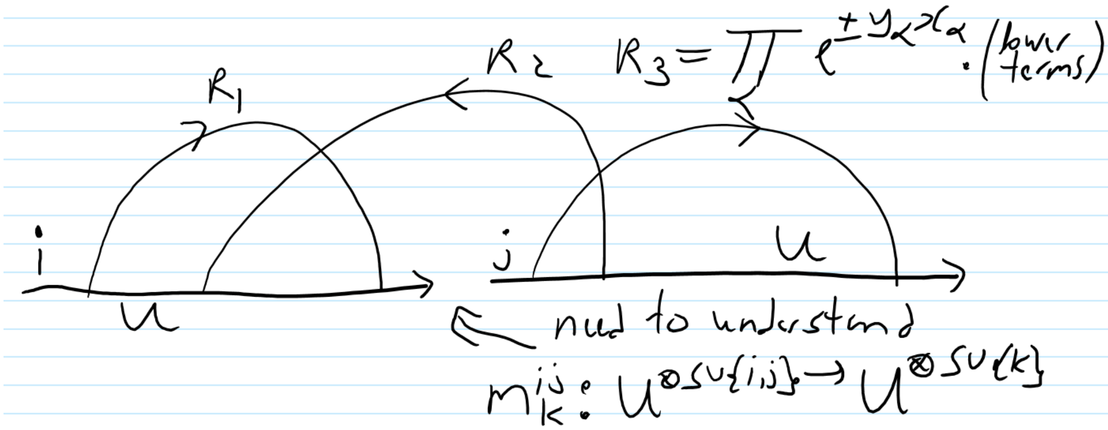
Dror Bar-Natan: Talks: Sydney-1708:

The Dogma is Wrong - Extra Details

Goal



Agenda

- Quantizing and de-quantizing $\mathfrak{sl}_2^\epsilon$.
- Some understanding of $\mathfrak{sl}_2^\epsilon$.
- A full understanding of $\mathfrak{sl}_2^\epsilon$ at $\epsilon = 0$.
- A full understanding of $\mathfrak{sl}_2^\epsilon$ at $\epsilon^2 = 0$.
- Pushforwards of distributions, 0-dimensional QFT, Feynman diagrams and what had really happened here.

Some Shortcuts

```
ME[x_] := MatrixExp[x]; MB[x_, y_] := x.y - y.x; MF[x_] := MatrixForm[x];
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Representing $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$

$$\rho h = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix};\ \rho e = \begin{pmatrix} 0 & 0 \\ -\epsilon & 0 \end{pmatrix};\ \rho l = \begin{pmatrix} -(1 + 1/\epsilon)/2 & 0 \\ 0 & (1 - 1/\epsilon)/2 \end{pmatrix};\ \rho f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

```
Simplify@{MB[ρe, ρl] == -ρe, MB[ρf, ρl] == ρf, MB[ρe, ρf] == ρh - 2 ϵ ρl}
{True, True, True}
```

The Main $\mathfrak{g}_0 := \mathfrak{g}^\epsilon / (\epsilon = 0)$ Theorem.

The $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega e^{L+Q} \mid \prod_{i\in S} e_i l_i f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q = \sum b_{ij} e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Proof. Indeed, as we shall see, the following lemmas hold, and the rest is straight-forward.

Lemma 0. $R^s = e^{s(h\otimes l+e\otimes f)} = \mathbb{O}\left(\exp\left(s\,\mathrm{hl} + \frac{e^{sh}-1}{h}\,\mathrm{ef}\right) \mid e\otimes \mathrm{lf}\right)$.

Lemma 1. $\mathbb{O}\left(e^{\gamma \mathrm{l}+\beta \mathrm{e}} \mid \mathrm{le}\right) = \mathbb{O}\left(e^{\gamma \mathrm{l}+e^{\gamma}\beta \mathrm{e}} \mid \mathrm{el}\right)$.

Lemma 2. $\mathbb{O}\left(e^{\gamma \mathrm{l}+\beta \mathrm{f}} \mid \mathrm{fl}\right) = \mathbb{O}\left(e^{\gamma \mathrm{l}+e^{\gamma}\beta \mathrm{f}} \mid \mathrm{lf}\right)$.

Lemma 3. $\mathbb{O}\left(e^{\beta \mathrm{e}+\alpha \mathrm{f}+\delta \mathrm{ef}} \mid \mathrm{fe}\right) = \mathbb{O}\left(\nu\, e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta e f)} \mid e\,f\right)$, with $\nu = (1+\mathrm{h}\delta)^{-1}$.

Some $\mathfrak{g}^\epsilon$ lemmas

Lemma 1. $\mathbb{O}\left(e^{\gamma l+\beta e} \mid l\,e\right) = \mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta e} \mid e\,l\right)$.

Lemma 2. $\mathbb{O}\left(e^{\gamma l+\beta f} \mid f\,l\right) = \mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta f} \mid l\,f\right)$.

Proofs.

```
MF /@ {ME[γ ρl].ME[β ρe], ME[e^γ β ρe].ME[γ ρl]}
```

$$\left\{\begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & 0 \\ -e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta\epsilon & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}, \begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & 0 \\ -e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta\epsilon & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}\right\}$$

```
MF /@ {ME[β ρf].ME[γ ρl], ME[γ ρl].ME[e^γ β ρf]}
```

$$\left\{\begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta \\ 0 & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}, \begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta \\ 0 & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}\right\}$$

Lemma 3 at $\delta = 0$. $\mathbb{O}\left(e^{\alpha f+\beta e} \mid f\,e\right) = \mathbb{O}\left(e^{c\,h+a\,e-2\,\epsilon\,c\,l+b\,f} \mid e\,l\,f\right)$, with $\left\{a \to -\frac{\beta}{-1+\alpha\beta\epsilon}, b \to -\frac{\alpha}{-1+\alpha\beta\epsilon}, c \to \frac{\mathrm{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$.

Derivation.

```
ME[α ρf].ME[β ρe] // Simplify // MF
```

```
eqn = ME[α ρf].ME[β ρe] == ME[a ρe].ME[c (ρh - 2 ϵ ρl)].ME[b ρf]
```

```
sol = Solve[Thread[Flatten /@ eqn], {a, b, c}][[1]]
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```
sol = sol /. C[1] → 0
```

Lemma 3 for $\mathfrak{g}_0$.

```
Limit[{a, b, c} /. sol, ϵ → 0]
```

And so in $\mathfrak{g}_0$, $\mathbb{O}\left(e^{\alpha f+\beta e} \mid f\,e\right) = \mathbb{O}\left(e^{\alpha f+\beta e-\alpha\beta h} \mid e\,l\,f\right)$. Hence

$\mathbb{O}\left(e^{\alpha f+\beta e+\delta e f} \mid f e\right) = e^{\delta\partial_\alpha\partial_\beta}\,\mathbb{O}\left(e^{\alpha f+\beta e} \mid f e\right) = e^{\delta\partial_\alpha\partial_\beta}\,\mathbb{O}\left(e^{\alpha f+\beta e-\alpha\beta h} \mid e l f\right) = \mathbb{O}(\psi \mid e l f)$, where $\psi = e^{\delta\partial_\alpha\partial_\beta}\,e^{\alpha f+\beta e-\alpha\beta h}$ satisfies $\psi_{\delta=0} = e^{\alpha f+\beta e-\alpha\beta h}$ and $\partial_\delta\psi = \partial_{\alpha,\beta}\psi$.

```
With[{ψ = ν e^(ν (δ e f - α β h + α f + β e)) /. ν → (1 + δ h)^-1}, Simplify@{∂_δ ψ - ∂_{α,β} ψ, ψ /. δ → 0}]
{0, e^(f α+e β-h α β)}
```

A Lemma 3 for $\mathfrak{g}_k := \mathfrak{g}^\epsilon/(\epsilon^{k+1}=0)$.

Lemma 3$_k$. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta e f} \mid f e\right) = \mathbb{O}\left(\nu\, e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta e f)}\,\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e f\right)$, with $\nu = (1+h\,\delta)^{-1}$ and where for any fixed k, $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ is a fixed polynomial of degree at most $4\,k$ in e, $\sqrt{l}$, f, α, β, with scalar coefficients.

Comment. Even better, $\log(\Lambda_k)$ is of degree at most $2\,k+2$ in said variables.

Comment. And hence the $\mathfrak{g}_k$ invariant is computable in polynomial time.

Proof of Lemma 3$_k$. We know that $\mathbb{O}\left(e^{\alpha f+\beta e} \mid f e\right) = \mathbb{O}\left(e^{c h+a e-2\epsilon c l+b f} \mid e l f\right)$, with $\left\{a \to -\frac{\beta}{-1+\alpha\beta\epsilon},\ b \to -\frac{\alpha}{-1+\alpha\beta\epsilon},\ c \to \frac{\mathrm{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$. Expanding in ϵ we get
$\mathbb{O}\left(e^{\alpha f+\beta e} \mid f e\right) = \mathbb{O}\left(\lambda_\epsilon(\alpha,\beta)\,e^{\alpha f+\beta e-\alpha\beta h}\right) = \mathbb{O}\left(\lambda_\epsilon(\partial_f,\partial_e)\,e^{\alpha f+\beta e-\alpha\beta h} \mid e l f\right)$ and so
$\mathbb{O}\left(e^{\alpha f+\beta e+\delta e f} \mid f e\right) = \mathbb{O}\left(\lambda_\epsilon(\partial_f,\partial_e)\,e^{\delta\partial_\alpha\partial_\beta}\,e^{\alpha f+\beta e-\alpha\beta h} \mid e l f\right) = \mathbb{O}\left(\lambda_\epsilon(\partial_f,\partial_e)\,\nu\,e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta e f)} \mid e l f\right)$.

```
DP_{α_→D_f_, β_→D_e_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :→ c D[λ, {f, m}, {e, n}]]

(* "D" for Detailed *)
DΛ_k_[h_, e_, l_, f_, α_, β_, δ_] := Module[
   {ρh, ρe, ρl, ρf, eqn, a, b, c, sol, λ, q, ν},
   ρh = ( -1  0 ; 0  -1 ); ρe = ( 0  0 ; -ϵ  0 ); ρl = ( -(1 + 1/ϵ)/2  0 ; 0  (1 - 1/ϵ)/2 ); ρf = ( 0  1 ; 0  0 );
   eqn = ME[α ρf].ME[β ρe] == ME[a ρe].ME[c (ρh - 2 ϵ ρl)].ME[b ρf];
   sol = Solve[Thread[Flatten /@ eqn], {a, b, c}][[1]] /. C[1] → 0;
   λ = Simplify[e^(-f α-e β+h α β) Normal@Series[e^(c h + a e - 2 ϵ c l + b f) /. sol, {ϵ, 0, k}]];
   q = e^(ν (f α+e β-h α β+e f δ));
   Collect[q^-1 DP_{α→D_f, β→D_e}[λ][q] /. ν → (1 + h δ)^-1, ϵ, Simplify]
  ];

DΛ_1[h, e, l, f, α, β, δ]

DΛ_2[h, e, l, f, α, β, δ]

Λ_k_[h_, e_, l_, f_, α_, β_, δ_] := Λ_k[h, e, l, f, α, β, δ] = Module[{λ},
   λ = Normal@Series[e^((f α+e β)/(1-α β ϵ)) (1 - α β ϵ)^(-2 l + h/ϵ), {ϵ, 0, k}] /. e- → 1;
   Collect[DP_{α→D_f, β→D_e}[λ][e^((f α+e β+e f δ)/(1+h δ))] /. e- → 1, ϵ, Simplify]];

Simplify[DΛ_2[h, e, l, f, α, β, δ] == Λ_2[h, e, l, f, α, β, δ]]
```

The Main $\mathfrak{g}_k$ Theorem

The $\mathfrak{g}_k$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega\,e^{L+Q+P} \mid \prod_{i\in S} e_i\,l_i\,f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij}, where $Q = \sum b_{ij} e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,\ e_i,\ l_i,\ f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\left\{e_i, \sqrt{l_i}\,,\ f_i\right\}$. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

0. $R^\pm = ?$, $n^\pm = ?$.

1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l+\beta e} \mid l\, e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta e} \mid e\, l\right)$,

2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l+\beta f} \mid f\, l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta f} \mid l\, f\right)$,

3. $\mathbb{O}\left(\mathcal{P}(e, f)\, e^{\beta e+\alpha f+\delta e f} \mid f\, e\right) = \mathbb{O}\left(v\,\mathcal{P}(\partial_\beta, \partial_\alpha)\, e^{v(-\alpha\beta h+\beta e+\alpha f+\delta e f)}\, \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f\right)$, with $v = (1+\mathrm{h}\delta)^{-1}$, and $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ as above.

Pushforwards of distributions, 0-dimensional QFT, Feynman diagrams and what had really happened here.