $e^{\beta u} e^{\gamma c} = e^{\gamma c} e^{e^{-\gamma}\beta u}$

2b. $N^{wc} := \mathbb{O}(e^{\gamma c + \alpha w} | wc) \stackrel{\rightarrow}{=} \mathbb{O}(e^{\gamma c + e^{\gamma}\alpha w} | cw)$ …in the $\{ax + b\}$ group)

3. $\mathbb{O}(e^{\alpha w + \beta u} | wu) = \mathbb{O}(e^{-b\alpha\beta + \alpha w + \beta u} | uw)$ (the Weyl relations)

4. $\mathbb{O}(e^{\delta uw} | wu) e^{\beta u} = e^{\nu\beta u} \mathbb{O}(e^{\delta uw} | wu)$, with $\nu = (1 + b\delta)^{-1}$

(a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)

5. $\mathbb{O}(e^{\delta uw} | wu) = \mathbb{O}(\nu e^{\nu\delta uw} | uw)$ (same techniques)

6. $N^{wu} := \mathbb{O}(e^{\beta u + \alpha w + \delta uw} | wu) \stackrel{\rightarrow}{=} \mathbb{O}(\nu e^{-b\nu\alpha\beta + \nu\alpha w + \nu\beta u + \nu\delta uw} | uw)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition

$$c_i u_i \overline{w_i c_j} u_j w_j \xrightarrow{N_k^{w_i c_j}} c_i \overline{u_i c_k} \overline{w_k u_j} w_j \xrightarrow{N_k^{u_i c_k} /\!/ N_k^{w_k u_j}} \overline{c_i c_k} \overline{u_k u_k} \overline{w_k w_j} \xrightarrow{N_k^{c_i c_k} /\!/ -- /\!/ N_k^{w_k w_j}} c_k u_k w_k$$

1-Smidgen Invariants. Much is the same:

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 2 are the same, yet

6. $\mathbb{O}\left(e^{\alpha w + \beta u + \delta uw} | wu\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\Lambda) e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} | cuw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:

$$\begin{aligned}\Lambda = &- b\nu\left(\nu^2\alpha^2\beta^2 + 4\delta\nu\alpha\beta + 2\delta^2\right)/2 - \delta\nu^3(3b\delta + 2)\beta^2 u^2/2 \\ &- b\delta^4\nu^3 u^2 w^2/2 - \delta^2\nu^3(2b\delta + 1)\beta u^2 w \\ &- \nu^2(2b\delta + 1)(\nu\alpha\beta + 2\delta)\beta u - 2b\delta^2\nu^2(\nu\alpha\beta + \delta)uw \\ &+ \delta\nu^3(b\delta + 2)\alpha^2 w^2/2 + 2(\nu\alpha\beta + \delta)c + 2\delta\nu\beta cu + 2\delta^2\nu cuw \\ &+ 2\delta\nu\alpha cw + \delta^2\nu^3\alpha u w^2 + \nu^2(\nu\alpha\beta + 2\delta)\alpha w.\end{aligned}$$

Proof. A brutal hell.

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c,u) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + e^{-\gamma}\beta u} | cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u,w) e^{\alpha w + \beta u + \delta uw} | wu\right) = \mathbb{O}\left(\epsilon P(\partial_\beta, \partial_\alpha) \nu e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} | cuw\right)$$

Note. Strand stitching requires a tiny extra step.

Finally, the values of the generators ⤫, ⤪, $\overrightarrow{n}$, $\overleftarrow{n}$, $\underset{\rightarrow}{u}$, and $\underset{\leftarrow}{u}$, are set by brutally solving many equations, non-uniquely.

Pragmatic Simplifications. Get rid of $\zeta = (e^b - 1)/b$ factors by rescaling $u \to \bar{u} = \zeta u$. Complement this with $\beta \to \bar{\beta} = \zeta^{-1}\beta$, $\delta \to \bar{\delta} = \zeta^{-1}\delta$, $\epsilon \to \bar{\epsilon} = \zeta^{-1}\epsilon$. Simplify further by naming $e^b \to t$; e.g., $\nu \to \bar{\nu} = (1 + (t-1)\delta)^{-1}$. Get confused by renaming $(\bar{u}, \bar{\beta}, \bar{\delta}, \bar{\nu}) \to (u, \beta, \delta, \nu)$, and more confused by working with $\mu = \nu^{-1}$ and $\mathbb{E}(\omega, L, Q, P) := \omega^{-1}(1 + \epsilon\omega^{-4}P)e^{L + \omega^{-1}Q}$, where $\omega \in R := \mathbb{Q}(t_k)$, $L = \sum l_{ij} b_i c_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij} u_i w_j$ with $q_{ij} \in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

Rough complexity estimate, after $t_k \to t$. n: xing number; w: width, maybe $\sim \sqrt{n}$.

$$\underset{A}{\underline{n}} \overbrace{\sum_{d=0}^{4} \underset{E}{\underline{w^{4-d}}}\, \underset{F}{\underline{w^d}}\, \underset{G}{\underline{n^2}}}^{B} = n^3 w^4 \in [n^5, n^7]$$

A: go over stitchings in order. B: multiplication ops per $N^{u_i w_j}$. d: deg of u_i, w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Expectation. Our invariant is the "1-higher diagonal" in the MMR expansion of the coloured Jones polynomial J_λ.


Melvin, Morton, Garoufalidis

Theorem ([BNG], conjectured [MM], elucidated [Ro]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of $sl(2)$. Writing

$$\left.\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot A(K)(e^\hbar) = 1$.

Demo Programs for 0-Co. ωεβ/Demo

The R-matrices

$$R^{+}_{\theta,i\_,j\_} := \mathbb{E}\left[b_i\, c_j + b_i^{-1}\left(e^{b_i}-1\right) u_i\, w_j\right];$$
$$R^{-}_{\theta,i\_,j\_} := \mathbb{E}\left[-b_i\, c_j + b_i^{-1}\left(e^{-b_i}-1\right) u_i\, w_j\right];$$

Utilities

$$\mathtt{CF}[\omega\_ .\ \mathbb{E}[Q\_]] := \mathtt{Simplify}[\omega]\ \mathbb{E}[\mathtt{Simplify}[Q]];$$
$$\mathbb{E}\ /:\ \mathbb{E}[Q1\_]\ \mathbb{E}[Q2\_] := \mathtt{CF@}\mathbb{E}[Q1+Q2];$$
$$\omega 1\_ .\ \mathbb{E}[Q1\_] \equiv \omega 2\_ .\ \mathbb{E}[Q2\_] := \mathtt{Simplify}[\omega 1 == \omega 2 \wedge Q1 == Q2];$$

Normal Ordering Operators

$$N_{u_{i\_}\,c_{j\_}\to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \mathtt{CF}\big[\omega\,\mathbb{E}[e^{-\gamma}\beta\, u_k + \gamma\, c_k + (Q\ /.\ c_j \mid u_i \to 0)]\ /.\ \{\gamma \to \partial_{c_j} Q,\ \beta \to \partial_{u_i} Q\}\big];$$
$$N_{w_{i\_}\,c_{j\_}\to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \mathtt{CF}\big[\omega\,\mathbb{E}[e^{\gamma}\alpha\, w_k + \gamma\, c_k + (Q\ /.\ c_j \mid w_i \to 0)]\ /.\ \{\gamma \to \partial_{c_j} Q,\ \alpha \to \partial_{w_i} Q\}\big];$$
$$\begin{aligned}N_{w_{i\_}\,u_{j\_}\to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \mathtt{CF}\big[&\nu\,\omega\,\mathbb{E}[-b_k\,\nu\,\alpha\,\beta + \nu\,\beta\, u_k + \nu\,\delta\, u_k\, w_k + \nu\,\alpha\, w_k + (Q\ /.\ w_i \mid u_j \to 0)]\ /.\\ &\nu \to (1+b_k\,\delta)^{-1}\ /.\\ &\{\alpha \to \partial_{w_i} Q\ /.\ u_j \to 0,\ \beta \to \partial_{u_j} Q\ /.\ w_i \to 0,\ \delta \to \partial_{w_i,u_j} Q\}\big];\end{aligned}$$

Stitching

$$\begin{aligned}m_{i\_,j\_\to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \mathtt{CF}\big[\mathtt{Module}\big[\{x\},\ &\big(\omega\,\mathbb{E}[Q]\ /.\ b_{i|j}\to b_k\ //\ N_{w_i c_j\to x}\ //\ N_{u_i c_x\to x}\ //\ N_{w_x u_j\to x}\big)\ /.\\ &\{c_i \to c_k,\ w_j \to w_k,\ y\_{x} \mapsto y_k\}\big]\big]\end{aligned}$$

Some calculations for T_0

$$T_{\theta,\theta} = R^{+}_{\theta,5,1}\, R^{+}_{\theta,2,4}\, R^{-}_{\theta,3,6}$$
$$\mathbb{E}\left[b_5\, c_1 + b_2\, c_4 - b_3\, c_6 + \frac{(-1+e^{b_5})\, u_5\, w_1}{b_5} + \frac{(-1+e^{b_2})\, u_2\, w_4}{b_2} + \frac{(-1+e^{-b_3})\, u_3\, w_6}{b_3}\right]$$
$$T_{\theta,1} = T_{\theta,\theta}\ //\ N_{u_3\, c_4\to 4}$$
$$\mathbb{E}\left[b_5\, c_1 + b_2\, c_4 - b_3\, c_6 + \frac{(-1+e^{b_5})\, u_5\, w_1}{b_5} + \frac{(-1+e^{b_2})\, u_2\, w_4}{b_2} + \frac{e^{-b_2}(-1+e^{-b_3})\, u_4\, w_6}{b_3}\right]$$
$$T_{\theta,2} = T_{\theta,1}\ //\ N_{w_4\, u_5\to 4}$$
$$\mathbb{E}\left[b_5\, c_1 + b_2\, c_4 + \frac{(-1+e^{b_5})\left(-(-1+e^{b_2})\, b_4\, u_2 + b_2\, u_4\right) w_1}{b_2\, b_5} + \frac{(-1+e^{b_2})\, u_2\, w_4}{b_2} - \frac{b_3^2\, c_6 + e^{-b_2-b_3}(-1+e^{b_3})\, u_4\, w_6}{b_3}\right]$$
$$T_{\theta,2}\ //\ N_{w_1\, u_2\to 1}$$
$$\begin{aligned}\frac{1}{1-\frac{(-1+e^{b_2})(-1+e^{b_5})\, b_1\, b_4}{b_2\, b_5}}\ \mathbb{E}\Big[&\frac{1}{b_3\left((-1+e^{b_2})(-1+e^{b_5})\, b_1\, b_4 - b_2\, b_5\right)}\\ &\big(b_3\, b_5\left((-1+e^{b_2})(-1+e^{b_5})\, b_1\, b_4 - b_2\, b_5\right) c_1 +\\ &\ b_2\, b_3\left((-1+e^{b_2})(-1+e^{b_5})\, b_1\, b_4 - b_2\, b_5\right) c_4 +\\ &\ (-1+e^{b_2})(-1+e^{b_5})\, b_3\, b_4\, u_1\, w_1 - (-1+e^{b_5})\, b_2\, b_3\, u_4\, w_1 -\\ &\ (-1+e^{b_2})\, b_3\, b_5\, u_1\, w_4 + (-1+e^{b_2})(-1+e^{b_5})\, b_1\, b_3\, u_4\, w_4 -\\ &\ \left((-1+e^{b_2})(-1+e^{b_5})\, b_1\, b_4 - b_2\, b_5\right)\\ &\quad \left(b_3^2\, c_6 + e^{-b_2-b_3}(-1+e^{b_3})\, u_4\, w_6\right)\big)\Big]\end{aligned}$$
$$T_{\theta,\theta}\ //\ m_{1,2\to 1}\ //\ m_{3,4\to 3}\ //\ m_{3,5\to 3}\ //\ m_{3,6\to 3}$$
$$\begin{aligned}\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\ \mathbb{E}\Big[&b_3\, c_1 + b_1\, c_3 - b_3\, c_3 +\\ &\frac{(-1+e^{b_1})(-1+e^{b_3})\, u_1\, w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\, b_1} - \frac{e^{-b_3}(-1+e^{b_1})\left(b_3\, u_1 - e^{b_3}(-1+e^{b_3})\, b_1\, u_3\right) w_3}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\, b_1\, b_3} +\\ &\frac{e^{-b_1}(-1+e^{b_3})\, u_3\left(-e^{b_1+b_3}\, w_1 + (e^{b_1}+e^{b_3}-e^{b_1+b_3})\, w_3\right)}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\, b_3}\Big]\end{aligned}$$

Verifying meta-associativity

$$Q0 = \mathbb{E}[\mathtt{Sum}[f_i\, c_i,\ \{i,3\}] + \mathtt{Sum}[f_{i,j}\, u_i\, w_j,\ \{i,3\},\ \{j,3\}]]$$
$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$
$$(Q0\ //\ m_{1,2\to 1}\ //\ m_{1,3\to 1}) \equiv (Q0\ //\ m_{2,3\to 2}\ //\ m_{1,2\to 1})$$

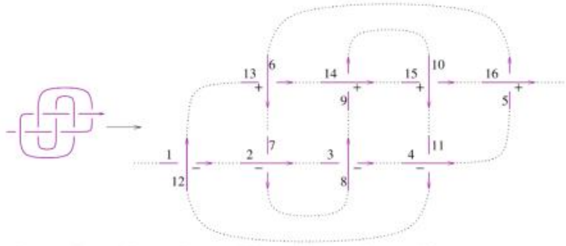
True

Testing R3

$$t1 = R^{+}_{\theta,1,2}\, R^{+}_{\theta,3,4}\, R^{+}_{\theta,5,6}\ //\ m_{3,5\to x}\ //\ m_{1,6\to y}\ //\ m_{2,4\to z}$$
$$\mathbb{E}\left[b_x\,(c_y + c_z) + \frac{(-1+e^{b_x})\, u_x\,(w_y + w_z)}{b_x} + \frac{b_y^2\, c_z + (-1+e^{b_y})\, u_y\, w_z}{b_y}\right]$$
$$t1 \equiv (R^{+}_{\theta,1,2}\, R^{+}_{\theta,3,4}\, R^{+}_{\theta,5,6}\ //\ m_{1,3\to x}\ //\ m_{2,5\to y}\ //\ m_{4,6\to z})$$

True

8_{17}



$$z1 = R^{-}_{\theta,12,1}\, R^{-}_{\theta,2,7}\, R^{-}_{\theta,8,3}\, R^{-}_{\theta,4,11}\, R^{+}_{\theta,16,5}\, R^{+}_{\theta,6,13}\, R^{+}_{\theta,14,9}\, R^{+}_{\theta,10,15};$$
$$\mathtt{Do}[z1 = (z1\ //\ m_{1,n\to 1})\ /.\ b\_ \to b,\ \{n, 2, 16\}];$$
$$\{\mathtt{CF@}z1,\ \mathtt{KnotData}[\{8, 17\},\ \text{"AlexanderPolynomial"}][t]\}$$
$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$$

Demo Programs for 1-Co. ωεβ/Demo

The Λόγος

$$\begin{aligned}\Lambda[k\_] := {}&(1-t_k)\left(\alpha^2\beta^2 + 4\,\alpha\,\beta\,\delta\,\mu + 2\,\delta^2\mu^2\right)/2 + 2\,\mu^2\,(\alpha\,\beta + \delta\,\mu)\, c_k -\\ &\beta\,(2\mu-1)\,(\alpha\,\beta + 2\,\delta\,\mu)\, u_k + 2\,\beta\,\delta\,\mu^2\, c_k\, u_k - \beta^2\,\delta\,(3\mu-1)\, u_k^2/2 +\\ &\alpha\,(\alpha\,\beta + 2\,\delta\,\mu)\, w_k + 2\,\alpha\,\delta\,\mu^2\, c_k\, w_k - 2\,(t_k-1)\,\delta^2\,(\alpha\,\beta + \delta\,\mu)\, u_k\, w_k +\\ &2\,\delta^2\mu^2\, c_k\, u_k\, w_k - \beta\,\delta^2\,(2\mu-1)\, u_k^2\, w_k + \alpha^2\,\delta\,(1+\mu)\, w_k^2/2 +\\ &\alpha\,\delta^2\, u_k\, w_k^2 - (t_k-1)\,\delta^4\, u_k^2\, w_k^2/2;\end{aligned}$$

Differential Polynomials

$$\begin{aligned}DP_{x\_\to D_{\alpha\_},\,y\_\to D_{\beta\_}}[P\_][f\_] := {}&(\ast\ \text{means } P[\partial_\alpha, \partial_\beta][f]\ \ast)\\ &\mathtt{Total}[\mathtt{CoefficientRules}[P,\ \{x, y\}]\ /.\\ &\quad (\{m\_, n\_\} \to c\_) \mapsto c\, D[f,\ \{\alpha, m\},\ \{\beta, n\}]]\end{aligned}$$

Utilities

$$\begin{aligned}\mathtt{CF}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := {}&\mathtt{Expand\ /@\ Together\ /@}\\ &\mathbb{E}[\omega\ /.\ b_{l\_} \mapsto \mathtt{Log}[t_l],\ L,\ Q\ /.\ b_{l\_} \mapsto \mathtt{Log}[t_l],\\ &\quad P\ /.\ b_{l\_} \mapsto \mathtt{Log}[t_l]];\end{aligned}$$
$$\begin{aligned}\mathbb{E}\ /:\ \mathbb{E}[\omega 1\_, L1\_, Q1\_, P1\_]\ \mathbb{E}[\omega 2\_, L2\_, Q2\_, P2\_] :=\\ \mathtt{CF@}\mathbb{E}\left[\omega 1\,\omega 2,\ L1 + L2,\ \omega 2\, Q1 + \omega 1\, Q2,\ \omega 2^4\, P1 + \omega 1^4\, P2\right];\end{aligned}$$

Normal Ordering Operators

$$\begin{aligned}N_{u_{i\_}\,c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathtt{With}\big[&\{q = e^{-\gamma}\beta\, u_k + \gamma\, c_k\},\ \mathtt{CF}\big[\\ &\mathbb{E}\big[\omega,\ \gamma\, c_k + (L\ /.\ c_j \to 0),\ \omega\, e^{-\gamma}\beta\, u_k + (Q\ /.\ u_i \to 0),\\ &\quad e^{-q}\, DP_{c_j\to D_\gamma,\,u_i\to D_\beta}[P][e^q]\big]\ /.\ \{\gamma \to \partial_{c_j} L,\ \beta \to \omega^{-1}\partial_{u_i} Q\}\big]\big];\end{aligned}$$
$$\begin{aligned}N_{w_{i\_}\,c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathtt{With}\big[&\{q = e^{\gamma}\alpha\, w_k + \gamma\, c_k\},\ \mathtt{CF}\big[\\ &\mathbb{E}\big[\omega,\ \gamma\, c_k + (L\ /.\ c_j \to 0),\ \omega\, e^{\gamma}\alpha\, w_k + (Q\ /.\ w_i \to 0),\\ &\quad e^{-q}\, DP_{c_j\to D_\gamma,\,w_i\to D_\alpha}[P][e^q]\big]\ /.\ \{\gamma \to \partial_{c_j} L,\ \alpha \to \omega^{-1}\partial_{w_i} Q\}\big]\big];\end{aligned}$$
$$\begin{aligned}&N_{w_{i\_}\,u_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] :=\\ &\mathtt{With}\big[\{q = (1-t_k)\,\mu^{-1}\alpha\,\beta + \mu^{-1}\beta\, u_k + \mu^{-1}\delta\, u_k\, w_k + \mu^{-1}\alpha\, w_k\},\ \mathtt{CF}\big[\\ &\quad \mathbb{E}\big[\mu\,\omega,\ L,\ \mu\,\omega\, q + \mu\,(Q\ /.\ w_i \mid u_j \to 0),\\ &\qquad \mu^4\, e^{-q}\, DP_{w_i\to D_\alpha,\,u_j\to D_\beta}[P][e^q] + \omega^4\,\Lambda[k]\big]\ /.\\ &\qquad \mu \to 1 + (t_k - 1)\,\delta\ /.\\ &\qquad \{\alpha \to \omega^{-1}(\partial_{w_i} Q\ /.\ u_j \to 0),\ \beta \to \omega^{-1}(\partial_{u_j} Q\ /.\ w_i \to 0),\\ &\qquad \delta \to \omega^{-1}\partial_{w_i,u_j} Q\}\big]\big];\end{aligned}$$

Stitching

$$\begin{aligned}m_{i\_,j\_\to k\_}[Z\_] := \mathtt{Module}\big[&\{x, y, z\},\\ &Z\ //\ N_{w_i c_j\to x}\ //\ N_{w_x u_j\to y}\ //\ \mathtt{ReplaceAll}[\{c_{x|y} \to c_x,\ w_j \to w_y\}]\ //\\ &\quad N_{u_i c_x\to x}\ //\ \mathtt{ReplaceAll}[z\_{i|j|x|y} \to z_k]\ //\ \mathtt{CF}\big]\end{aligned}$$

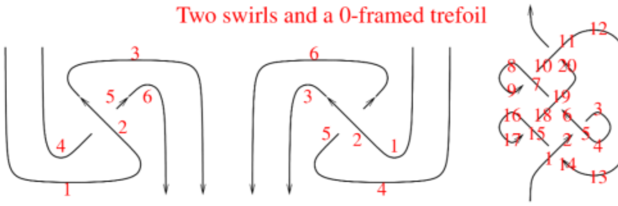
The Generators

$$R^{+}_{i\_,j\_} := \mathbb{E}\big[1,\ b_i c_j,\ u_i w_j,$$
$$-c_i (t_i-1)^2/2 - c_i^2 (t_i-1)^2/2 + c_i c_j \left(t_j^2 - t_i - 2\right)/2 -$$
$$c_j u_i w_i/2 + c_i (1-t_i) u_i w_i - u_i^2 w_i^2/2 + u_i w_j + c_j t_i u_i w_j/2 +$$
$$c_i (t_i-2) t_i u_i w_j + c_i (1+t_j) u_j w_j/2 + (t_i-1) u_i^2 w_i w_j -$$
$$(t_i-2) t_i u_i^2 w_j^2/2\big];$$

$$R^{-}_{i\_,j\_} := \mathbb{E}\big[1,\ -b_i c_j,\ -t_i^{-1} u_i w_j,$$
$$c_i (t_i-1)^2/2 + c_i^2 (t_i-1)^2/2 + c_i c_j \left(2 + t_i - t_j^2\right)/2 +$$
$$c_j u_i w_i/2 + c_i (t_i-1) u_i w_i + u_i^2 w_i^2/2 + \left(1 - t_i^{-1}\right) u_i w_j/2 +$$
$$c_i \left(2 t_i - 5 + 3 t_i^{-1}\right) u_i w_j/2 + c_j \left(t_i^{-1} + 1 - t_i^{-1} t_j^2\right) u_i w_j/2 -$$
$$c_i (t_j+1) u_j w_j/2 + \left(2 - 3 t_i^{-1}\right) u_i^2 w_i w_j/2 +$$
$$\left(1 + 2 t_i^{-2} - 3 t_i^{-1}\right) u_i^2 w_j^2/2 - t_i^{-1} (1+t_j) u_i u_j w_j^2/2\big];$$

$$ur_{i\_} := \mathbb{E}\left[t_i^{-1/4}, 0, 0, c_i t_i/4 + u_i w_i/8\right];$$
$$nr_{i\_} := \mathbb{E}\left[t_i^{1/4}, 0, 0, -c_i t_i^3/4 - t_i^2 u_i w_i/8\right];$$
$$ul_{i\_} := \mathbb{E}\left[t_i^{1/4}, 0, 0, c_i t_i (4+t_i)/4 - t_i^2 u_i w_i/8\right];$$
$$nl_{i\_} := \mathbb{E}\left[t_i^{-1/4}, 0, 0, -c_i \left(1 + 4 t_i^{-1}\right)/4 + u_i w_i/8\right];$$

Two swirls and a 0-framed trefoil



$$\text{t2} = ur_1 R^{-}_{2,5} nr_3 ur_4 nr_6 \;//\; m_{1,2\to 1} \;//\; m_{1,3\to 1} \;//\; m_{4,5\to 4} \;//\; m_{4,6\to 4}$$

$$\mathbb{E}\Big[1,\ -b_1 c_4,\ -\frac{u_1 w_4}{t_1},$$
$$\frac{c_1}{2} + \frac{c_1^2}{2} + c_1 c_4 - c_1 t_1 - c_1^2 t_1 + \frac{1}{2} c_1 c_4 t_1 + \frac{1}{2} c_1 t_1^2 + \frac{1}{2} c_1^2 t_1^2 -$$
$$\frac{1}{2} c_1 c_4 t_4^2 - c_1 u_1 w_1 + \frac{1}{2} c_4 u_1 w_1 + c_1 t_1 u_1 w_1 + \frac{1}{2} u_1^2 w_1^2 + \frac{3 u_1 w_4}{8} -$$
$$\frac{5}{2} c_1 u_1 w_4 + \frac{1}{2} c_4 u_1 w_4 - \frac{u_1 w_4}{2 t_1} + \frac{3 c_1 u_1 w_4}{2 t_1} + \frac{c_4 u_1 w_4}{2 t_1} - \frac{1}{8} t_1 u_1 w_4 +$$
$$c_1 t_1 u_1 w_4 + \frac{t_4 u_1 w_4}{8 t_1} + \frac{t_4^2 u_1 w_4}{8 t_1} - \frac{c_4 t_4^2 u_1 w_4}{2 t_1} - \frac{1}{2} c_1 u_4 w_4 - \frac{1}{2} c_1 t_4 u_4 w_4 +$$
$$u_1^2 w_1 w_4 - \frac{3 u_1^2 w_1 w_4}{2 t_1} + \frac{1}{2} u_1^2 w_4^2 + \frac{u_1^2 w_4^2}{t_1^2} - \frac{3 u_1^2 w_4^2}{2 t_1} - \frac{u_1 u_4 w_4^2}{2 t_1} - \frac{t_4 u_1 u_4 w_4^2}{2 t_1}\Big]$$

$$\text{t2} == \left(ul_1 R^{-}_{2,5} nl_3 ul_4 nl_6 \;//\; m_{1,2\to 1} \;//\; m_{1,3\to 1} \;//\; m_{4,5\to 4} \;//\; m_{4,6\to 4}\right)$$

True

$$\text{z2} = R^{+}_{1,14} R^{-}_{5,2} nr_3 ul_4 R^{+}_{19,6} R^{-}_{7,10} nl_8 ur_9 R^{+}_{11,20} nr_{12} ul_{13} R^{-}_{15,18}$$
$$nl_{16} ur_{17};$$

$$(\text{Do}[\text{z2} = \text{z2} \;//\; m_{1,k\to 1},\ \{k, 2, 20\}];\ \text{z2} = \text{z2}\ /.\ a\_{}_1 :\!\to a)$$

$$\mathbb{E}\Big[-1 + \frac{1}{t} + t,\ 0,\ 0,$$
$$-16 + \frac{9c}{2} - \frac{2c}{t^4} + \frac{1}{t^3} + \frac{11c}{2t^3} - \frac{4}{t^2} - \frac{8c}{t^2} + \frac{10}{t} + \frac{4c}{t} + 18t - 10ct - 14t^2 +$$
$$8ct^2 + 7t^3 - \frac{3ct^3}{2} - 2t^4 - 2ct^4 + 2ct^5 - \frac{ct^6}{2} - 4uw + \frac{2uw}{t^4} -$$
$$\frac{7uw}{2t^3} + \frac{9uw}{2t^2} + \frac{uw}{2t} + 6tuw - 2t^2uw - \frac{1}{2}t^3uw + \frac{3}{2}t^4uw - \frac{1}{2}t^5uw\Big]$$

```
FromCoefficientRules[
 CoefficientRules[z2[[4]], {c, u, w}] /.
  {(e_ → a_) :→ (e → Simplify[a])}, {c, u, w}]
```

$$-\frac{(1-t+t^2)^2(-1+2t-3t^2+2t^3)}{t^3} -$$
$$\frac{c(1-t+t^2)^3(4+t-5t^2-t^3+t^4)}{2t^4} - \frac{(1-t+t^2)^3(-4-5t+t^3)uw}{2t^4}$$

Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $b^+ \leftrightarrow b^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

References.

[Al] J. W. Alexander, *Topological invariants of knots and link,* Trans. Amer. Math. Soc. **30** (1928) 275–306.

[BN1] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BN2] D. Bar-Natan, *Polynomial Time Knot Polynomial,* research proposal for the 2017 Killam Fellowship, ωεβ/K17.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I, II, IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO4, arXiv:1405.1956, arXiv:1405.1955, arXiv:1511.05624.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras,* Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, Sep. 2002, arXiv:math.QA/0211031.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[PV] M. Polyak and O. Viro, *Gauss Diagram Formulas for Vassiliev Invariants,* Inter. Math. Res. Notices **11** (1994) 445–453.

[Ro] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Se] P. Ševera, *Quantization of Lie Bialgebras Revisited,* Sel. Math., NS, to appear, arXiv:1401.6164.

Disclaimer. This is all quite new. The overall picture is correct, yet some details might be somewhat off. Many pieces are certainly not in their final form yet. \\ Still Help needed!

Repartition!