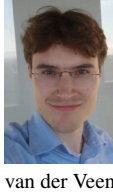# The Strongest Genuinely Computable Knot Invariant in 2024

Abstract. "Genuinely computable" means we have computed it for random knots with over 300 crossings. "Strongest" means it separates prime knots with up to 15 crossings better than the less-computable HOMFLY-PT and Khovanov homology taken together. And hey, it's also meaningful and fun.

Continues Rozansky, Garoufalidis, Kricker, and Ohtsuki, joint with van der Veen.

van der Veen

Strongest. Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis): (ρ_1: [Ro1, Ro2, Ro3, Ov, BV1])

|	knots	(H, Kh)	(Δ, ρ_1)	$\Theta = (\Delta, \theta)$	together
reign		2005-22	2022-24	2024-	
xing ≤ 10	249	248 (1)	249 (0)	249 (0)	249 (0)
xing ≤ 11	801	771 (30)	787 (14)	798 (3)	798 (3)
xing ≤ 12	2,977	(214)	(95)	(19)	(18)
xing ≤ 13	12,965	(1,771)	(959)	(194)	(185)
xing ≤ 14	59,937	(10,788)	(6,253)	(1,118)	(1,062)
xing ≤ 15	313,230	(70,245)	(42,914)	(6,758)	(6,555)

Genuinely Computable. Here's Θ on a random 300 crossing knot (from [DHOEBL]). For almost every other invariant, that's science fiction.

Fun. There's so much more to see in 2D pictures than in 1D ones! Yet almost nothing of the patterns you see we know how to prove. We'll have fun with that over the next few years. Would you join?

Meaningful. θ gives a genus bound (unproven yet with confidence). We hope (with reason) it says something about ribbon knots.

Conventions. T, T_1, and T_2 are indeterminates and $T_3 := T_1 T_2$.

Preparation. Draw an n-crossing knot K as a diagram D as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \ldots, 2n + 1\}$ and with rotation numbers φ_k.
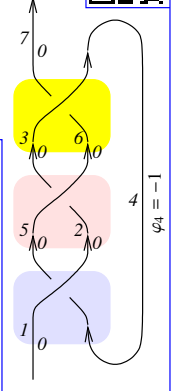
Model T Traffic Rules. Cars always drive forward. When a car crosses over a sign-s bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0$. At the very end, cars fall off and disappear. On various edges *traffic counters* are placed. See also [Jo, LTW].
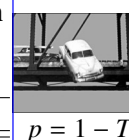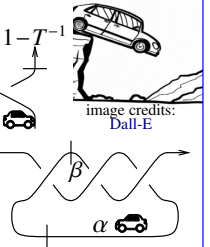
image credits: diamondtraffic.com

$$p = 1 - T^s$$

$1-T \qquad T \qquad 1 \qquad 0 \qquad 0 \qquad 1 \qquad T^{-1} \qquad 1-T^{-1}$
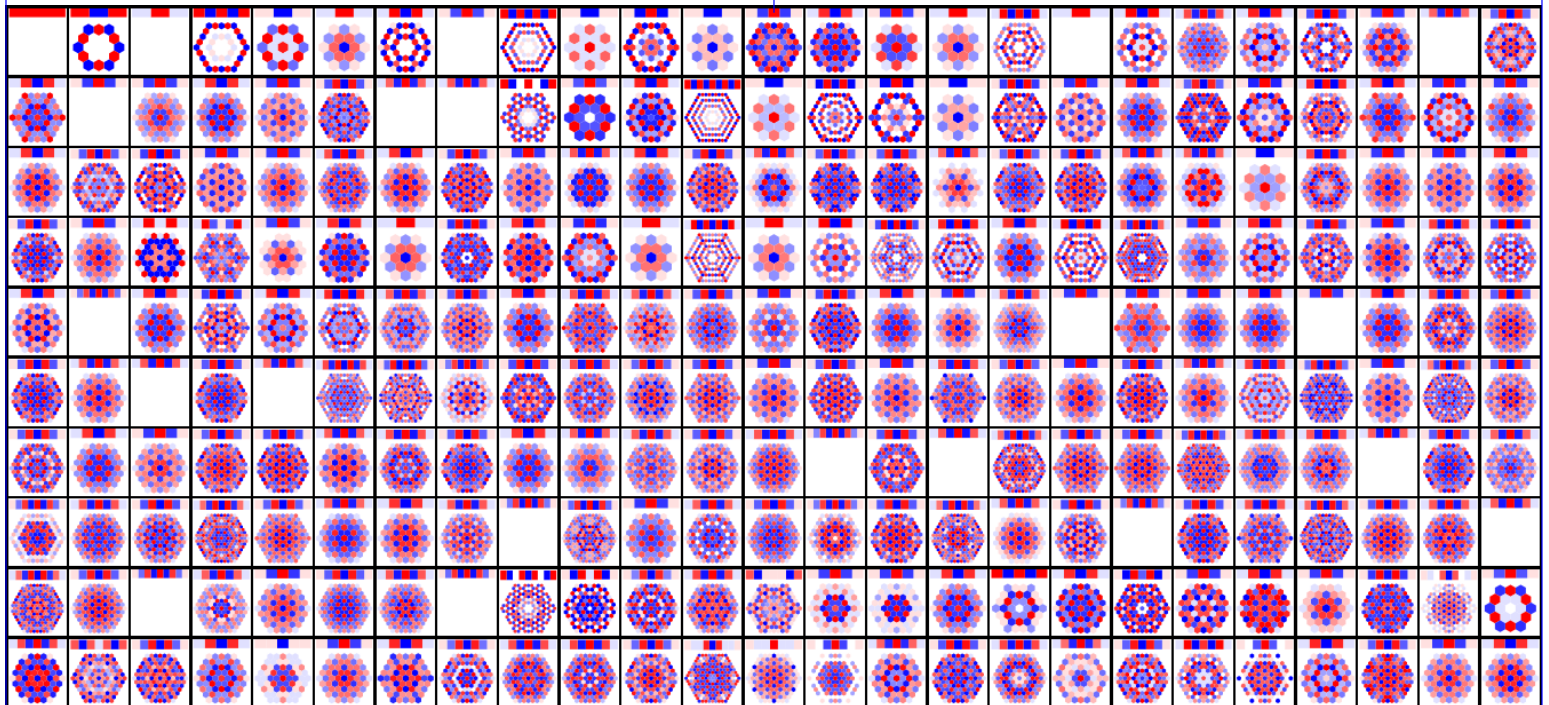
image credits: Dall-E

Definition. The *traffic function* $G = (g_{\alpha\beta})$ (also, the *Green function* or the *two-point function*) is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point). There are also model-T_ν traffic functions $G_\nu = (g_{\nu\alpha\beta})$ for $\nu = 1, 2, 3$.

Example.

$$\sum_{p \geq 0}(1-T)^p = T^{-1} \qquad T^{-1} \qquad \qquad 0$$

$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

**Don't Look.**

$$R_{11}(c) = s\Big[1/2 - g_{3ii} + T_2^s g_{1ii}g_{2ji} - T_2^s g_{3jj}g_{2ji} - (T_2^s-1)g_{3ii}g_{2ji}$$
$$+ (T_3^s-1)g_{2ji}g_{3ji} - g_{1ii}g_{2jj} + 2g_{3ii}g_{2jj} + g_{1ii}g_{3jj} - g_{2ii}g_{3jj}\Big]$$
$$+ \frac{s}{T_2^s-1}\Big[(T_1^s-1)T_2^s\big(g_{3jj}g_{1ji} - g_{2jj}g_{1ji} + T_2^s g_{1ji}g_{2ji}\big)$$
$$+ (T_3^s-1)\big(g_{3ji} - T_2^s g_{1ii}g_{3ji} + g_{2ij}g_{3ji} + (T_2^s-2)g_{2jj}g_{3ji}\big)$$
$$- (T_1^s-1)(T_2^s+1)(T_3^s-1)g_{1ji}g_{3ji}\Big]$$

$$R_{12}(c_0, c_1) = \frac{s_1(T_1^{s_0}-1)(T_3^{s_1}-1)g_{1j_1i_0}g_{3j_0i_1}}{T_2^{s_1}-1}\Big(T_2^{s_0}g_{2i_1i_0} + g_{2j_1j_0} - T_2^{s_0}g_{2j_1i_0} - g_{2i_1j_0}\Big)$$

$$\Gamma_1(\varphi, k) = \varphi(-1/2 + g_{3kk})$$

Theorem. With $c = (s, i, j)$, $c_0 = (s_0, i_0, j_0)$, and $c_1 = (s_1, i_1, j_1)$ denoting crossings, there is a quadratic $R_{11}(c) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i, j\}]$, a cubic $R_{12}(c_0, c_1) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i_0, j_0, i_1, j_1\}]$, and a linear $\Gamma_1(\varphi, k)$ such that the following is a knot invariant:

$$\theta(D) := \underbrace{\Delta_1 \Delta_2 \Delta_3}_{\substack{\text{normalization,} \\ \text{see later}}} \left(\sum_c R_{11}(c) + \sum_{c_0, c_1} R_{12}(c_0, c_1) + \sum_k \Gamma_1(\varphi_k, k) \right),$$

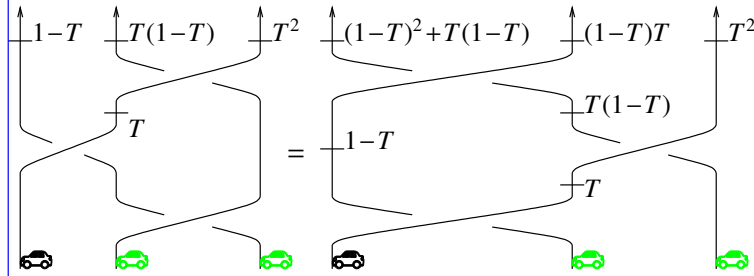

This picture gave the invariant its name
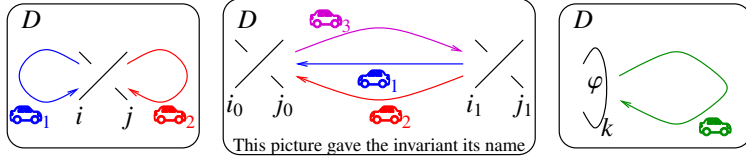
If these pictures remind you of Feynman diagrams, it's because they are Feynman diagrams [BN2].

Lemma 1. The traffic function $g_{\alpha\beta}$ is a "relative invariant":
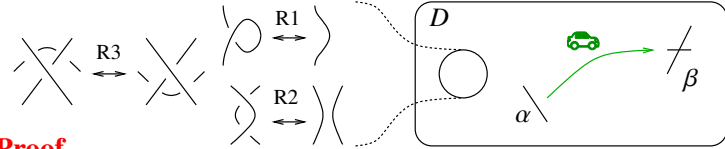
Proof.

Lemma 2. With $k^+ := k + 1$, the "g-rules" hold near a crossing $c = (s, i, j)$:

$$g_{j\beta} = g_{j^+\beta} + \delta_{j\beta} \quad g_{i\beta} = T^s g_{i^+\beta} + (1 - T^s)g_{j^+\beta} + \delta_{i\beta} \quad g_{2n^+,\beta} = \delta_{2n^+,\beta}$$

$$g_{\alpha i^+} = T^s g_{\alpha i} + \delta_{\alpha i^+} \quad g_{\alpha j^+} = g_{\alpha j} + (1 - T^s)g_{\alpha i} + \delta_{\alpha j^+} \quad g_{\alpha,1} = \delta_{\alpha,1}$$

Corollary 1. G is easily computable, for $AG = I (= GA)$, with A the $(2n+1)\times(2n+1)$ identity matrix with additional contributions:

$c = (s, i, j) \mapsto$	A	col i^+	col j^+
row i		$-T^s$	$T^s - 1$
row j		0	-1

For the trefoil example, we have:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{T^2-T+1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note. The Alexander polynomial Δ is given by
$$\Delta = T^{(-\varphi - w)/2} \det(A), \qquad \text{with } \varphi = \sum_k \varphi_k, w = \sum_c s.$$
We also set $\Delta_\nu := \Delta(T_\nu)$ for $\nu = 1, 2, 3$.

Questions, Conjectures, Expectations, Dreams.

Question 1. What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

Conjecture 2. On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

Conjecture 3. θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinery on the double $\mathcal{D}(\mathfrak{b}, b, \epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

Conjecture 4. θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $2g(K) \geq \deg_{T_1} \theta(K)$.

Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

Expectation 8. There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

Dream 9. These invariants can be explained by something less foreign than semisimple Lie algebras.

Dream 10. θ will have something to say about ribbon knots.

References.

[BN1] D. Bar-Natan, *Everything around sl_{2+}^ϵ is DoPeGDO. So what?*, talk in Da Nang, May 2019. Handout and video at ωεβ/DPG.

[BN2] —, *Knot Invariants from Finite Dimensional Integration,* talks in Beijing (July 2024, ωεβ/icbs24) and in Geneva (August 2024, ωεβ/ge24).

[BV1] —, R. van der Veen, *A Perturbed-Alexander Invariant,* Quantum Topology **15** (2024) 449–472, ωεβ/APAI.

[BV2] —, —, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* arXiv:2109.02057.

[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharya, D. Lei, and others, *Random Knots: A Preliminary Report,* lecture notes at ωεβ/DHOEBL. Also a data file at ωεβ/DD.

[GK] S. Garoufalidis, R. Kashaev, *Multivariable Knot Polynomials from Braided Hopf Algebras with Automorphisms,* arXiv:2311.11528.

[GL] —, S. Y. Li, *Patterns of the V_2-polynomial of knots,* arXiv:2409.03557.

[GR] —, L. Rozansky, *The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[Jo] V. F. R. Jones, *Hecke Algebra Representations of Braid Groups and Link Polynomials,* Annals Math., **126** (1987) 335-388.

[Kr] A. Kricker, *The Lines of the Kontsevich Integral and Rozansky's Rationality Conjecture,* arXiv:math/0005284.

[LTW] X-S. Lin, F. Tian, Z. Wang, *Burau Representation and Random Walk on String Links,* Pac. J. Math., **182-2** (1998) 289–302, arXiv:q-alg/9605023.

[Oh] T. Ohtsuki, *On the 2–loop Polynomial of Knots,* Geom. Top. **11** (2007) 1357–1475.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, Aug. 2013, ωεβ/Ov.

[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] —, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] —, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Corollary 2. Proving invariance is easy:
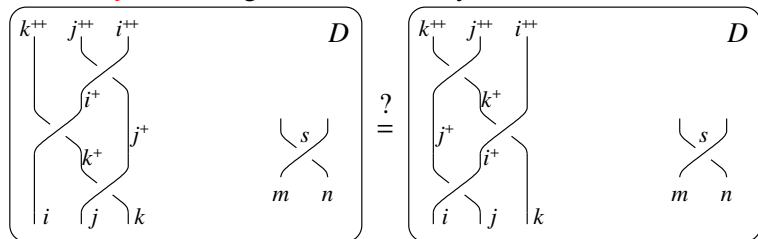
Invariance under R3

This is Theta.nb of http://drorbn.net/to24/ap.

```
☺ Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];

☺ T₃ = T₁ T₂;

☺ CF[ℰ_] :=
    Module[{vs = Union@Cases[ℰ, g__, ∞], ps, c},
      Total[CoefficientRules[Expand[ℰ], vs] /.
        (ps_ → c_) ⇒ Factor[c] (Times @@ vs^ps)]];

☺ R₁₁[{s_, i_, j_}] =
    CF[
      s (1 / 2 - g₃ᵢᵢ + T₂ˢ g₁ᵢᵢ g₂ⱼᵢ - g₁ᵢᵢ g₂ⱼⱼ -
        (T₂ˢ - 1) g₂ⱼᵢ g₃ᵢᵢ + 2 g₂ⱼⱼ g₃ᵢᵢ - (1 - T₃ˢ) g₂ⱼᵢ g₃ⱼᵢ -
        g₂ᵢᵢ g₃ⱼⱼ - T₂ˢ g₂ⱼᵢ g₃ⱼⱼ + g₁ᵢᵢ g₃ⱼⱼ +
        ((T₁ˢ - 1) g₁ⱼᵢ (T₂²ˢ g₂ⱼᵢ - T₂ˢ g₂ⱼⱼ + T₂ˢ g₃ⱼⱼ) +
          (T₃ˢ - 1) g₃ⱼᵢ
          (1 - T₂ˢ g₁ᵢᵢ - (T₁ˢ - 1) (T₂ˢ + 1) g₁ⱼᵢ +
            (T₂ˢ - 2) g₂ⱼⱼ + g₂ᵢⱼ)) / (T₂ˢ - 1))];

☺ R₁₂[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
    CF[s1 (T₁^s0 - 1) (T₂^s1 - 1)^(-1) (T₃^s1 - 1) g₁,ⱼ₁,ᵢ₀ g₃,ⱼ₀,ᵢ₁
      ((T₂^s0 g₂,ᵢ₁,ᵢ₀ - g₂,ᵢ₁,ⱼ₀) - (T₂^s0 g₂,ⱼ₁,ᵢ₀ - g₂,ⱼ₁,ⱼ₀))]

☺ Γ₁[φ_, k_] = - φ / 2 + φ g₃ₖₖ;

☺ δᵢ_,ⱼ_ := If[i === j, 1, 0];
  gRₛ_,ᵢ_,ⱼ_ := {
    g_v_ⱼβ_ ⇒ g_vⱼ⁺β + δⱼβ,
    g_v_ᵢβ_ ⇒ T_v^s g_vᵢ⁺β + (1 - T_v^s) g_vⱼ⁺β + δᵢβ,
    g_v_α_i⁺ ⇒ T_v^s g_vαi + δαi⁺,
    g_v_α_j⁺ ⇒ g_vαj + (1 - T_v^s) g_vαi + δαj⁺
  }

☺ DSum[Cs___] := Sum[R₁₁[c], {c, {Cs}}] +
    Sum[R₁₂[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
  lhs = DSum[{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺},
      {s, m, n}] //. gR₁,ⱼ,ₖ ⋃ gR₁,ᵢ,ₖ⁺ ⋃ gR₁,ᵢ⁺,ⱼ⁺;
  rhs = DSum[{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺},
      {s, m, n}] //. gR₁,ᵢ,ⱼ ⋃ gR₁,ᵢ⁺,ₖ ⋃ gR₁,ⱼ⁺,ₖ⁺;
  Simplify[lhs == rhs]

⌨ True
```

The Main Program

```
☺ Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    A = IdentityMatrix[2 n + 1];
    Cases[Cs, {s_, i_, j_} ⇒
      (A⟦{i, j}, {i + 1, j + 1}⟧ += (-Tˢ  Tˢ - 1
                                      0    -1 ))];
    Δ = T^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A];
    G = Inverse[A];
    ev[ℰ_] :=
      Factor[ℰ /. g_v_,α_,β_ ⇒ (G⟦α, β⟧ /. T → T_v)];
    θ = ev[∑ₖ₁₌₁ⁿ ∑ₖ₂₌₁ⁿ R₁₂[Cs⟦k1⟧, Cs⟦k2⟧]];
    θ += ev[∑ₖ₌₁ⁿ R₁₁[Cs⟦k⟧]];
    θ += ev[∑ₖ₌₁²ⁿ Γ₁[φ⟦k⟧, k]];
    Factor@
      {Δ, (Δ /. T → T₁) (Δ /. T → T₂) (Δ /. T → T₃) θ}];
```

The Trefoil, Conway, and Kinoshita-Terasaka

```
☺ Θ[Knot[3, 1]] // Expand
```

$$\left\{-1 + \frac{1}{T} + T, \; -\frac{1}{T_1^2} - T_1^2 - \frac{1}{T_2^2} - \frac{1}{T_1^2 T_2^2} + \frac{1}{T_1 T_2^2} + \right.$$
$$\left. \frac{1}{T_1^2 T_2} + \frac{T_1}{T_2} + \frac{T_2}{T_1} + T_1^2 T_2 - T_2^2 + T_1 T_2^2 - T_1^2 T_2^2\right\}$$

```
☺ GraphicsRow[PolyPlot[Θ[Knot[#]]] & /@
    {"3_1", "K11n34", "K11n42"}]
```

⌨
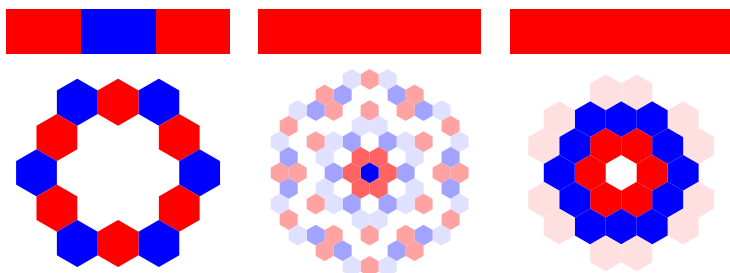
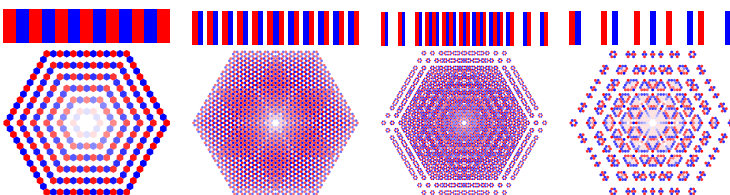(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

Some Torus Knots
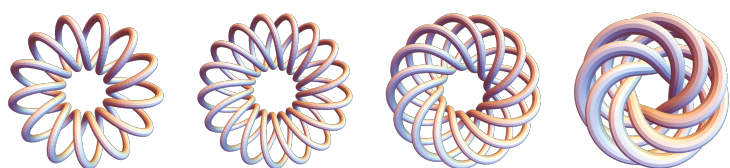
```
☺ TKs = {{13, 2}, {17, 3}, {13, 5}, {7, 6}};
  GraphicsRow[PolyPlot[Θ[TorusKnot @@ #]] & /@ TKs]
  GraphicsRow[TubePlot[TorusKnot @@ #] & /@ TKs]
```

⌨

⌨

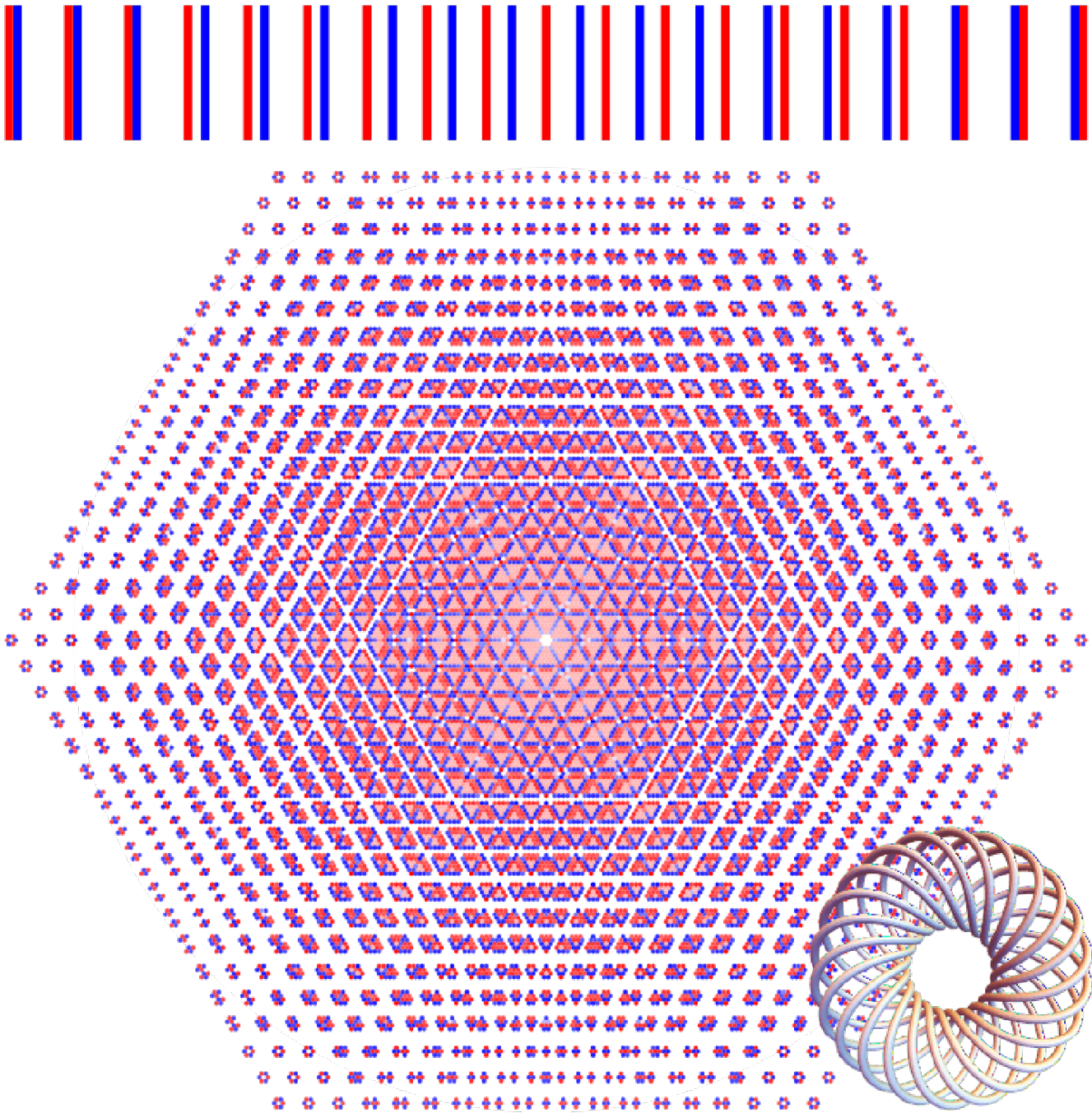The 132-crossing torus knot $T_{22/7}$:

Random knots from [DHOEBL], with 50-73 crossings: