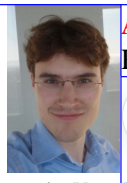
Chern-Simons-Witten Theory Near The Co-Commutative Limit

Abstract. Perhaps every algebra meeting should have one analysis talk (and vice versa), lest we forget that the other exists. In my role as the outsider, I will tell you today about the other – perturbative – evaluation of path integrals, where instead of hoping that nature will help us compute faster, we approximate nature by things we already can compute quickly.

Specifically I will tell you how in the Chern-Simons-Witten theory you can perturb the base Lie algebra from where it's easy towards where it's strong, leading to the strongest genuinely computable knot invariant we presently have.

I wish I could give my talk in the language of the Kabbalah, but I ain't smart enough for that. So I'll highlight the Kabbalistic points that we're still missing, and then stick to the Talmud.



van der Veen

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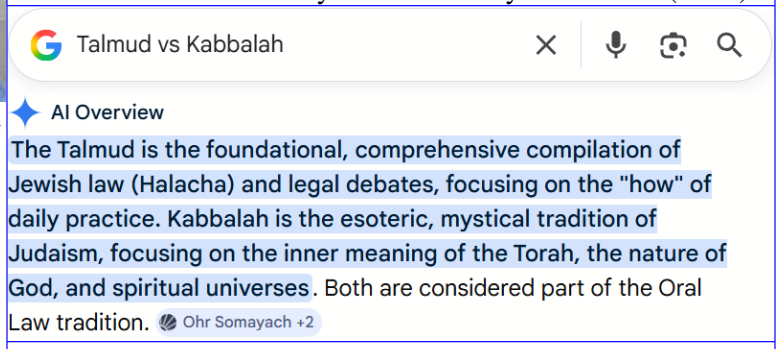


Chern-Simons-Witten with Lie algebra $\mathfrak{g}$ and representation V:

$$\int_{A\in\Omega^1(\mathbb{R}^3;\mathfrak{g})} \mathcal{D}A\, e^{\frac{i}{4\pi}\int_{\mathbb{R}} A\wedge dA+\frac{2\hbar}{3}A\wedge A\wedge A}\, \mathrm{tr}_V\, \mathcal{P}\exp_\gamma(\hbar A)$$

Same as old

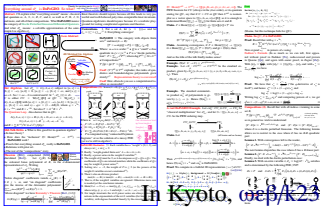
Missing Kabbalah:
1. Exact evaluation at $\epsilon = 0$ giving Δ.
2. Then perturbation theory.

Old Kabbalah

Old Talmud

Universal Reshetikhin-Turaev for $\mathfrak{g}_\epsilon$ with $q = e^{\hbar^2\epsilon}$

ϵ-expansion: perturbation theory, Feynman diagrams



In Kyoto, ~~ωεβ/k23~~

Today's invariants:

$$\left(\begin{array}{c|c|c|c} \epsilon=0 & \epsilon^2=0 & \epsilon^3=0 & \dots \\ \text{Alexander's } \Delta & \rho_1,\theta & \rho_2,\dots & \dots \end{array}\right)$$

Solvable Approximation:
$\mathfrak{g} \to \mathfrak{g}_\epsilon$

Chern-Simons-Witten with Lie algebra $\mathfrak{g}_\epsilon$:

$$\int_{A\in\Omega^1(\mathbb{R}^3;\mathfrak{g}_\epsilon)} \mathcal{D}A\, e^{\frac{i}{4\pi}\int_{\mathbb{R}} A\wedge dA+\frac{2\hbar}{3}A\wedge A\wedge A}\mathcal{P}\exp_\gamma(\hbar A)$$

Same as old

Missing Kabbalah:
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New Kabbalah

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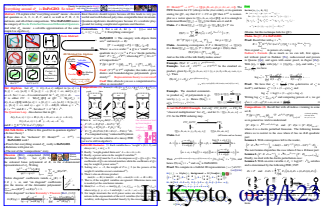

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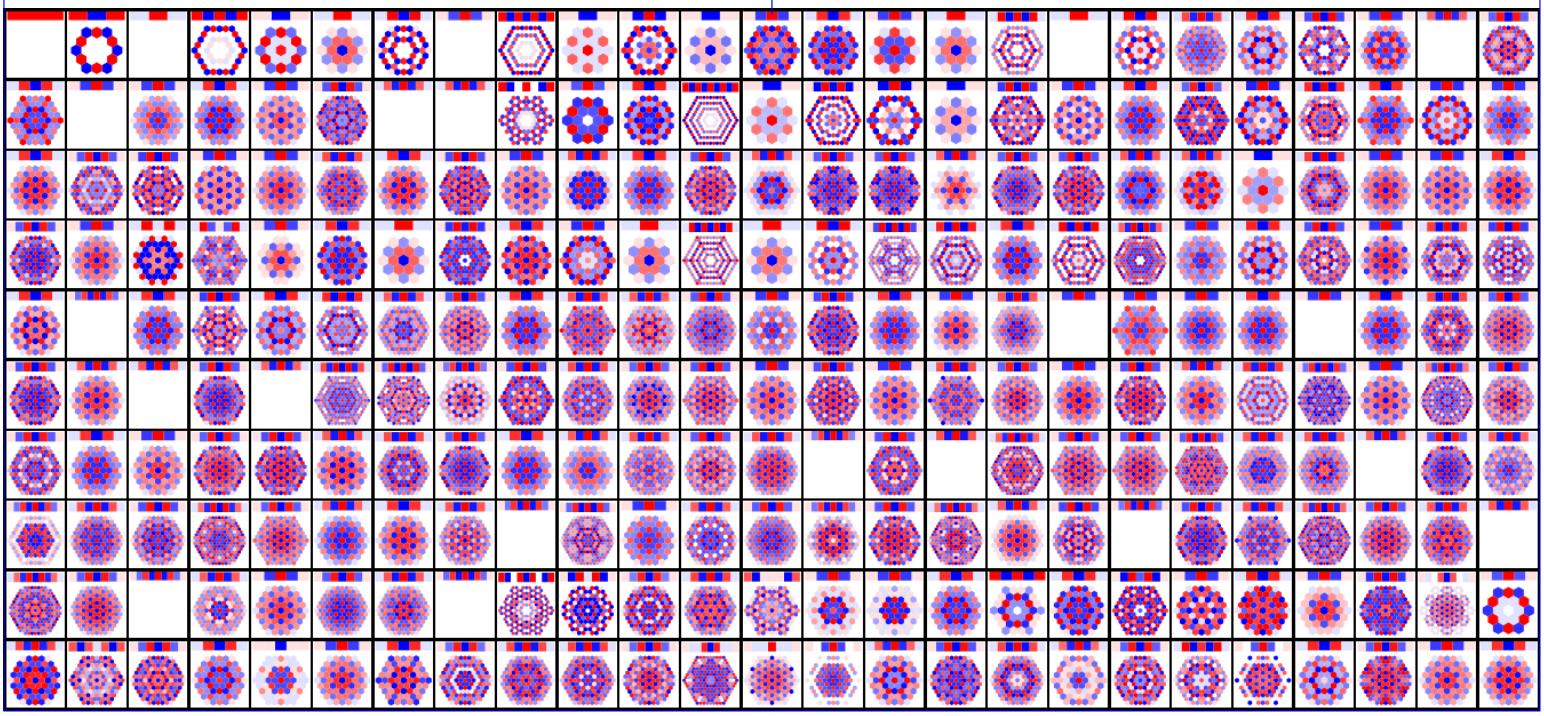

TBW.

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[BN3] —, *Homework,* talk in Les Diablerets, January 2026. Handout and video at ωεβ/HW.

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