Do Not Turn Over Until Instructed

Dror Bar-Natan: Talks: MAASeaway-1810:

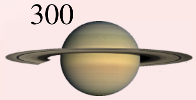
Thanks for inviting me to the fall 2018 MAA Seaway Section meeting!

My Favourite First-Year Analysis Theorem

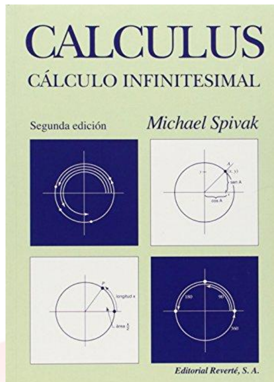
Handout, video, links at ωεβ:=http://drorbn.net/maa18/

Abstract. Whatever it may be, it should say something useful and exciting and it should not be *about* rigour, yet it should *demand* rigour. You can't guess. You probably think it the dreariest. You are wrong.

Contents Ⓢ



Several excerpts here are from Spivak's "Calculus" Ⓢ. I believe they fall under "fair use".



6 Continuous Functions.

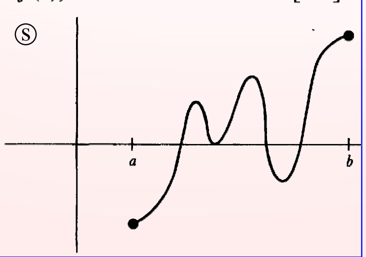
for every $\varepsilon > 0$ there is $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Ⓢ

If f and g are continuous at a, then

(1) $f + g$ is continuous at a,
(2) $f \cdot g$ is continuous at a.

7 Three Hard Theorems.

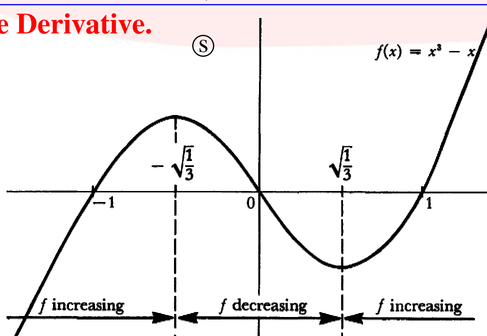
If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$. Ⓢ



11 Significance of the Derivative.

$y = x^3 - x$
$y' = 3x^2 - 1$
$= (\sqrt{3}x + 1)(\sqrt{3}x - 1)$
$= \begin{cases} >0 & x > \sqrt{1/3} \\ <0 & -\sqrt{1/3} < x < \sqrt{1/3} \\ >0 & x < -\sqrt{1/3} \end{cases}$

Ⓢ



14 The Fundamental Theorem of Calculus.

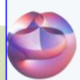
If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a).$$

Ⓢ

*16 π is Irrational.

Tweets   Tweets & replies



Dror Bar-Natan @drorbarnatan · 2 Apr 2013
π=a/b, f(x)=x^n(a-bx)^n/n!, n large => 0<V=∫(0,π)f(x)sin(x)dx<1. Repeated integration by parts & f(x)=f(π-x) => V∈Z. So π is irrational.
1

20 Approximation by Polynomial Functions.

Suppose that f is a function for which

$$f'(a), \ldots, f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \le k \le n,$$

and define

$$P_{n,a}(x) = a_0 + a_1(x - a) + \cdots + a_n(x - a)^n.$$

Then

$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x - a)^n} = 0.$$

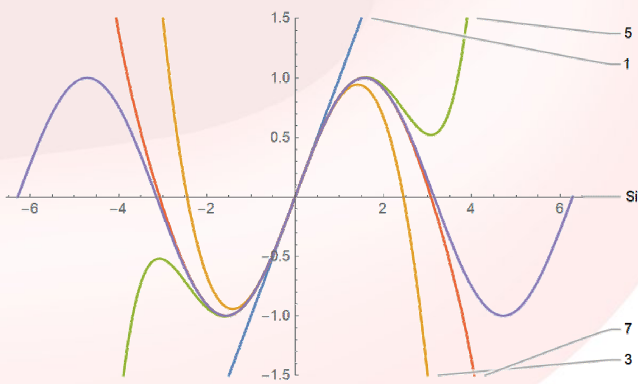
Ⓢ

For example for $f(x) = \sin(x)$ at $a = 0$, $f^{(k)} = \sin, \cos, -\sin, -\cos, \sin, \ldots$, so

$$a_k = \begin{cases} \frac{(-1)^{(k-1)/2}}{k!} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

```
In[1]:= a_k_ := { (-1)^((k-1)/2) / k!   OddQ[k]
                  0                     EvenQ[k] };
        Plot[Evaluate@Append[
            Table[Labeled[Σ_{k=0}^{n} a_k x^k, n], {n, {1, 3, 5, 7}}],
            Labeled[Sin[x], Sin]
          ], {x, -2π, 2π}, PlotRange → {-1.5, 1.5}]
```

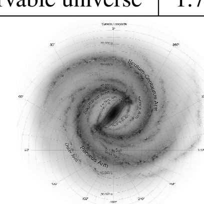
Out[2]=



```
In[3]:= Column@Table[k → N[a_k 157^k], {k, {0, 3, 9, 13, 29, 35, 157, 223, 457}}]

Out[3]= 0 → 0.
        3 → -644982.
        9 → 1.59711×10^14
        13 → 5.65477×10^18
        29 → 5.42689×10^32
        35 → -6.95433×10^36
        157 → 4.86366×10^66
        223 → -1.94045×10^61
        457 → 4.87404×10^-16

In[4]:= {N[Σ_{k=0}^{457} a_k 157^k], Σ_{k=0}^{457} N[a_k 157^k]}

Out[4]= {-0.0795485, 5.10624×10^50}

In[5]:= N@Sin[157]

Out[5]= -0.0795485
```

Some sizes (in multiples of the diameter of a Hydrogen atom:

A red blood cell	1.56×10^5
The CN Tower	1.11×10^{13}
The rings of Saturn	5.6×10^{18}
The Milky Way galaxy	1.89×10^{31}
The observable universe	1.76×10^{37}



Do Not Turn Over Until Instructed

The Taylor Remainder Formulas. Let f be a smooth function, let $P_{n,a}(x)$ be the nth order Taylor polynomial of f around a and evaluated at x, so with $a_k = f^{(k)}(a)/k!$,

Brook Taylor

$$P_{n,a}(x) := \sum_{k=0}^{n} a_k (x-a)^k,$$

and let $R_{n,a}(x) := f(x) - P_{n,a}(x)$ be the "mistake" or "remainder term". Then

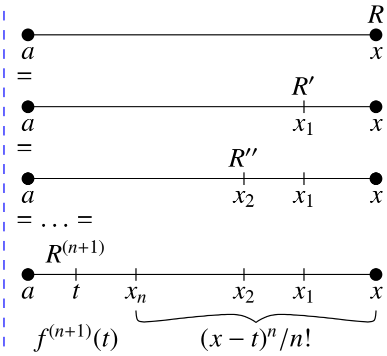
$$R_{n,a}(x) = \int_a^x dt\, \frac{f^{(n+1)}(t)}{n!}(x-t)^n, \tag{1}$$

or alternatively, for some t between a and x,

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}. \tag{2}$$

(In particular, the Taylor expansions of sin, cos, exp, and of several other lovely functions converges to these functions *everywhere*, no matter the odds.)

Proof of (1) (for adults; I learned it from my son Itai). The fundamental theorem of calculus says that if $g(a) = 0$ then $g(x) = \int_a^x dx_1 g(x_1)$. By design, $R^{(k)}_{n,a}(a) = 0$ for $0 \le k \le n$. Therefore



$$R_{n,a}(x) = \int_a^x dx_1 R'_{n,a}(x_1)$$

$$= \int_a^x dx_1 \int_a^{x_1} dx_2 R''_{n,a}(x_2)$$

$$= \ldots = \int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_n} dx_n \int_a^t dt\, R^{(n+1)}_{n,a}(t)$$

$$= \int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_n} dx_n \int_a^t dt\, f^{(n+1)}(t),$$

when $x > a$, and with similar logic when $x < a$,

$$= \int_{a\le t\le x_n\le\ldots\le x_1\le x} f^{(n+1)}(t) = \int_a^t dt\, f^{(n+1)}(t) \int_{t\le x_n\le\ldots\le x_1\le x} 1$$

$$= \int_a^t dt \frac{f^{(n+1)}(t)}{n!} \int_{(x_1,\ldots,x_n)\in[t,x]^n} 1 = \int_a^x dt\, \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

□

de-Fubini (obfuscation in the name of simplicity). Prematurely aborting the above chain of equalities, we find that for any $1 \le k \le n+1$,

Guido Fubini

$$R(x) = \int_a^x dt\, R^{(k)}(t)\frac{(x-t)^{k-1}}{(k-1)!}.$$

But these are easy to prove by induction using integration by parts, and there's no need to invoke Fubini.

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Partial Derivatives Commute. Make Fubini Smile Again!

If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is C^2 near $a \in \mathbb{R}^2$, then $f_{12}(a) = f_{21}(a)$.

Proof. Let $x \in \mathbb{R}^2$ be small, and let $R := [a_1, a_1+x_1]\times[a_2, a_2+x_2]$.

$$f_{12}(a) \sim \boxed{\int f_{12}} = \boxed{\sum f} = \boxed{\int f_{21}} \sim f_{21}(a)$$

$$f_{12}(a) \sim \frac{1}{|R|}\int_R f_{12} = \frac{1}{|R|}\int_{a_1}^{a_1+x_1} dt_1\,(f_1(t_1, a_2+x_2) - f_1(t_1, a_2))$$

$$= \frac{1}{|R|}\begin{pmatrix} f(a_1+x_1, a_2+x_2) - f(a_1+x_1, a_2) \\ -f(a_1, a_2+x_2) + f(a_1, a_2)\end{pmatrix}.$$

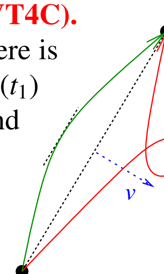
But the answer here is the same as in

$$f_{21}(a) \sim \frac{1}{|R|}\int_R f_{21} = \frac{1}{|R|}\int_{a_2}^{a_2+x_2} dt_2\,(f_2(a_1+x_1, t_2) - f_2(a_1, t_2))$$

$$= \frac{1}{|R|}\begin{pmatrix} f(a_1+x_1, a_2+x_2) - f(a_1, a_2+x_2) \\ -f(a_1+x_1, a_2) + f(a_1, a_2)\end{pmatrix},$$

and both of these approximations get better and better as $x \to 0$. □

The Mean Value Theorem for Curves (MVT4C). If $\gamma\colon [a,b] \to \mathbb{R}^2$ is a smooth curve, then there is some $t_1 \in (a,b)$ for which $\gamma(b) - \gamma(a)$ and $\dot\gamma(t_1)$ are linearly dependent. If also $\gamma(a) = 0$, and $\gamma = \begin{pmatrix}\xi \\ \eta\end{pmatrix}$ and $\eta \ne 0 \ne \dot\eta$ on (a,b), then



$$\frac{\xi(b)}{\eta(b)} = \frac{\dot\xi(t_1)}{\dot\eta(t_1)} \quad \left(\text{when lucky, } = \frac{\ddot\xi(t_2)}{\ddot\eta(t_2)}\ldots\right).$$

de l'Hôpital

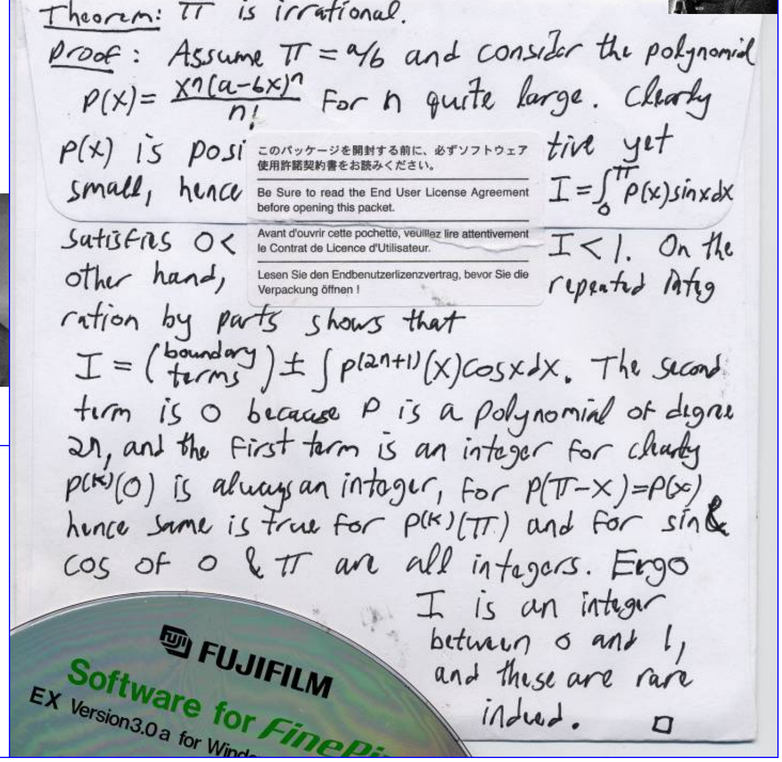
Proof of (2). Iterate the lucky MVT4C as follows:

$$\frac{R_{n,a}(x)}{(x-a)^{n+1}} = \frac{R'_{n,a}(t_1)}{(n+1)(t_1-a)^n} = \ldots = \frac{R^{(n+1)}_{n,a}(t_{n+1})}{(n+1)!} = \frac{f^{(n+1)}(t)}{(n+1)!}.$$

□

J.H. Lambert

π is Irrational following Ivan Niven, Bull. Amer. Math. Soc. (1947) pp. 509:



Theorem: π is irrational.
Proof: Assume $\pi = a/b$ and consider the polynomial $p(x) = \frac{x^n(a-bx)^n}{n!}$ for n quite large. Clearly $p(x)$ is positive yet small, hence $I = \int_0^\pi p(x)\sin x\,dx$ satisfies $0 < I < 1$. On the other hand, repeated integration by parts shows that $I = \left(\begin{smallmatrix}\text{boundary}\\ \text{terms}\end{smallmatrix}\right) \pm \int p^{(2n+1)}(x)\cos x\,dx$. The second term is 0 because p is a polynomial of degree $2n$, and the first term is an integer for clearly $p^{(k)}(0)$ is always an integer, for $p(\pi - x) = p(x)$ hence same is true for $p^{(k)}(\pi)$ and for sin & cos of 0 & π are all integers. Ergo I is an integer between 0 and 1, and these are rare indeed. □