Pensieve header: Implementing Θ - the main notebook accompanying Talks/MonteVerita-2604.

Invariance under R3

exec

```
nb2tex$TeXFileName = "Invariance.tex";
```

This is Theta.nb of http://drorbn.net/mv26/ap.

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\MonteVerita-2604"];
```

pdf

In[•]:=
```
T3 = T1 T2;
```

pdf

In[•]:=
```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

pdf

In[•]:=
```
F1[{s_, i_, j_}] =
  CF[s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj - (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
      g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj + ((T1^s - 1) g1ji (T2^2 s g2ji - T2^s g2jj + T2^s g3jj) +
        (T3^s - 1) g3ji (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji + (T2^s - 2) g2jj + g2ij)) / (T2^s - 1))];
```

pdf

In[•]:=
```
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] := CF[
  s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1 ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
```

pdf

In[•]:=
```
F3[φ_, k_] = -φ/2 + φ g3kk;
```

pdf

```
δi_,j_ := If[i === j, 1, 0];
gRs_,i_,j_ := {
  gν_jβ_ :> gνj⁺β + δjβ, gν_iβ_ :> T2^s gνi⁺β + (1 - T2^s) gνj⁺β + δiβ,
  gν_α_i⁺ :> T2^s gναi + δαi⁺, gν_α_j⁺ :> gναj + (1 - T2^s) gναi + δαj⁺
 }
```

Proof of Reidemeister 3:

pdf

In[•]:=
```
DSum[Cs___] := Sum[F1[c], {c, {Cs}}] + Sum[F2[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}, {s, m, n}] //. gR1,j,k ⋃ gR1,i,k⁺ ⋃ gR1,i⁺,j⁺;
rhs = DSum[{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}, {s, m, n}] //. gR1,i,j ⋃ gR1,i⁺,k ⋃ gR1,j⁺,k⁺;
Simplify[lhs == rhs]
```

Out[•]=
pdf

True

exec

```
nb2tex$TeXFileName = "Program.tex";
nb2tex$PDFWidth = 4.2
```

The Main Program

tex

{\red\bf The Main Program.}

pdf

In[•]:=
```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/v25/ap to compute rotation numbers.
```

pdf

```
Loading PolyPlot.m from http://drorbn.net/mv26/ap to plot 2-variable polynomials.
```

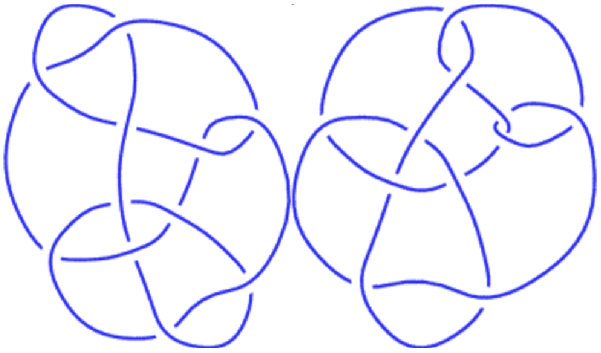
pdf

In[•]:=
```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, Θ},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
  Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
  G = Inverse[A];
  ev[ℰ_] := Factor[ℰ /. g_(ν_, α_, β_) :> (G[[α, β]] /. T -> T_ν)];
  Θ = ev[∑_(k=1)^n F_1[Cs[[k]]]];
  Θ += ev[∑_(k1=1)^n ∑_(k2=1)^n F_2[Cs[[k1]], Cs[[k2]]]];
  Θ += ev[∑_(k=1)^(2 n) F_3[φ[[k]], k]];
  Factor@{Δ, (Δ /. T -> T_1) (Δ /. T -> T_2) (Δ /. T -> T_3) Θ}]
```

The Trefoil, Conway, and Kinoshita-Terasaka Knots

tex

\vskip 3mm
{\red\bf The Trefoil, Conway, and Kinoshita-Terasaka Knots.}
\parpic[r]{\parbox{15mm}{
 \includegraphics[width=15mm]{../PhuQuoc-2506/K11n34.png}
 \vskip 1mm
 \includegraphics[width=15mm]{../PhuQuoc-2506/K11n42.png}
}}



pdf

In[•]:=
```
Θ[Knot[3, 1]] // Expand
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

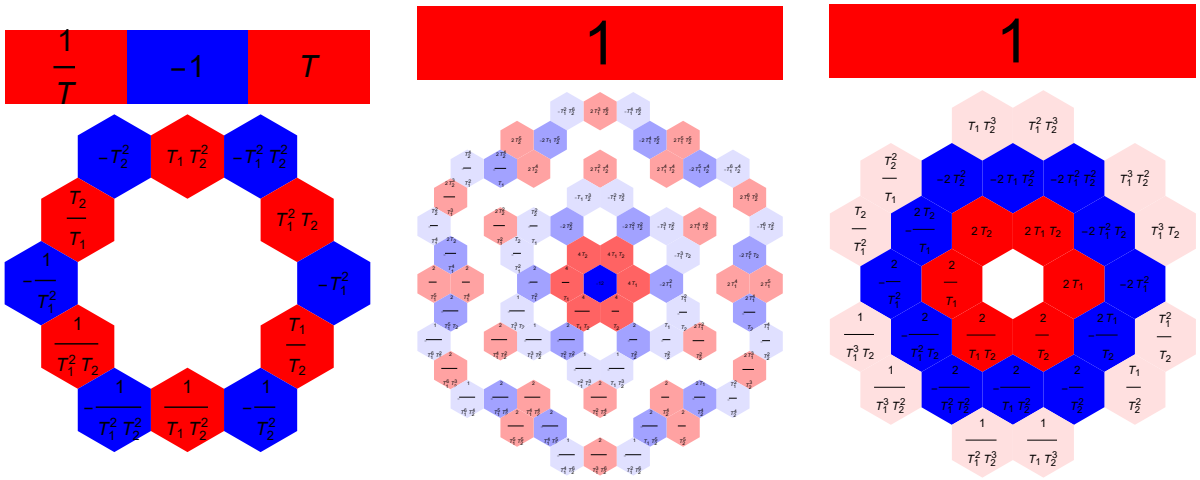
Out[•]=
pdf

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2\,T_2^2}+\frac{1}{T_1\,T_2^2}+\frac{1}{T_1^2\,T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2\,T_2-T_2^2+T_1\,T_2^2-T_1^2\,T_2^2\right\}$$

pdf

In[•]:=
```
GraphicsRow[PolyPlot[Θ[Knot[#]], Labeled → True] & /@ {"3_1", "K11n34", "K11n42"}]
```

Out[•]=
pdf



tex

(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

Some Torus Knots

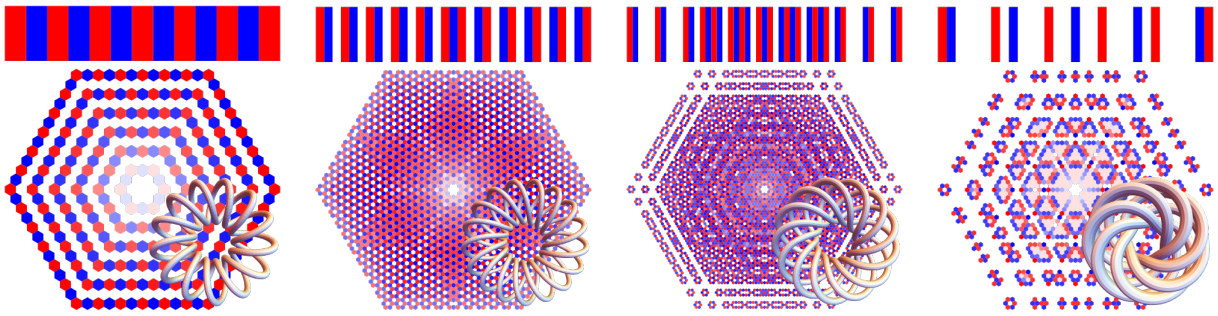
tex

{\red\bf Some Torus Knots.}

pdf

In[•]:=
```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
   ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```

Out[•]=
pdf



The Rank 2 Formulas

exec

```
nb2tex$PDFWidth = 4.625
```

pdf:LX

In[•]:=
```
ℒ[X_{i_,j_}[s_]] := T3^s 𝔼[CF@Plus[
    Σ_{ν=1}^{3} (x_{νi} (p_{νi'} - p_{νi}) + x_{νj} (p_{νj'} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi'} - p_{νj'})),
    (T1^s - 1) p_{3j} x_{1i} (T2^s x_{2i} - x_{2j}),
    ϵ s (T3^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T2^s - 1),
    ϵ s (1/2 + T2^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} - (T2^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} +
        (T3^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} + 2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} - T2^s p_{2j} p_{3j} x_{2i} x_{3j} +
        ((T1^s - 1) p_{1j} x_{1i} (T2^{2 s} p_{2j} x_{2i} - T2^s p_{2j} x_{2j} - (T2^s + 1) (T3^s - 1) p_{3j} x_{3i} + T2^s p_{3j} x_{3j}) +
            (T3^s - 1) p_{3j} x_{3i} (1 - T2^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T2^s - 2) p_{2j} x_{2j})) / (T2^s - 1))]]
```

pdf:LC

In[•]:=
```
ℒ[C_{i_}[φ_]] := T3^φ 𝔼[Σ_{ν=1}^{3} x_{νi} (p_{νi'} - p_{νi}) + ϵ φ (p_{3i} x_{3i} - 1/2)]
```

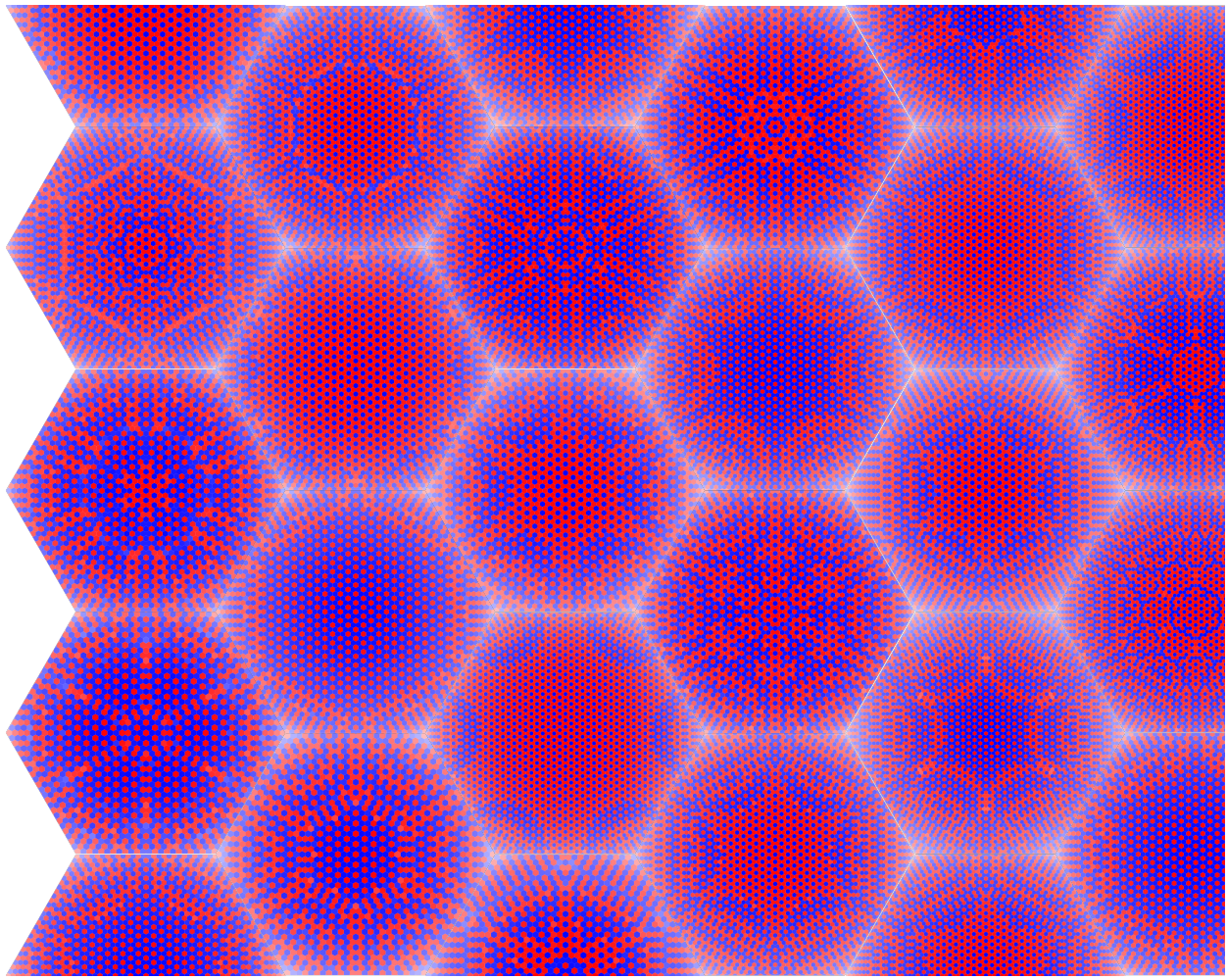
The Beehive

In[•]:=
```
Data51to77 =
  (Get["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank\\DunfieldKnots\\D0" <>
        ToString[#] <> ".m"][[2, 2]] /. {T1 → T_1, T2 → T_2}) & /@ Range[51, 77];
```

In[•]:=
```
Images51to77 = PolyPlot2 /@ Data51to77;
```

In[•]:=
```
Show[img = 0;
 Beehive = Graphics[{
    {0, 0}, {1, 0}, {2, 0}, {3, 0}, {4, 0},
       {0, 1}, {1, 1}, {2, 1}, {3, 1},
    {-1, 2}, {0, 2}, {1, 2}, {2, 2}, {3, 2},
       {-1, 3}, {0, 3}, {1, 3}, {2, 3},
    {-2, 4}, {-1, 4}, {0, 4}, {1, 4}, {2, 4},
       {-2, 5}, {-1, 5}, {0, 5}, {1, 5}
   } /. {x_, y_} ↦
    Inset[
     SetAlphaChannel[Images51to77[[++img]], Graphics[{White,
        Polygon[Table[{Cos[t], Sin[t]}, {t, 0, 5 π / 3, π / 3}]]}, Background → Black]],
     x {0, 1} + y {√3 / 2, 1 / 2}, Center, 1.2
    ],
   PlotRange → {{-0.6, 4.91}, {0, 4}}, ImageSize → 1200],
 ImageSize → Small]
```

Out[•]=



In[•]:=
```
Export["Beehive.pdf", ImageCrop[Beehive]]
```

Out[•]=

Beehive.pdf