A Seifert Dream

Thanks for inviting me to Pitzer College!

Abstract. Given a knot K with a Seifert surface Σ, I dream that the well-known Seifert linking form Q, a quadratic form on $H_1(\Sigma)$, has plenty docile local perturbations P_ϵ such that the formal Gaussian integrals of $\exp(Q+P_\epsilon)$ are invariants of K.
In my talk I will explain what the above means, why this dream is oh so sweet, and why it is in fact closer to a plan than to a delusion. Joint with Roland van der Veen.

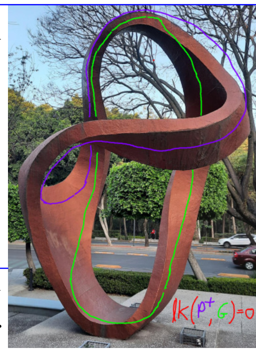
The Seifert-Alexander Formula. With $P,Q\in H_1(\Sigma)$,

$$Q(P,G)=T^{1/2}lk(P^+,G)+T^{-1/2}lk(P,G^+)$$

$$\Delta(K)=\det(Q)$$

$$\int_{2H_1(\Sigma)} dp\,dx\,\exp Q(p,x)\doteq\det(Q)^{-1}$$

(where $\doteq$ means "ignoring silly factors").

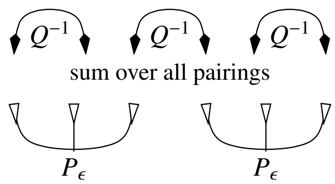


From Mexico City, tariffs exempt

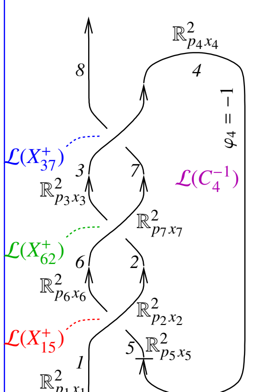
Perturbed Gaussian Integration. We say that $P_\epsilon\in\mathbb{Q}[x_1,\dots x_n][\![\epsilon]\!]$ is M-docile (for some $M\colon\mathbb{N}\to\mathbb{N}$) if for every monomial m in P_ϵ we have $\deg_{x_1,\dots,x_n}(m)\le M(\deg_\epsilon(m))$.

Theorem (Feynman). If Q is a quadratic in $x_1,\dots,x_n$ and P_ϵ is docile, set $Z_\epsilon=\int_{\mathbb{R}^n}dx_1\cdots x_n\exp(Q+P_\epsilon)$. Then every coefficient in the ϵ-expansion of Z_ϵ is computable in polynomial time in n. in fact,

$$\Delta^{1/2}Z_\epsilon\doteq\left\langle\exp Q^{-1}(\partial_{x_i}),\exp P_\epsilon\right\rangle=$$



$\theta(T,1)$ is like that! With $\epsilon^2=0$,



$$\longrightarrow Z\doteq\int_{2E=\mathbb{R}^{14}_{p_ix_i}}\mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

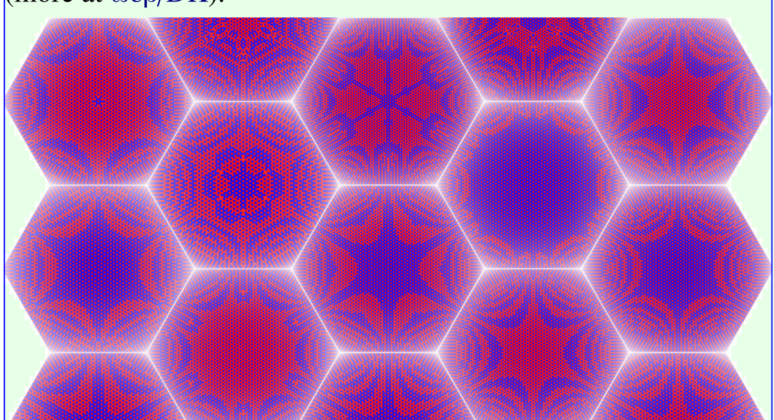
where $\mathcal{L}(X^s_{ij})\doteq\mathbb{e}^{L(X^s_{ij})}$, $\mathcal{L}(C^\varphi_i)\doteq\mathbb{e}^{L(C^\varphi_i)}$,

$$L(X^s_{ij})=x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j)+(T^s-1)x_i(p_{i+1}-p_{j+1})+\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\+2(1-x_jp_j)\end{pmatrix}-1\right)$$

$$L(C^\varphi_i)=x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

$\theta(T_1,T_2)$ is likewise, with harder formulas and integration over $6E$.

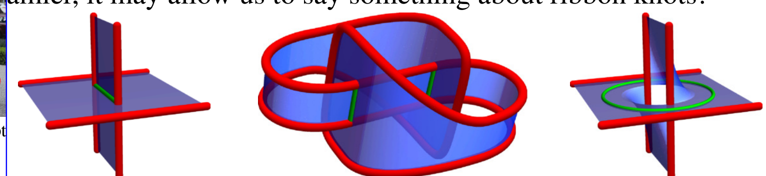
Right. The 132-crossing torus knot $T_{22/7}$ (more at ωεβ/TK).
Below. Random knots from [DHOEBL], with 101-115 crossings (more at ωεβ/DK).



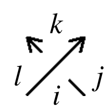
Dream. There is a similar perturbed Gaussian integral formula for θ, but with integration over $6H_1(\Sigma)$. The quadratic Q will be the same as in the Seifert-Alexander formula (but repeated 3 times, for each T_ν). The perturbation P_ϵ will be given by low-degree finite type invariants of curves on Σ (possibly also dependent on the intersection points of such curves, or on other information coming from Σ).

Evidence. Experimentally (yet undeniably), $\deg\theta$ is bounded by the genus of Σ. How else could such a genus bound arise? Further very strong evidence comes from the conjectural (yet undeniable) understanding of θ as the two-loop contribution to the Kontsevich integral [Oh] and/or as the "solvable approximation" of the universal sl_3 invariant [BN1, BV2].

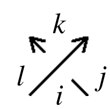
Why so sweet? It will allow us to prove the aforementioned genus bound and likely, the hexagonal symmetry. Sweeter and dreamier, it may allow us to say something about ribbon knots!



What's "local"? How will we compute? The Będlewo Alexander formula: Let F be the faces of a knot diagram. Make an $F\times F$ matrix A by adding for each crossing contributions

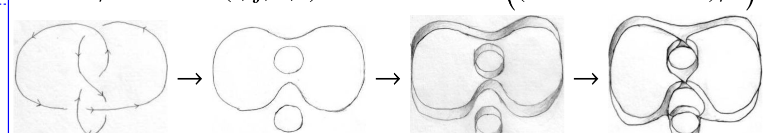


$$\to\begin{pmatrix}-1&-1&2&0\\0&0&0&0\\0&1&-1&0\\1&0&-1&0\end{pmatrix}$$

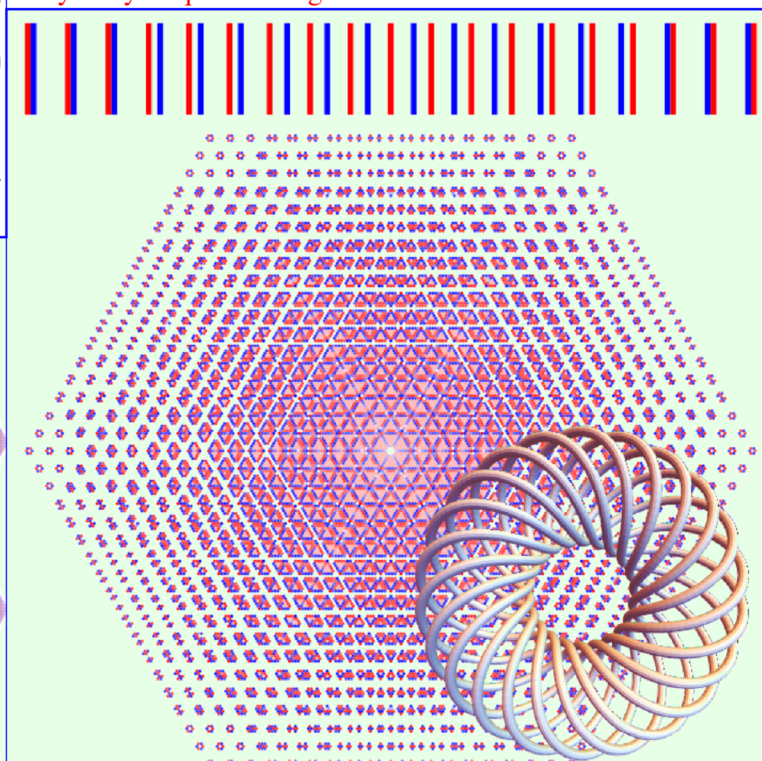


$$\to\begin{pmatrix}1&-1&0&0\\0&0&0&0\\-2&1&1&0\\1&0&-1&0\end{pmatrix}$$

at rows / columns (i,j,k,l). Then $\Delta=\det'\left((T^{1/2}A-T^{-1/2}A)/2\right)$.


(the Seifert algorithm by Emily Redelmeier)

Expect the like for θ! Expect more like θ! Topology first! Resist the tyranny of quantum algebra!



The Strongest Genuinely Computable Knot Invariant in 2024

van der Veen

Abstract. "Genuinely computable" means we have computed it for random knots with over 300 crossings. "Strongest" means it separates prime knots with up to 15 crossings better than the less-computable HOMFLY-PT and Khovanov homology taken together. And hey, it's also meaningful and fun.

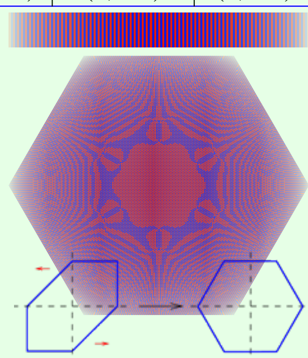
Continues Rozansky, Garoufalidis, Kricker, and Ohtsuki, joint with van der Veen.

Acknowledgement. This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Strongest. Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis): (ρ_1: [Ro1, Ro2, Ro3, Ov, BV1])

	knots	(H, Kh)	(Δ, ρ_1)	$\Theta = (\Delta, \theta)$	together
reign		2005-22	2022-24	2024-	
xing ≤ 10	249	248 (1)	249 (0)	249 (0)	249 (0)
xing ≤ 11	801	771 (30)	787 (14)	798 (3)	798 (3)
xing ≤ 12	2,977	(214)	(95)	(19)	(18)
xing ≤ 13	12,965	(1,771)	(959)	(194)	(185)
xing ≤ 14	59,937	(10,788)	(6,253)	(1,118)	(1,062)
xing ≤ 15	313,230	(70,245)	(42,914)	(6,758)	(6,555)

Genuinely Computable. Here's Θ on a random 300 crossing knot (from [DHOEBL]). For almost every other invariant, that's science fiction.

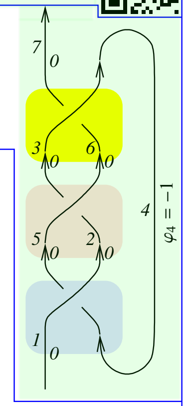


Fun. There's so much more to see in 2D pictures than in 1D ones! Yet almost nothing of the patterns you see we know how to prove. We'll have fun with that over the next few years. Would you join?

Meaningful. θ gives a genus bound (unproven yet with confidence). We hope (with reason) it says something about ribbon knots.

Conventions. T, T_1, and T_2 are indeterminates and $T_3 := T_1T_2$.

Preparation. Draw an n-crossing knot K as a diagram D as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \ldots, 2n+1\}$ and with rotation numbers φ_k.



Model T Traffic Rules. Cars always drive forward. When a car crosses over a sign-s bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0$. At the very end, cars fall off and disappear. On various edges *traffic counters* are placed. See also [Jo, LTW].

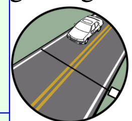


image credits: diamondtraffic.com

$p = 1 - T^s$

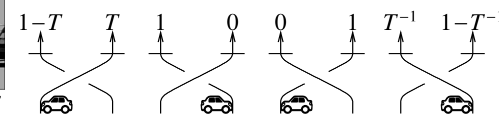
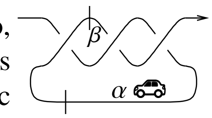


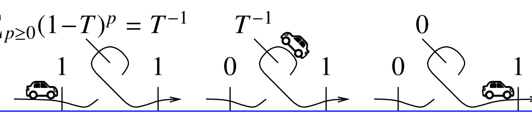
image credits: Dall-E

Definition. The *traffic function* $G = (g_{\alpha\beta})$ (also, the *Green function* or the *two-point function*) is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point). There are also model-T_ν traffic functions $G_\nu = (g_{\nu\alpha\beta})$ for $\nu = 1, 2, 3$.



Example.

$\sum_{p\geq 0}(1-T)^p = T^{-1}$



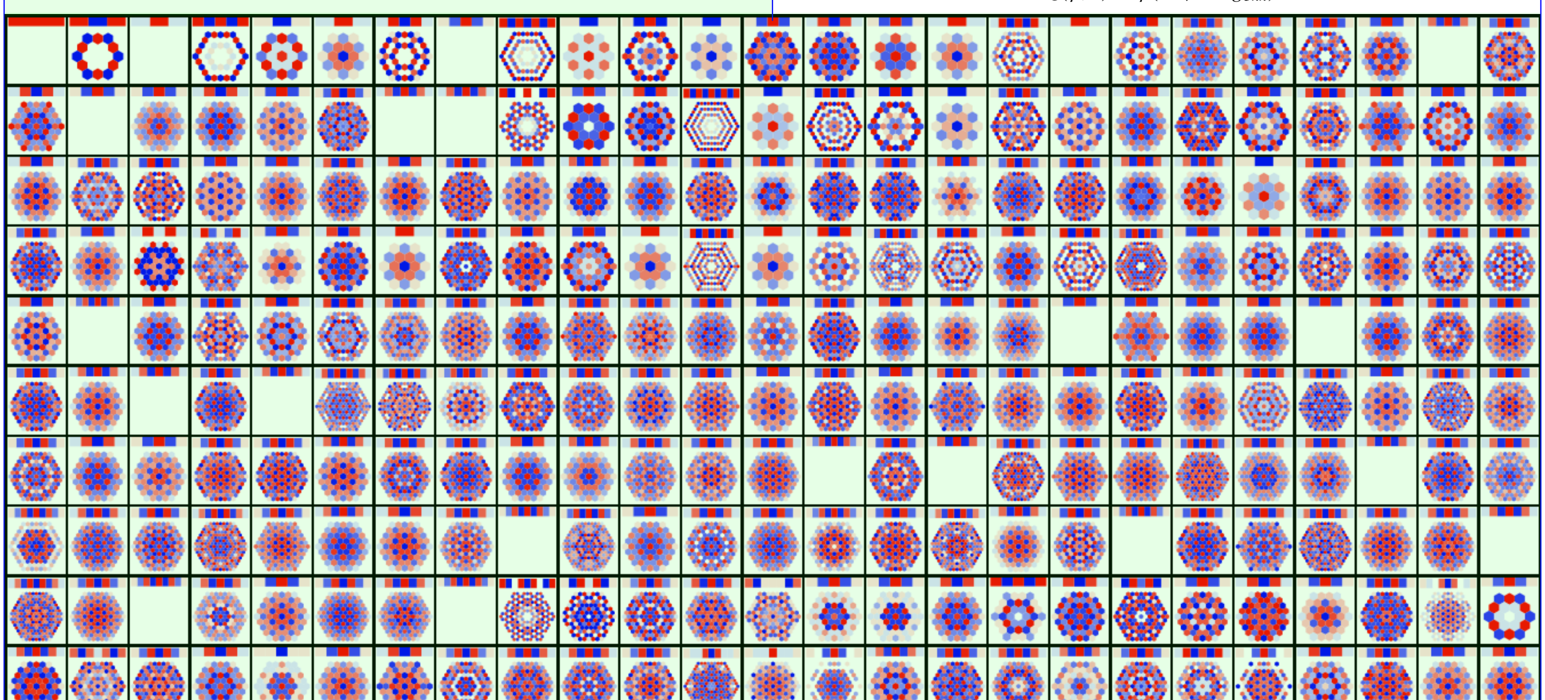
$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Don't Look.

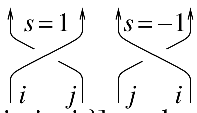
$$\begin{aligned} R_{11}(c) = s\Big[&1/2 - g_{3ii} + T_2^s g_{1ii} g_{2ji} - T_2^s g_{3jj} g_{2ji} - (T_2^s - 1) g_{3ii} g_{2ji} \\ &+ (T_3^s - 1) g_{2ji} g_{3ji} - g_{1ii} g_{2jj} + 2 g_{3ii} g_{2jj} + g_{1ii} g_{3jj} - g_{2ii} g_{3jj}\Big] \\ &+ \frac{s}{T_2^s - 1}\Big[(T_1^s - 1) T_2^s \left(g_{3jj} g_{1ji} - g_{2jj} g_{1ji} + T_2^s g_{1ji} g_{2ji}\right) \\ &+ (T_3^s - 1)\left(g_{3ji} - T_2^s g_{1ii} g_{3ji} + g_{2ij} g_{3ji} + (T_2^s - 2) g_{2jj} g_{3ji}\right) \\ &- (T_1^s - 1)(T_2^s + 1)(T_3^s - 1) g_{1ji} g_{3ji}\Big] \end{aligned}$$

$$R_{12}(c_0, c_1) = \frac{s_1 (T_1^{s_0} - 1)(T_3^{s_1} - 1) g_{1j_1i_0} g_{3j_0i_1}}{T_2^{s_1} - 1}\left(T_2^{s_0} g_{2i_1i_0} + g_{2j_1j_0} - T_2^{s_0} g_{2j_1i_0} - g_{2i_1j_0}\right)$$

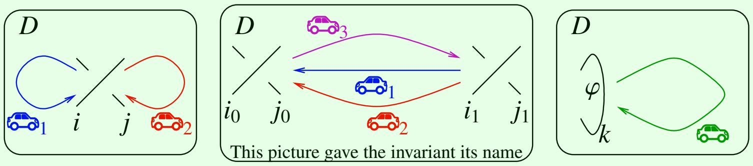
$$\Gamma_1(\varphi, k) = \varphi(-1/2 + g_{3kk})$$



Theorem. With $c = (s, i, j)$, $c_0 = (s_0, i_0, j_0)$, and $c_1 = (s_1, i_1, j_1)$ denoting crossings, there is a quadratic $R_{11}(c) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i, j\}]$, a cubic $R_{12}(c_0, c_1) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i_0, j_0, i_1, j_1\}]$, and a linear $\Gamma_1(\varphi, k)$ such that the following is a knot invariant:

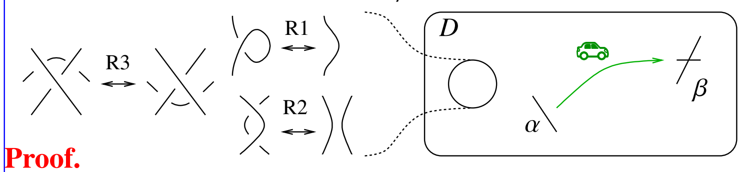


$$\theta(D) := \underbrace{\Delta_1\Delta_2\Delta_3}_{\text{normalization, see later}} \left(\sum_c R_{11}(c) + \sum_{c_0,c_1} R_{12}(c_0, c_1) + \sum_k \Gamma_1(\varphi_k, k) \right),$$

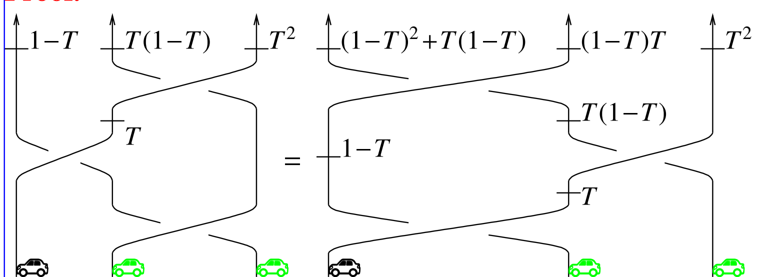


If these pictures remind you of Feynman diagrams, it's because they are Feynman diagrams [BN2].

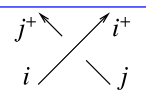
Lemma 1. The traffic function $g_{\alpha\beta}$ is a "relative invariant":



Proof.



Lemma 2. With $k^+ := k + 1$, the "g-rules" hold near a crossing $c = (s, i, j)$:



$g_{j\beta} = g_{j^+\beta} + \delta_{j\beta}$ $\quad g_{i\beta} = T^s g_{i^+\beta} + (1 - T^s) g_{j^+\beta} + \delta_{i\beta}$ $\quad g_{2n^+,\beta} = \delta_{2n^+,\beta}$

$g_{\alpha i^+} = T^s g_{\alpha i} + \delta_{\alpha i^+}$ $\quad g_{\alpha j^+} = g_{\alpha j} + (1 - T^s) g_{\alpha i} + \delta_{\alpha j^+}$ $\quad g_{\alpha,1} = \delta_{\alpha,1}$

Corollary 1. G is easily computable, for $AG = I\ (= GA)$, with A the $(2n+1)\times(2n+1)$ identity matrix with additional contributions:

$c = (s, i, j) \mapsto$	A	col i^+	col j^+
	row i	$-T^s$	$T^s - 1$
	row j	0	-1

For the trefoil example, we have:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2} \det(A), \qquad \text{with } \varphi = \textstyle\sum_k \varphi_k,\ w = \sum_c s.$$

We also set $\Delta_\nu := \Delta(T_\nu)$ for $\nu = 1, 2, 3$.

Questions, Conjectures, Expectations, Dreams.

Question 1. What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

Conjecture 2. On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

Conjecture 3. θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinery on the double $\mathcal{D}(\mathfrak{b}, b, \epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

Conjecture 4. θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $g(K) < \lceil \deg_{T_1} \theta(K) \rceil$.

Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

Expectation 8. There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

Dream 9. These invariants can be explained by something less foreign than semisimple Lie algebras.

Dream 10. θ will have something to say about ribbon knots.

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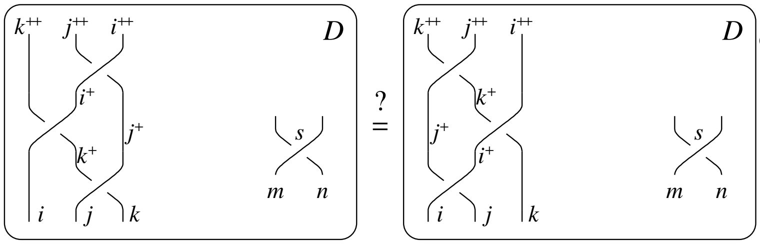
[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Corollary 2. Proving invariance is easy:



Invariance under R3

This is Theta.nb of http://drorbn.net/to24/ap.

```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

```
T3 = T1 T2;
CF[ℰ_] :=
  Module[{vs = Union@Cases[ℰ, g__, ∞], ps, c},
   Total[CoefficientRules[Expand[ℰ], vs] /.
     (ps_ → c_) :> Factor[c] (Times @@ vs^ps)]];
R11[{s_, i_, j_}] =
  CF[
   s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj -
       (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
       g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
       ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
          (T3^s - 1) g3ji
           (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji +
             (T2^s - 2) g2jj + g2ij)) / (T2^s - 1))];
R12[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g_{1,j1,i0} g_{3,j0,i1}
    ((T2^s0 g_{2,i1,i0} - g_{2,i1,j0}) - (T2^s0 g_{2,j1,i0} - g_{2,j1,j0}))]
Γ1[φ_, k_] = -φ/2 + φ g3kk;
```

```
δ_{i_,j_} := If[i === j, 1, 0];
gR_{s_,i_,j_} := {
   g_{ν_ j β_} :> g_{ν j+ β} + δ_{j β},
   g_{ν_ i β_} :> T_ν^s g_{ν i+ β} + (1 - T_ν^s) g_{ν j+ β} + δ_{i β},
   g_{ν_ α_ i+} :> T_ν^s g_{ν α i} + δ_{α i+},
   g_{ν_ α_ j+} :> g_{ν α j} + (1 - T_ν^s) g_{ν α i} + δ_{α j+}
  }
DSum[Cs___] := Sum[R11[c], {c, {Cs}}] +
   Sum[R12[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k+}, {1, i+, j+},
     {s, m, n}] //. gR_{1,j,k} ∪ gR_{1,i,k+} ∪ gR_{1,i+,j+};
rhs = DSum[{1, i, j}, {1, i+, k}, {1, j+, k+},
     {s, m, n}] //. gR_{1,i,j} ∪ gR_{1,i+,k} ∪ gR_{1,j+,k+};
Simplify[lhs == rhs]
```

```
True
```

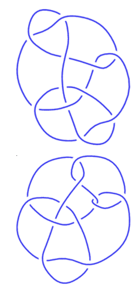
The Main Program

```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :>
     (A[[{i, j}, {i + 1, j + 1}]] += {{-T^s, T^s - 1}, {0, -1}})];
   Δ = T^((-Total[φ] - Total[Cs[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] :=
    Factor[ℰ /. g_{ν_,α_,β_} :> (G[[α, β]] /. T → T_ν)];
   θ = ev[Σ_{k1=1}^{n} Σ_{k2=1}^{n} R12[Cs[[k1]], Cs[[k2]]]];
   θ += ev[Σ_{k=1}^{n} R11[Cs[[k]]]];
   θ += ev[Σ_{k=1}^{2 n} Γ1[φ[[k]], k]];
   Factor@
    {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}];
```

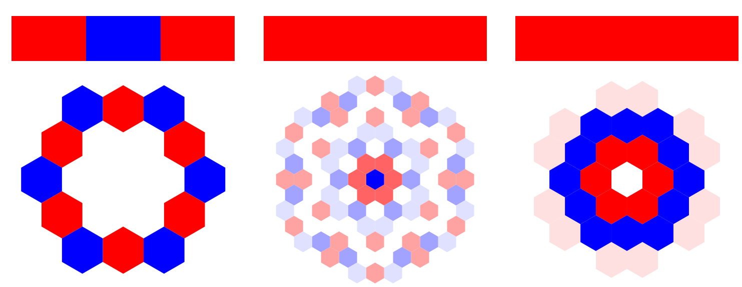
The Trefoil, Conway, and Kinoshita-Terasaka

```
Θ[Knot[3, 1]] // Expand
```

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$



```
GraphicsRow[PolyPlot[Θ[Knot[#]]] & /@
  {"3_1", "K11n34", "K11n42"}]
```



(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

Some Torus Knots

```
TKs = {{13, 2}, {17, 3}, {13, 5}, {7, 6}};
GraphicsRow[PolyPlot[Θ[TorusKnot @@ #]] & /@ TKs]
GraphicsRow[TubePlot[TorusKnot @@ #] & /@ TKs]
```