Pensieve header: Implementing Θ - the main notebook accompanying Talks/PhuQuoc-2506.

pdf

Invariance under R3

pdf

This is Theta.nb of http://drorbn.net/v25/ap.

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\PhuQuoc-2506"];
```

pdf

In[•]:=
```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/v25/ap to compute rotation numbers.
```

pdf

```
Loading PolyPlot.m from http://drorbn.net/v25/ap to plot 2-variable polynomials.
```

pdf

In[•]:=
```
T3 = T1 T2;
```

pdf

In[•]:=
```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

pdf

In[•]:=
```
F1[{s_, i_, j_}] =
  CF[s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj - (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
      g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj + ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
        (T3^s - 1) g3ji (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji + (T2^s - 2) g2jj + g2ij))/(T2^s - 1))];
```

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In[•]:=
```
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] := CF[
  s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1 ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
```

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In[•]:=
```
F3[φ_, k_] = -φ/2 + φ g3kk;
```

pdf

In[•]:=
```
δi_,j_ := If[i === j, 1, 0];
gRs_,i_,j_ := {
  gv_jβ_ :> gvj⁺β + δjβ, gv_iβ_ :> T^s_v gvi⁺β + (1 - T^s_v) gvj⁺β + δiβ,
  gv_α_i⁺ :> T^s_v gvαi + δαi⁺, gv_α_j⁺ :> gvαj + (1 - T^s_v) gvαi + δαj⁺
 }
```

Proof of Reidemeister 3:

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In[•]:=
```
DSum[Cs___] := Sum[F1[c], {c, {Cs}}] + Sum[F2[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k+}, {1, i+, j+}, {s, m, n}] //. gR1,j,k ⋃ gR1,i,k+ ⋃ gR1,i+,j+;
rhs = DSum[{1, i, j}, {1, i+, k}, {1, j+, k+}, {s, m, n}] //. gR1,i,j ⋃ gR1,i+,k ⋃ gR1,j+,k+;
Simplify[lhs == rhs]
```

Out[•]=
pdf

True

tex

\needspace{30mm}

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The Main Program

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In[•]:=
```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, Θ},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
  Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
  G = Inverse[A];
  ev[ℰ_] := Factor[ℰ /. g_ν_,α_,β_ :> (G[[α, β]] /. T → T_ν)];
  Θ = ev[∑_{k=1}^{n} F1[Cs[[k]]]];
  Θ += ev[∑_{k1=1}^{n} ∑_{k2=1}^{n} F2[Cs[[k1]], Cs[[k2]]]];
  Θ += ev[∑_{k=1}^{2 n} F3[φ[[k]], k]];
  Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) Θ}];
```

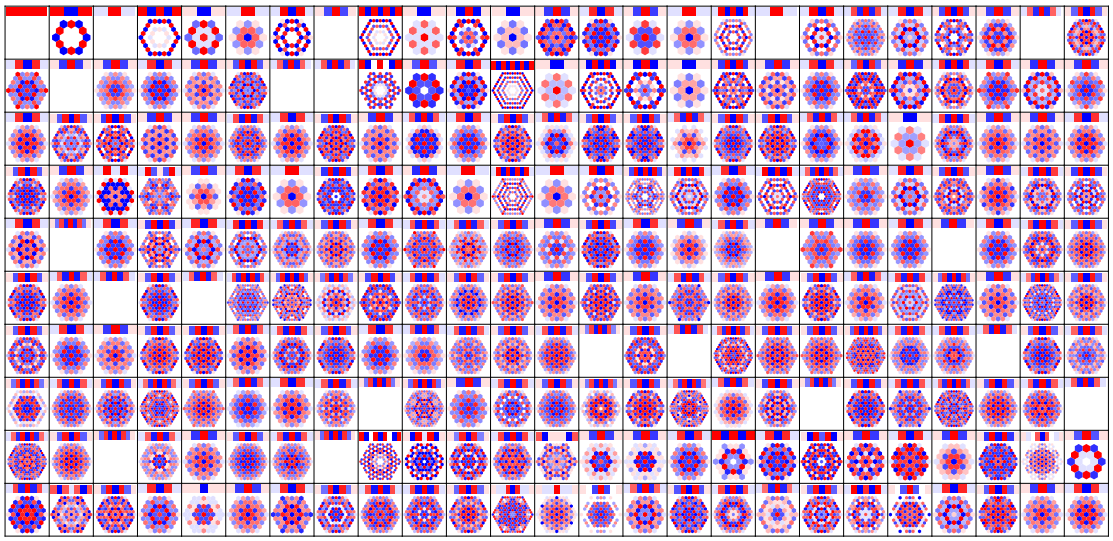
The Rolfsen Table

In[•]:=
```
tab250 = {{1, 0}} ~Join~ Table[Θ[K], {K, AllKnots[{3, 10}]}];
```

KnotTheory: Loading precomputed data in PD4Knots`.

In[•]:=
```
g250 = GraphicsGrid[Partition[PolyPlot /@ tab250, 25], Spacings → 0, Dividers → All]
```

Out[•]=



In[•]:=
```
Export["g250.png", ImageCrop@g250]
```

Out[•]=

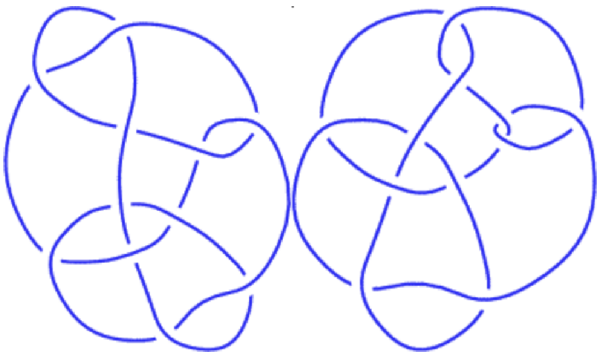
g250.png

pdf

The Trefoil, Conway, and Kinoshita-Terasaka

tex

```
\parpic[r]{\parbox{15mm}{
 \includegraphics[width=15mm]{K11n34.png}
 \vskip 1mm
 \includegraphics[width=15mm]{K11n42.png}
}}
```



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In[•]:=
```
Θ[Knot[3, 1]] // Expand
```

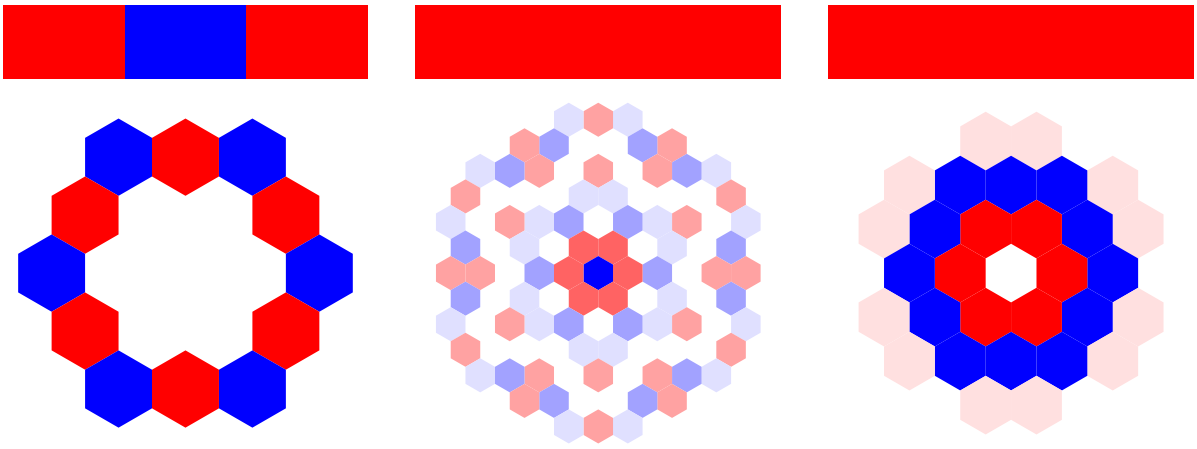
Out[•]=
pdf

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2\,T_2^2}+\frac{1}{T_1\,T_2^2}+\frac{1}{T_1^2\,T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2\,T_2-T_2^2+T_1\,T_2^2-T_1^2\,T_2^2\right\}$$

pdf

In[•]:=
```
GraphicsRow[PolyPlot[Θ[Knot[#]]] & /@ {"3_1", "K11n34", "K11n42"}]
```

Out[•]=
pdf



tex

(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

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Some Torus Knots

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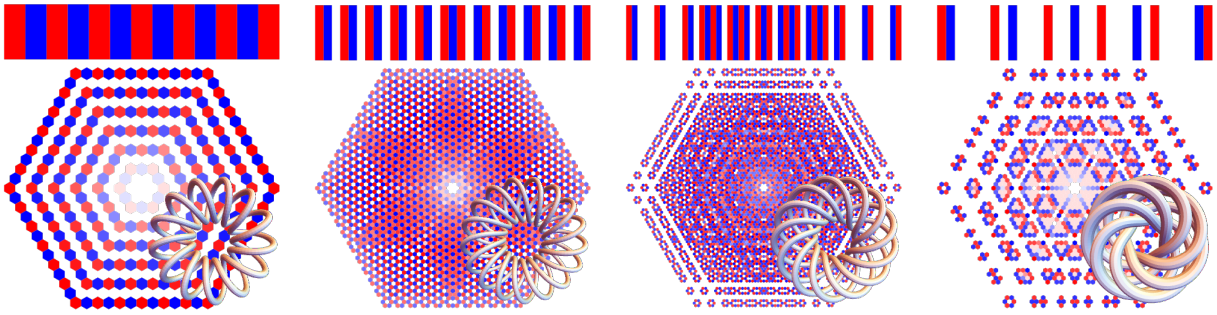
In[•]:=
```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
   ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```

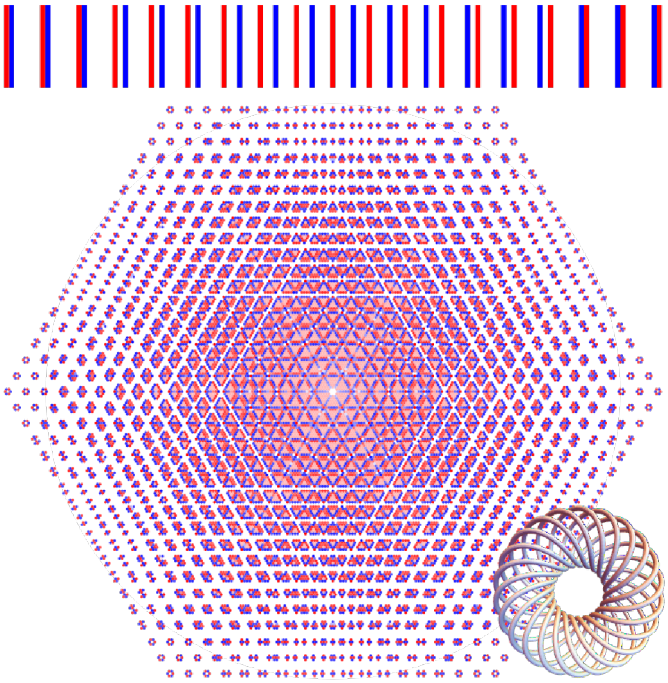
Out[•]=
pdf



The 132-crossing Torus Knot $T_{22/7}$

```
In[•]:= AbsoluteTiming[T227 = Θ[TorusKnot[22, 7]];]
Out[•]=
{933.273, Null}

In[•]:= T227Plot = ImageCompose[
  PolyPlot[T227],
  TubePlot[TorusKnot[22, 7], ImageSize → 240],
  {Right, Bottom}, {Right, Bottom}
 ]
Out[•]=
```



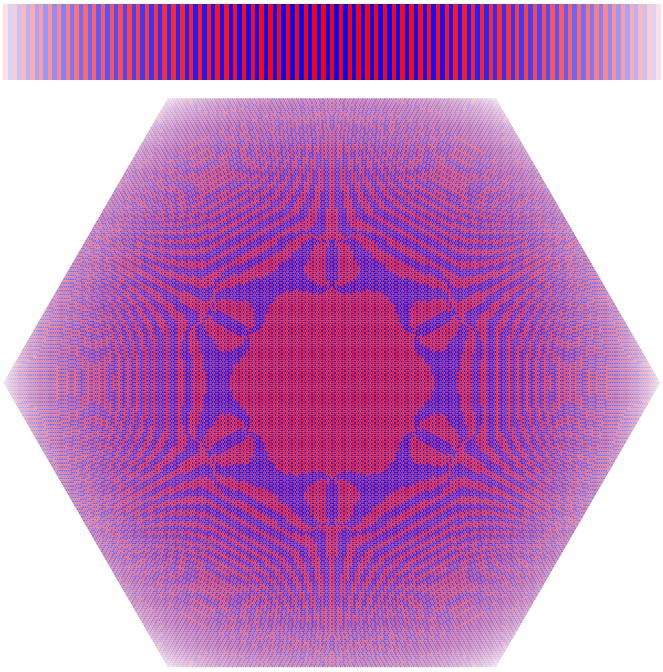
```
In[•]:= Export["T227Plot.pdf", T227Plot]
Out[•]=
T227Plot.pdf
```

Large Random Knots

See also https://drorbn.net/AcademicPensieve/Projects/HigherRank/DunfieldKnots/.

```
In[•]:= DK300 = PolyPlot@Expand[
           Get["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank\\DunfieldKnots\\D300.m"][[2]] /.
            {T1 → T1, T2 → T2}]
```

Out[•]=



```
In[•]:= Export["DK300.pdf", DK300]
```

Out[•]=

DK300.pdf

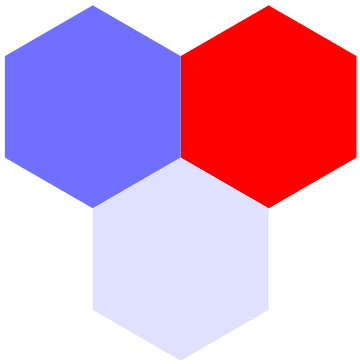
Some Virtual Knots

In[•]:=
```
pd = PD[X[2, 4, 3, 1], X[3, 1, 4, 2]];
Rot[pd]
th = PowerExpand[Θ[pd]]
PolyPlot[th]
```

Out[•]=
```
{{{-1, 4, 2}, {-1, 1, 3}}, {0, 0, 1, 0}}
```

Out[•]=
$$\left\{\frac{1}{\sqrt{T}}, \frac{-1-2\,T_2+4\,T_1\,T_2}{2\,T_1\,T_2}\right\}$$

Out[•]=



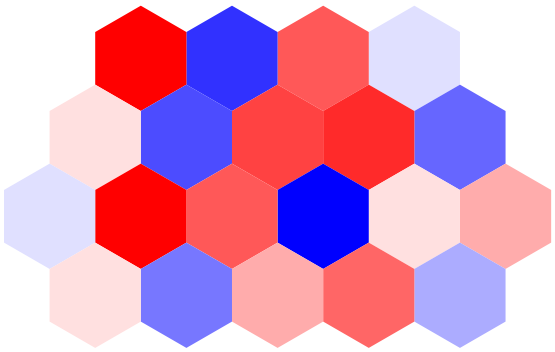
In[•]:=
```
pd = PD[X[3, 8, 4, 1], X[1, 6, 2, 7], X[5, 2, 6, 3], X[7, 5, 8, 4]];
Rot[pd]
th = PowerExpand[⊖[pd]]
PolyPlot[th]
```

Out[•]=
```
{{{-1, 8, 3}, {-1, 6, 1}, {-1, 2, 5}, {1, 4, 7}}, {0, 0, 0, 0, 1, -1, -1, -1}}
```

Out[•]=

$$\left\{-\frac{1-3\,T+T^2}{T},\ -\frac{\left(1-3\,T_1+T_1^2\right)\,\left(-1+T_1+2\,T_1^2+T_2-7\,T_1^2\,T_2-2\,T_1^3\,T_2-T_1\,T_2^2+4\,T_1^2\,T_2^2+5\,T_1^3\,T_2^2-3\,T_1^2\,T_2^3+T_1^3\,T_2^3\right)}{T_1^2}\right\}$$

Out[•]=



In[•]:=
```
pd = PD[X[2, 8, 3, 1], X[3, 1, 4, 2], X[6, 4, 7, 5], X[7, 5, 8, 6]];
Rot[pd]
th = PowerExpand[⊖[pd]]
PolyPlot[th]
```

Out[•]=
```
{{{-1, 8, 2}, {-1, 1, 3}, {-1, 4, 6}, {-1, 5, 7}}, {0, 0, 1, 0, 0, 1, 1, 0}}
```

Out[•]=

$$\left\{\frac{1}{\sqrt{T}},\ \frac{1}{2\,T_1^3\,T_2^4}\left(4-4\,T_1-6\,T_2+2\,T_1\,T_2+6\,T_1^2\,T_2+\right.\right.$$
$$\left.\left.4\,T_1\,T_2^2-8\,T_1^2\,T_2^2+2\,T_2^3+2\,T_1\,T_2^3+T_1^2\,T_2^3-4\,T_1^3\,T_2^3-2\,T_1\,T_2^4-6\,T_1^2\,T_2^4+10\,T_1^3\,T_2^4\right)\right\}$$

Out[•]=