Do Not Turn Over Until Instructed

Dror Bar-Natan: Talks: MAASeaway-1810:

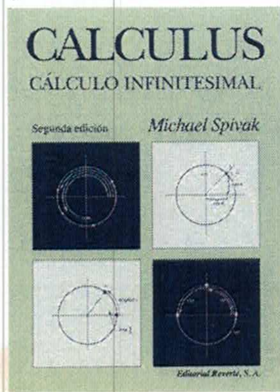
Thanks for inviting me to the MAA Seaway meeting!

ωεβ:=http://drorbn.net/maa18/

My Favourite First-Year Analysis Theorem

Abstract. Whatever it may be, it should say something useful and exciting and it should not be *about* rigour, yet it should *demand* rigour. You can't guess. You probably think it the dreariest. You are wrong.

Many excerpts here are from Spivak's "Calculus". I believe they fall under "fair use".



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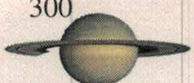
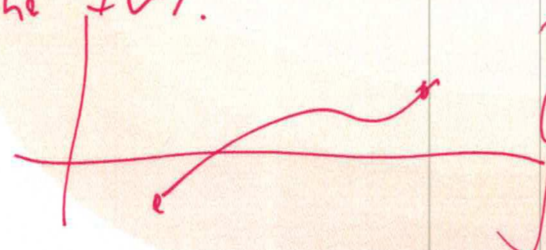


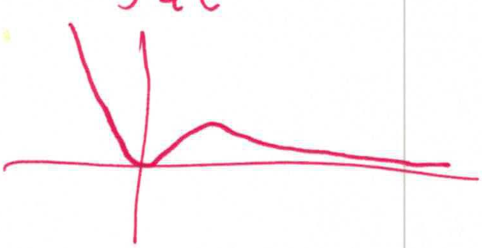
$\forall\epsilon\ \exists\delta$.
Thm f, g cont $\Rightarrow f\cdot g$ cont. } spivak snip.

The IVT.



} snip

graphing xe^{-x}



snip.

The Fundamental Thm of Calculus } snip.

My "π is irrational" tweet

Taylor's Theorem, "strong formulation"

$f \sim T_n f$ } snip if available.

$\Longrightarrow$

$\sin 157 = \sum$ ______

Mathematica, highlighting

1. 1, 2, 3, 4
2. red blood cell
3. Jupiter
4. Milky Way
5. Observable Universe.
6. Way outside.
7. Back to subatomic
8. Mathematica: Sum all, got sin 157

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Brook Taylor

The Taylor Remainder Formula. Let f be a smooth function, let $T_nf(x_0;x)$ be the nth order Taylor polynomial of f around x_0 and evaluated at x,

$$T_nf(x_0;x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,$$

and let $R_n(x) = R_nf(x_0;x) := f(x) - T_nf(x_0;x)$ be the "mistake" or "remainder term". Then

$$R_n(x) = \int_{x_0}^{x} dt\, \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

(In particular, the Taylor expansions of sin, cos, exp, and of several other lovely functions converges to these functions *everywhere*, no matter the odds.)

Proof (for adults; I learned it from my son Itai). The fundamental theorem of calculus says that if $g(x_0) = 0$ then $g(x) = \int_{x_0}^{x} dx_1 g(x_1)$. By design, $R_n^{(k)}(x_0) = 0$ for $0 \le k \le n$. Therefore

$$\begin{aligned} R_n(x) &= \int_{x_0}^{x} dx_1 R_n'(x_1) = \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 R_n''(x_2) \\ &= \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \ldots \int_{x_0}^{x_n} dx_n \int_{x_0}^{t} dt\, R_n^{(n+1)}(t) \\ &= \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \ldots \int_{x_0}^{x_n} dx_n \int_{x_0}^{t} dt\, f^{(n+1)}(t) \end{aligned}$$

[Should be some volume of some simplex, explaining the factorial and the $(side)^n$ factors. Indeed,] when $x > x_0$, and with similar logic when $x < x_0$,

$$\begin{aligned} &= \int_{x_0 \le t \le x_n \le \ldots \le x_1 \le x} f^{(n+1)}(t) = \int_{x_0}^{t} dt\, f^{(n+1)}(t) \int_{t \le x_n \le \ldots \le x_1 \le x} 1 \\ &= \int_{x_0}^{t} dt \frac{f^{(n+1)}(t)}{n!} \int_{(x_1,\ldots,x_n)\in[t,x]^n} 1 = \int_{x_0}^{x} dt\, \frac{f^{(n+1)}(t)}{n!}(x-t)^n. \end{aligned}$$

Fubini

□

~~Children~~
Students & Purists may complain about the use of integration & Fubini.

The MVT w/ pictures

Proof 2 (for students & purists, I learned it from spivak)

Spi

Thm ∂ commute.

Pf 1 (Fubini)

Fubini

Pf 2 (MVT)

π is irrational, untweaked.

Pf 1

Whatever (D'Alembert)

Recycling. Proof (by the trailhead sign method: iteratively use the fundamental theorem of calculus, then Fubini at the crucial time).

$$f(x) = f(x_0) + \int_{x_0}^{x} dx_1 f'(x_1)$$

$$= f(x_0) + \int_{x_0}^{x} dx_1 \left[f'(x_0) + \int_{x_0}^{x_1} dx_2 f''(x_2) \right]$$