

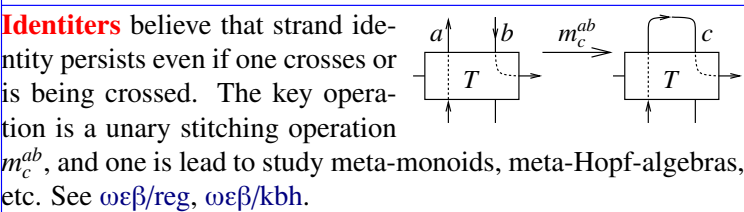
Geography vs. Identity

Thanks for inviting me to Macquarie University!

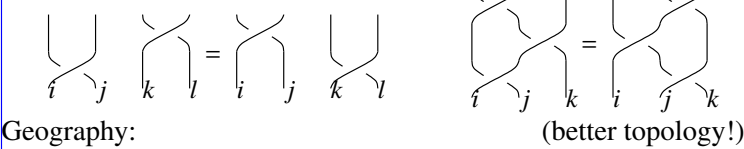


Abstract. When you describe one manifold embedded in another (e.g., when you talk about knots) you can do it from the perspective of the ambient manifold or from the perspective of the embedded manifold. For reason I will explain in my talk, the former is “geography” and the latter is “identity”. After making the distinction, I will say that almost all appearances of category theory within low-dimensional topology are within the “geography” perspective. But then I will tell a few stories from the “identity” perspective for which the only generalization is algebraic structures appear, but they are not a generalization of categories. So in some sense, category theory is not the right category for talking about something topology.

Geographers believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation m_c^{ab} , and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See $\omega\epsilon\beta/\text{reg}$, $\omega\epsilon\beta/\text{kbbh}$.



Braids.



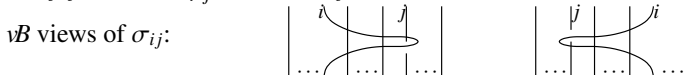
$$GB := \langle \gamma_i \rangle \left(\begin{array}{l} \gamma_i \gamma_k = \gamma_k \gamma_i \text{ when } |i - k| > 1 \\ \gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1} \end{array} \right) = B.$$

Identity: (captures quantum algebra!)

$$IB := \langle \sigma_{ij} \rangle \left(\begin{array}{l} \sigma_{ij} \sigma_{kl} = \sigma_{kl} \sigma_{ij} \text{ when } \{|i, j, k, l|\} = 4 \\ \sigma_{ij} \sigma_{ik} \sigma_{jk} = \sigma_{jk} \sigma_{ik} \sigma_{ij} \text{ when } \{|i, j, k|\} = 3 \end{array} \right) = PB.$$

Theorem. Let $S = \{\tau\}$ be the symmetric group. Then vB is both $PB \rtimes S \cong B * S / (\gamma_i \tau = \tau \gamma_j \text{ when } \tau i = j, \tau(i+1) = (j+1))$ (and so PB is “bigger” than B , and hence quantum algebra doesn’t see topology very well).

Proof. Going left, $\gamma_i \mapsto \sigma_{i,i+1}(i \ i+1)$. Going right, if $i < j$ map $\sigma_{ij} \mapsto (j-1 \ j-2 \ \dots \ i) \gamma_{j-1}(i \ i+1 \ \dots \ j)$ and if $i > j$ use $\sigma_{ij} \mapsto (j \ j+1 \ \dots \ i) \gamma_j(i \ i-1 \ \dots \ j+1)$.



The Burau Representation of PB_n acts on $R^n := \mathbb{Z}[t^{\pm 1}]^n = R\langle v_1, \dots, v_n \rangle$ by

$$\sigma_{ij} v_k = v_k + \delta_{kj}(t-1)(v_j - v_i).$$



Werner Burau

$\delta := \delta_{i,j} := \text{If}[i = j, 1, 0];$
 $B_{i,j}[\underline{\epsilon}] := \underline{\epsilon} / \cdot v_{k-} \mapsto v_k + \delta_{k,j} (t-1) (v_j - v_i) // \text{Expand}$

$(\text{bas3} = \{v_1, v_2, v_3\}) // B_{1,2}$
 $\{v_1, v_1 - t v_1 + t v_2, v_3\}$
 $\text{bas3} // B_{1,2} // B_{1,3} // B_{2,3}$
 $\{v_1, v_1 - t v_1 + t v_2, v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3\}$
 $\text{bas3} // B_{2,3} // B_{1,3} // B_{1,2}$
 $\{v_1, v_1 - t v_1 + t v_2, v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3\}$

S_n acts on R^n by permuting the v_i so the Burau representation extends to vB_n and restricts to B_n . With this, γ_i maps $v_i \mapsto v_{i+1}$, $v_{i+1} \mapsto t v_i t(1-t) v_{i+1}$, and otherwise $v_k \mapsto v_k$.

Geography view:

$$\gamma_1 = \begin{array}{c} \diagup \diagdown \\ | \end{array} \quad \gamma_2 = \begin{array}{c} \diagdown \diagup \\ | \end{array} \quad \gamma_3 = \begin{array}{c} | \diagup \diagdown \end{array} \dots$$

so x is γ_2 .

Identity view:

At x strand 1 crosses strand 3, so x is σ_{13} .



The Gold Standard is set by the “T-calculus” Alexander formulas ($\omega\epsilon\beta/\text{mac}$). An S -component tangle T has

$$\Gamma(T) \in R_S \times M_{S \times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\} \text{ with } R_S := \mathbb{Z}(\{T_a : a \in S\}):$$

$$(a \nearrow b, b \nwarrow a) \rightarrow \begin{array}{c|c|c} 1 & a & b \\ \hline a & 1 & 1 - T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array} \quad T_1 \sqcup T_2 \rightarrow \begin{array}{c|c|c} \omega_1 \omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|c|c} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[m_c^{ab}]{T_a, T_b \rightarrow T_c} \left(\begin{array}{c|c} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array} \right)$$

The Gassner Representation of PB_n acts on $V = R^n := \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^n = R\langle v_1, \dots, v_n \rangle$ by

$$\sigma_{ij} v_k = v_k + \delta_{kj}(t_i - 1)(v_j - v_i).$$

$G_{i,j}[\underline{\epsilon}] := \underline{\epsilon} / \cdot v_{k-} \mapsto v_k + \delta_{k,j} (t_i - 1) (v_j - v_i) // \text{Expand}$

$(\text{bas3} // G_{1,2} // G_{1,3} // G_{2,3}) = (\text{bas3} // G_{2,3} // G_{1,3} // G_{1,2})$

True

S_n acts on R^n by permuting the v_i and the t_i , so the Gassner representation extends to vB_n and then restricts to B_n as a $\mathbb{Z}$ -linear ∞ -dimensional representation. It then descends to PB_n as a finite-rank R -linear representation, with lengthy non-local formulas.

Geographers: Gassner is an obscure partial extension of Burau.

Identifiers: Burau is a trivial silly reduction of Gassner.

The Turbo-Gassner Representation. With the same R and V , TG acts on $V \oplus (R^n \otimes V) \oplus (S^2 V \otimes V^*) = R\langle v_k, v_{lk}, u_i u_j w_k \rangle$ by

$TG_{i,j}[\underline{\epsilon}] := \underline{\epsilon} / \cdot \{$
 $v_{k-} \mapsto v_k + \delta_{k,j} ((t_i - 1) (v_j - v_i) + v_{i,j} - v_{i,i}) +$
 $\delta_{k,i} (u_j - u_i) u_i w_j,$
 $v_{L-,k-} \mapsto v_{L-,k} + (t_i - 1) \times$
 $(\delta_{k,j} (v_{L-,j} - v_{L-,i}) + (\delta_{L-,i} - \delta_{L-,j} t_i^{-1} t_j)$
 $(u_k + \delta_{k,j} (t_i - 1) (u_j - u_i)) u_i w_j),$
 $u_{k-} \mapsto u_k + \delta_{k,j} (t_i - 1) (u_j - u_i),$
 $w_{k-} \mapsto w_k + (\delta_{k,j} - \delta_{k,i}) (t_i^{-1} - 1) w_j // \text{Expand}$

$\text{bas3} = \{v_1, v_2, v_3, v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,1},$
 $v_{3,2}, v_{3,3}, u_1^2 w_1, u_1^2 w_2, u_1^2 w_3, u_1 u_2 w_1, u_1 u_2 w_2, u_1 u_2 w_3,$
 $u_1 u_3 w_1, u_1 u_3 w_2, u_1 u_3 w_3, u_2^2 w_1, u_2^2 w_2, u_2^2 w_3, u_2 u_3 w_1,$
 $u_2 u_3 w_2, u_2 u_3 w_3, u_3^2 w_1, u_3^2 w_2, u_3^2 w_3\};$

$(\text{bas3} // TG_{1,2} // TG_{1,3} // TG_{2,3}) = (\text{bas3} // TG_{2,3} // TG_{1,3} // TG_{1,2})$

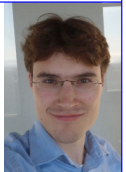
True

Like Gassner, TG is also a representation of PB_n .

I have no idea where it belongs!



Betty Jane Gassner deserves to be more famous



With Roland van der Veen

Gassner motifs

Adjoint-Gassner