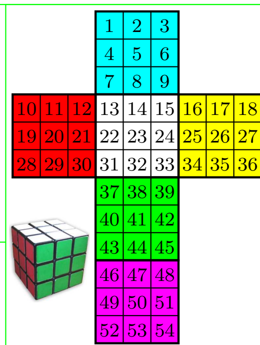
Non-Commutative Gaussian Elimination and Rubik's Cube

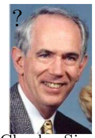
Joint study with
Itai Bar-Natan

The Problem. Let $G = \langle g_1, \dots, g_\alpha \rangle$ be a subgroup of S_n, with $n = O(100)$. Before you die, understand G:

1. Compute $|G|$.
2. Given $\sigma \in S_n$, decide if $\sigma \in G$.
3. Write a $\sigma \in G$ in terms of $g_1, \dots, g_\alpha$.
4. Produce *random* elements of G.



Based on an algorithm by



Otto Schreier Charles Sims

See also *Permutation Group Algorithms* by Ákos Seress.

The Commutative Analog. Let $V = \text{span}(v_1, \dots, v_\alpha)$ be a subspace of $\mathbb{R}^n$. Before you die, understand V.

Solution: Gaussian Elimination. Prepare an empty table,

1	2	3	4	$\cdots$	$n-1$	n

Space for a vector $u_4 \in V$, of the form $u_4 = (0, 0, 0, 1, *, \dots, *)$; 1 :="the pivot".

Feed $v_1, \dots, v_\alpha$ in order. To feed a non-zero v, find its pivotal position i.

1. If box i is empty, put v there.
2. If box i is occupied, find a combination v' of v and u_i that eliminates the pivot, and feed v'.

The Generators

```
In[1]:= gs = {
  purple = P[18,27,36,4,5,6,7,8,9,3,11,12,13,14,15,16,17,
   45,2,20,21,22,23,24,25,26,44,1,29,30,31,32,33,34,35,43,
   37,38,39,40,41,42,10,19,28,52,49,46,53,50,47,54,51,48],
  white = P[1,2,3,4,5,6,16,25,34,10,11,9,15,24,33,39,17,
   18,19,20,8,14,23,32,38,26,27,28,29,7,13,22,31,37,35,36,
   12,21,30,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54],
  green = P[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,
   19,20,21,22,23,24,25,26,27,31,32,33,34,35,36,48,47,46,
   39,42,45,38,41,44,37,40,43,30,29,28,49,50,51,52,53,54],
  blue = P[3,6,9,2,5,8,1,4,7,54,53,52,10,11,12,13,14,15,
   19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,
   37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,18,17,16],
  red = P[13,2,3,22,5,6,31,8,9,12,21,30,37,14,15,16,17,
   18,11,20,29,40,23,24,25,26,27,10,19,28,43,32,33,34,35,
   36,46,38,39,49,41,42,52,44,45,1,47,48,4,50,51,7,53,54],
  yellow = P[1,2,48,4,5,51,7,8,54,10,11,12,13,14,3,18,27,
   36,19,20,21,22,23,6,17,26,35,28,29,30,31,32,9,16,25,34,
   37,38,15,40,41,24,43,44,33,46,47,39,49,50,42,52,53,45]
 };
```
Enter

Non-Commutative Gaussian Elimination

Prepare a mostly-empty table,

$(1,1)$ I				
$(1,2)$	$(2,2)$ I			
$(1,3)$	$(2,3)$	$(3,3)$ I		
$\vdots$		(i,j)	$\ddots$	
$(1,n)$	$(2,n)$	$(3,n)$	$\cdots$	(n,n) I

Space for a $\sigma_{i,j} \in S_n$ of the form
$(1, 2, \dots, i-2, i-1, j, *, *, \dots, *)$
So $\sigma_{i,j}$ fixes $1, \dots, i-1$,
sends "the pivot" i to j and
goes wild afterwards, and
$\sigma_{i,j}^{-1}$ "does sticker j".

Feed $g_1, \dots, g_\alpha$ in order. To feed a non-identity σ, find its pivotal position i and let $j := \sigma(i)$.

1. If box (i, j) is empty, put σ there.
2. If box (i, j) contains $\sigma_{i,j}$, feed $\sigma' := \sigma_{i,j}^{-1}\sigma$.

The Twist. When done, for every occupied (i, j) and (k, l), feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

Claim. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table T.

Claim. Anything fed in T is a monotone product in T:

f was fed $\Rightarrow$ $f \in M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \ \& \ \sigma_{i,j_i} \in T\}$

Theorem. $G = M_1$. *G^{-1} is more fun!*

$$G = M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$$

Proof. The inclusions $M_1 \subset G$ and $\{g_1, \dots, g_\alpha\} \subset M_1$ are obvious. The rest follows from the following

Lemma. M_1 is closed under multiplication.

Proof. By backwards induction. Let

$$M_k := \{\sigma_{k,j_k}\cdots\sigma_{n,j_n} : \forall i \geq k, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$$

Clearly $M_n M_n \subset M_n$. Now assume that $M_5 M_5 \subset M_5$ and show that $M_4 M_4 \subset M_4$. Start with $\sigma_{8,j} M_4 \subset M_4$:

$$\sigma_{8,j}(\sigma_{4,j_4}M_5) \overset{1}{=} (\sigma_{8,j}\sigma_{4,j_4})M_5 \overset{2}{\subset} M_4 M_5$$

$$\overset{3}{=} \sigma_{4,j_4}(M_5 M_5) \overset{4}{\subset} \sigma_{4,j_4} M_5 \subset M_4$$

(1: associativity, 2: thank the twist, 3: associativity and tracing i_4, 4: induction). Now the general case

$$(\sigma_{4,j_4'}\sigma_{5,j_5'}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$

falls like a chain of dominos.

Problem Solved!

A Demo Program

```
In[2]:= ($RecursionLimit = 2^16;
  n = 54;
  P /: p_P ** P[a___] := p[[{a}]];
  Inv[p_P] := P @@ Ordering[p];
  Feed[P @@ Range[n]] := Null;
  Feed[p_P] := Module[{i, j},
    For[i = 1, p[[i]] == i, ++i];
    j = p[[i]];
    If[Head[s[i, j]] === P,
      Feed[Inv[s[i, j]] ** p],
    (* Else *) s[i, j] = p;
      Do[If[Head[s[k, l]] == P,
        Feed[s[i, j] ** s[k, l]];
        Feed[s[k, l] ** s[i, j]]
      ], {k, n}, {l, n}]
    ]]);
```
Enter

www.geocities.com/jaapsch/puzzles/

www.powerstrike.net/puzzles/

http://lodelnic.free.fr/

Homework Problem 1.

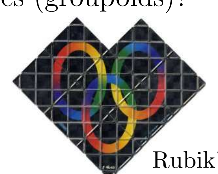
Can you do cosets?

www.powerstrike.net/puzzles/

Homework Problem 2.

Can you do categories (groupoids)?

7	9	2	5
1	4	8	3
6	10	11	12
13	14	15	



Rubik's magic

The Results

```
In[3]:= (Feed[#]; Product[1 + Length[Select[Range[n], Head[s[i, #]] === P &]], {i, n}]) & /@ gs
Out[3]= {4, 16, 159993501696000, 21119142223872000, 43252003274489856000, 43252003274489856000}
```
Enter — that's cool!