

I'm not a category theory person! So I cannot give a talk of any depth on category theory. The only idea I have is to repeat with some additions a short talk I gave in Toronto in 2019, which was titled "Geography vs. Identity" (see <http://drorbn.net/to19/>). Here's the gist:

When you describe one manifold embedded in another (e.g., when you talk about knots) you can do it from the perspective of the ambient manifold or from the perspective of the embedded manifold. For reason I will explain in my talk, if it will be given, the former is "geography" and the latter is "identity". After making the distinction, I will say that almost all appearances of category theory within low dimensional topology are within the "geography" perspective. But then I will tell a few stories from the "identity" perspective, in which some generalized algebraic structures appear, but they are not, generally speaking, categories. So in some sense my talk will be, "here is category theory; I'm talking about something else".

Add Geography $\longrightarrow$ "Hopf algebra objects in a monoidal category"

Identity $\longrightarrow$ "Mate-Hopf algebra"

Add AI statement.



Geography vs. Identity

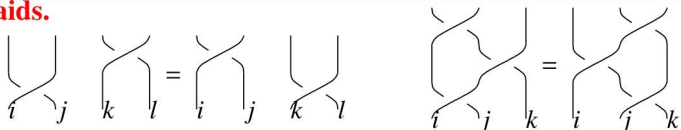
Thanks for inviting me to the *Topology* session!

Abstract. Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

Geographers care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".

Identifiers believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation m_c^{ab} , and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See ωεβ/reg, ωεβ/kbh.

Braids.



Geography:

$$GB := \langle \gamma_i \rangle \left(\begin{array}{l} \gamma_i \gamma_k = \gamma_k \gamma_i \text{ when } |i - k| > 1 \\ \gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1} \end{array} \right) = B.$$

Identity: (captures quantum algebra!)

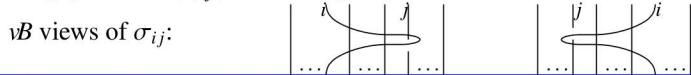
$$IB := \langle \sigma_{ij} \rangle \left(\begin{array}{l} \sigma_{ij} \sigma_{kl} = \sigma_{kl} \sigma_{ij} \text{ when } \{|i, j, k, l|\} = 4 \\ \sigma_{ij} \sigma_{ik} \sigma_{jk} = \sigma_{jk} \sigma_{ik} \sigma_{ij} \text{ when } \{|i, j, k|\} = 3 \end{array} \right) = PB.$$

Theorem. Let $S = \{\tau\}$ be the symmetric group. Then vB is both

$$PB \rtimes S \cong B * S \left(\gamma_i \tau = \tau \gamma_j \text{ when } \tau i = j, \tau(i+1) = (j+1) \right)$$

(and so PB is "bigger" than B , and hence quantum algebra doesn't see topology very well).

Proof. Going left, $\gamma_i \mapsto \sigma_{i,i+1}(i \ i+1)$. Going right, if $i < j$ map $\sigma_{ij} \mapsto (j-1 \ j-2 \ \dots \ i) \gamma_{j-1}(i \ i+1 \ \dots \ j)$ and if $i > j$ use $\sigma_{ij} \mapsto (j \ j+1 \ \dots \ i) \gamma_j(i \ i-1 \ \dots \ j+1)$.



The Burau Representation of PB_n acts on $R^n := \mathbb{Z}[t^{\pm 1}]^n = R\langle v_1, \dots, v_n \rangle$ by

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ωεβ/code

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$(\text{bas3} = \{v_1, v_2, v_3\}) // B_{1,2}$

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S_n acts on R^n by permuting the v_i so the Burau representation extends to vB_n and restricts to B_n . With this, γ_i maps $v_i \mapsto v_{i+1}$, $v_{i+1} \mapsto t v_i + (1-t) v_{i+1}$, and otherwise $v_k \mapsto v_k$.



Werner Burau

Geography view:

$$\gamma_1 = \begin{array}{c} \diagup \diagdown \\ | \end{array} \quad \gamma_2 = \begin{array}{c} \diagdown \diagup \\ | \end{array} \quad \gamma_3 = \begin{array}{c} | \diagup \diagdown \end{array} \dots$$

so x is γ_2 .

Identity view:

At x strand 1 crosses strand 3, so x is σ_{13} .



The Gold Standard is set by the "T-calculus" Alexander formulas (ωεβ/mac). An S -component tangle T has

$$\Gamma(T) \in R_S \times M_{S \times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\} \text{ with } R_S := \mathbb{Z}(\{T_a : a \in S\}):$$

$$(a \nearrow b, b \nwarrow a) \rightarrow \begin{array}{c|c} a & b \\ \hline b & 0 \end{array} \begin{array}{c|c} 1 & 1 - T_a^{\pm 1} \\ \hline 0 & T_a^{\pm 1} \end{array} \quad T_1 \sqcup T_2 \rightarrow \begin{array}{c|c} \omega_1 \omega_2 & S_1 \ S_2 \\ \hline S_1 & A_1 \ 0 \\ S_2 & 0 \ A_2 \end{array}$$

$$\begin{array}{c|c|c|c} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow{T_a, T_b \rightarrow T_c} \begin{array}{c|c|c|c} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}$$

The Gassner Representation of PB_n acts on $V = R^n := \mathbb{Z}[t^{\pm 1}]^n = R\langle v_1, \dots, v_n \rangle$ by

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$G_{i,j}[\underline{\varepsilon}] := \underline{\varepsilon} / \cdot v_{k-} \mapsto v_k + \delta_{k,j} (t-1) (v_j - v_i) // \text{Expand}$

$(\text{bas3} // G_{1,2} // G_{1,3} // G_{2,3}) = (\text{bas3} // G_{2,3} // G_{1,3} // G_{1,2})$

True

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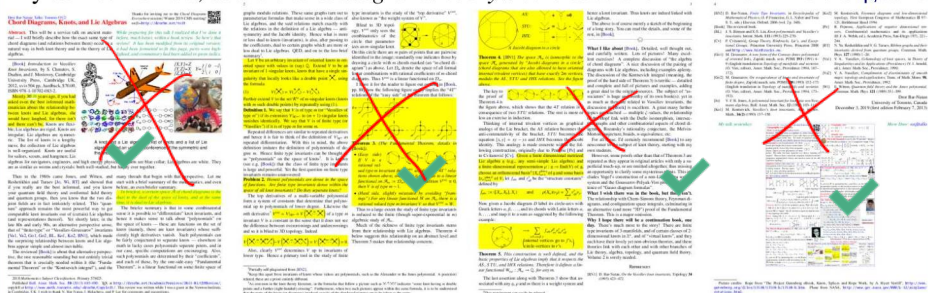
With Roland van der Veen

Gassner motifs

Adjoint-Gassner

My talk tomorrow, at the *chord diagrams everywhere* session:

More Dror: ωεβ/talks





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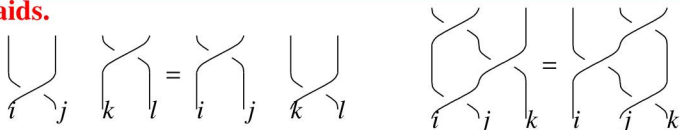
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νB views of σ_{ij} :



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