

Homework 1-25

ωεβ:=<http://drorbn.net/ld26>



Strong. Θ vs. a slew of other reasonably-computable invariants (deficits shown):

Abstract. I'll start with a review of my recent paper with van der Veen, "A **Fast, Strong, Topologically Meaningful**, and **Fun** Knot Invariant" [BV3], and then assign some homework. Much of what I'll say follows earlier work by Rozansky, Krickner, Garoufalidis, and Ohtsuki [Ro1, Ro2, Ro4, Kr, GR, Oh2].



van der Veen

Acknowledgement. This work was supported by NSERC grants RGPIN-2018-04350 and RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

A. With T an indeterminate, start from a presentation matrix A for the Alexander module of K , coming from the Wirtinger presentation of $\pi_1(K)$: $A := I_{2n+1} + \sum_c A_c$, where

$$i \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} j \\ s=+1 \end{matrix} \rightarrow \begin{matrix} A_c & i+1 & j+1 \\ i & -T^s & T^s-1 \\ j & 0 & -1 \end{matrix}$$

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Delta \doteq \det(A)$

G. Let $G = (g_{\alpha\beta}) := A^{-1}$:

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{T^2-T+1}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



ωεβ/pi

Let T_1 and T_2 be new indeterminates, let $T_3 = T_1 T_2$, and let $G_\nu = (g_{\nu\alpha\beta})$ be G with $T \rightarrow T_\nu$, for $\nu = 1, 2, 3$.

$$\theta \sim \Delta_1 \Delta_2 \Delta_3 \sum_{C_0, C_1} g_{1i_0 i_1} g_{2i_0 i_1} g_{3i_1 i_0} + \text{l.o.}$$

$$\Theta = (\Delta, \theta) \in \mathbb{Z}[T^{\pm 1}] \times \mathbb{Z}[T_1^{\pm 1}, T_2^{\pm 1}]$$

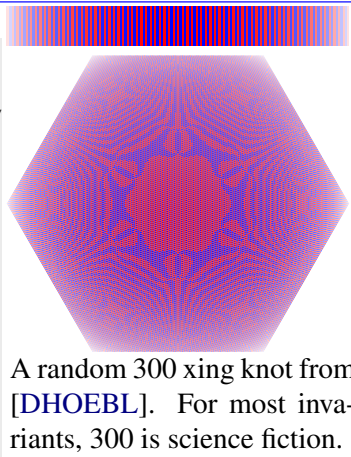
Fast.

```

F1([s_-, e_-, j_-]) := CF
S := (1/2 - B11 + T1 B21 - B11 B21 - (T1-1) B21 B11 + 2 B21 B11 -
(1-T1) B21 B11 - B21 B11 - T1 B21 B11 + B11 B21 +
((T1-1) B21 (T1-1) B21 - T1 B21 + T1 B21) +
(T1-1) B21 (1-T1 B11 + B21) + (T1-2) B21 - (T1-1) (T1+1) B21)/
(T1-1)
F2([s0, i0, j0], [s1, i1, j1]) :=
CF[s1 (T1-1) (T1-1) (T1-1) (T1-1) B21, i0, s0, j0, i1]
((T1-1) B21, i0 - B21, i0) - (T1-1) B21, j0, i0]]
F3([phi_-, h_-]) = phi B31 - phi/2;
F4 := T1 T2;
CF[phi_] := ExpandCollect[phi, g_-, F] /. F -> Factor;
phi[phi_] := phi[K] = Module[{X, phi, n, A, delta, G, ev, phi, k1, k2},
{X, phi} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
Cases[X, {s_-, e_-, j_-} >> {A[[i, j]], (i+1, j+1)] += {
-T1 T1-1-1,
0, -1}}];
delta = (Total[phi]-Total[phi[[All, 1]]])/2 Det[A];
G = Inverse[A];
ev[phi_] := Factor[phi /. G_-, phi_];
ev[phi_] := Sum[Sum[F2[X[[k1]], X[[k2]]], {k1, n}, {k2, n}];
phi += ev[Sum[F2[X[[k1]], X[[k2]]], {k1, n}, {k2, n}];
phi += ev[Sum[F2[X[[k1]], k1], {k1, Length[phi]}];
Factor[phi, (delta, phi, T -> T1) (delta, phi, T -> T2) (delta, phi, T -> T3) phi];
];
                    
```

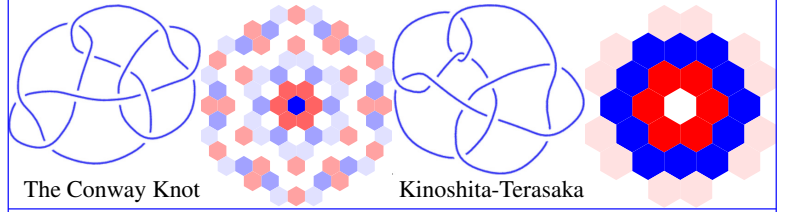
Data
(ouch)

Program



A random 300 xing knot from [DHOEBL]. For most invariants, 300 is science fiction.

n	≤ 10	≤ 11	≤ 12	≤ 13	≤ 14	≤ 15
knots	249	801	2,977	12,965	59,937	313,230
Δ	(38)	(250)	(1,204)	(7,326)	(39,741)	(236,326)
σ_{LT}	(108)	(356)	(1,525)	(7,736)	(40,101)	(230,592)
J	(7)	(70)	(482)	(3,434)	(21,250)	(138,591)
Kh	(6)	(65)	(452)	(3,226)	(19,754)	(127,261)
H	(2)	(31)	(222)	(1,839)	(11,251)	(73,892)
Vol	(~6)	(~25)	(~113)	(~1,012)	(~6,353)	(~43,607)
(Kh, H, Vol)	(~0)	(~14)	(~84)	(~911)	(~5,917)	(~41,434)
(Δ, ρ_1)	(0)	(14)	(95)	(959)	(6,253)	(42,914)
(Δ, ρ_1, ρ_2)	(0)	(14)	(84)	(911)	(5,926)	(41,469)
$(\rho_1, \rho_2, Kh, H, Vol)$	(0)	(~14)	(~84)	(~911)	(~5,916)	(~41,432)
Θ	(0)	(3)	(19)	(194)	(1,118)	(6,758)
(Θ, ρ_2)	(0)	(3)	(10)	(169)	(982)	(6,341)
(Θ, σ_{LT})	(0)	(3)	(19)	(194)	(1,118)	(6,758)
(Θ, Kh)	(0)	(3)	(18)	(185)	(1,062)	(6,555)
(Θ, H)	(0)	(3)	(18)	(185)	(1,064)	(6,563)
(Θ, Vol)	(0)	(~3)	(~10)	(~169)	(~973)	(~6,308)
$(\Theta, \rho_2, Kh, H, Vol)$	(0)	(~3)	(~10)	(~169)	(~972)	(~6,304)



Topologically Meaningful. θ is near Δ and we dream that anything Δ can do, θ does too (sometimes better). The following two conjectures are verified for knots with ≤ 13 crossings:

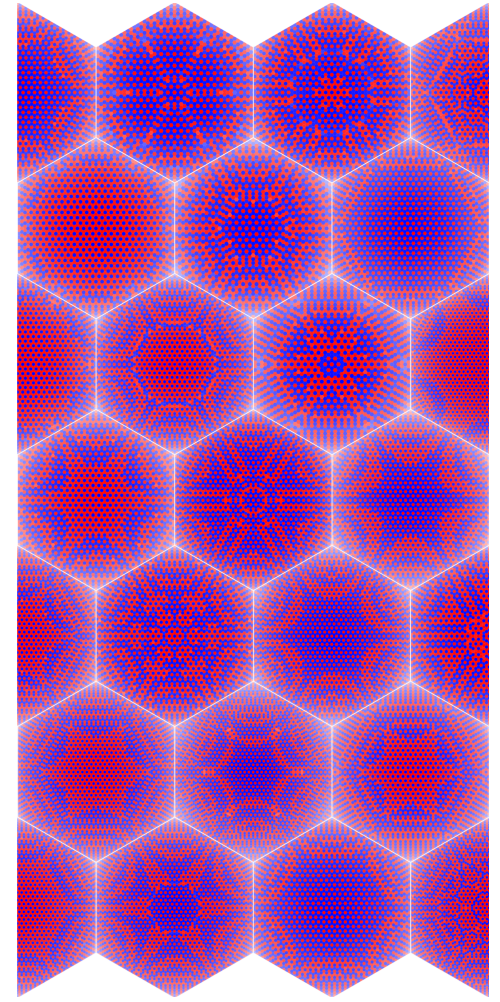
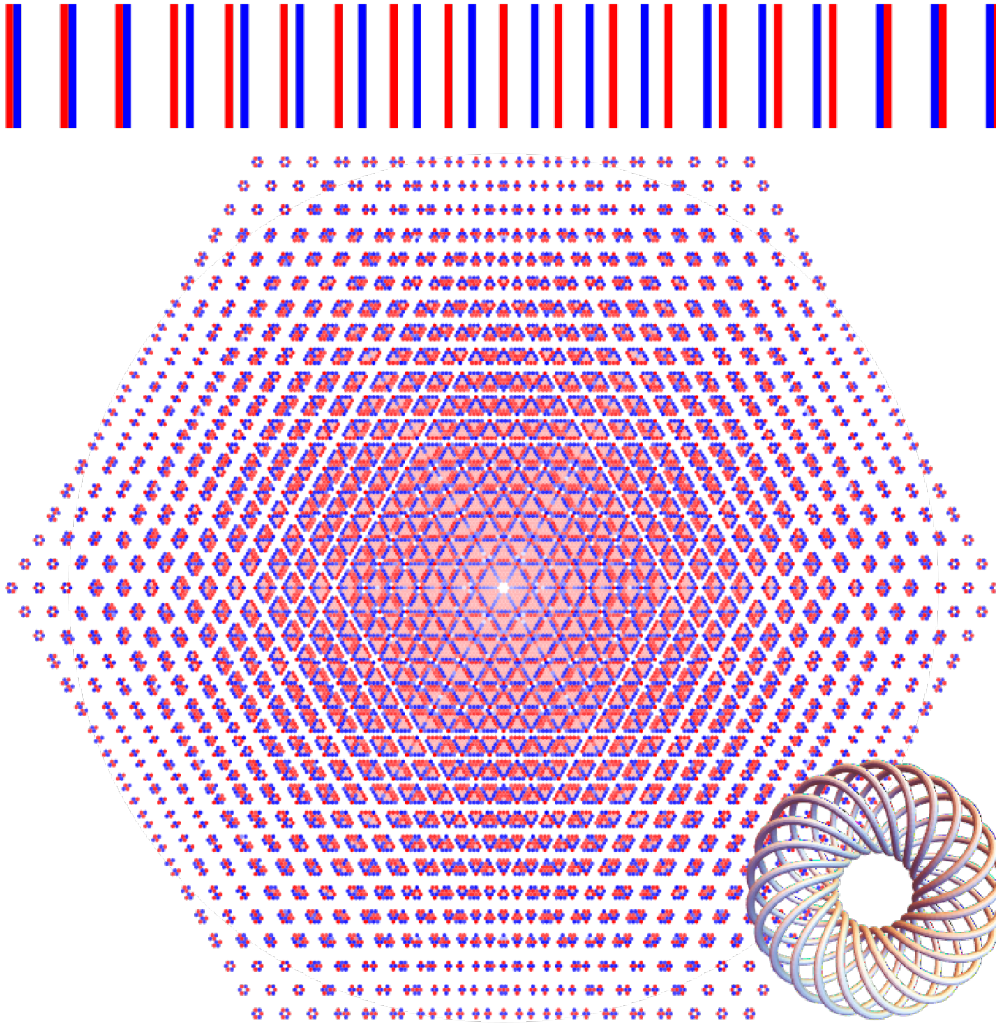
- Conjecture 1.** $\deg_{T_1} \theta(K) \leq 2g(K)$.
- Conjecture 2.** If K is a fibered knot and d is the degree of $\Delta(K)$ (the highest power of T), then the coefficient of T_1^{2d} in $\theta(K)$, which is a polynomial in T_1 , is an integer multiple of $T_1^{2d} \Delta(K)|_{T \rightarrow T_1}$.
- Dream.** θ has something to say about ribbon knots.

Fun. Θ on Rolfsen's Table:

The 132-crossing torus knot $T_{22/7}$:

(many more at $\omega\epsilon\beta/\text{TK}$)

Random knots from [DHOEBL] with 51 – 75 crossings: (many more at $\omega\epsilon\beta/\text{DK}$)



Moral. We must come to terms with Θ !

Task 1. Make the “data” formulas human friendly.

Task 2. Prove the hexagonal symmetry of $\theta(K)$, and that $\theta(K) = \theta(-K) = -\theta(\bar{K})$.

That’s harder than it seems! The formulas don’t naively show any of that. Δ has a palindromic symmetry first conjectured in Alexander’s original paper [A1] — it is invariant under $T \rightarrow T^{-1}$. Proving this took a few years, and the proof starting from the Wirtinger presentation is quite involved (e.g. [CF, Chapter IX]).

Task 3. Show that θ dominates the Rozansky-Overbay invariant ρ_1 [Ro1, Ro2, Ro4, Ov, BV1]. Precisely, show that $\rho_1 = -\theta|_{T_1 \rightarrow T, T_2 \rightarrow 1}$.

This one should be easy with techniques from [BV3, Section 4.2].

Task 4. Explain the “Chladni patterns”. Are there “dominant parts” of θ that can be computed in isolation?



left: © Whipple Museum of the History of Science, University of Cambridge; right: CC-BY-SA 4.0 / Wikimedia / Matematica (IME USP) / Rodrigo Tetsuo Argenton

Task 5. Prove the genus bound of Conjecture 1.

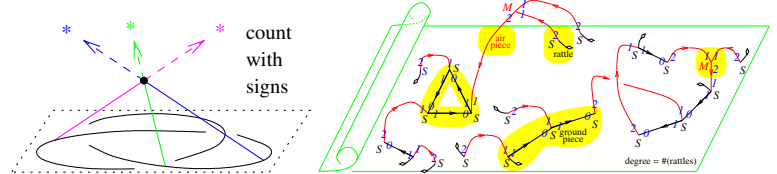
This is probably coming. One can bound the degree of $\Delta = \det(A)$ in terms of $g(K)$ using the Seifert presentation of the Alexander module. Pushing further, likely one can bound the degree of

$(g_{\alpha\beta}) = A^{-1}$ in terms of $g(K)$, and that’s probably enough.

Task 6. Find a 3D interpretation of the $g_{\alpha\beta}$ ’s.

They must be closely related to the equivariant linking numbers of [KY, GK, GT, Oh3, Le1].

Task 7. Find a formula $\mathcal{F}$ for $\Theta(K)$ that starts from a Seifert surface Σ of K . Better if $\mathcal{F}$ is completely 3D! Assuming Task 13, it is known that Θ depends only of invariants of type ≤ 3 of Σ . Maybe $\mathcal{F}$ is about configuration space integrals / chopstick towers? See CS: [Th, Le2, BN1], BF: [CR, BN2]



Task 8. Is there an intrinsic theory of finite type invariants for Seifert surfaces? For task 11, does its gr map to functions on H_1 ?

My current best understanding of finite type invariants for Seifert surfaces goes through thick graphs.



Task 9. Prove the the fibered condition of Conjecture 2.

If K is fibered, $\deg \Delta(K) = g(K)$ and $\Delta(K)$ is monic. Indeed, K is then the mapping cylinder of a diffeomorphism $f: \Sigma \rightarrow \Sigma$. The Alexander module of K is generated by $H_1(\Sigma)$ with relations

$\{\gamma = T f_* \gamma : \gamma \in H_1(\Sigma)\}$. Thus the highest monomial in Δ is $T^g \det(f_*)$ and $\det(f_*) = \pm 1$ as f_* preserves the intersection pairing. If only we had a formula for θ in terms of $f \dots$

Task 10. In general, find a formula for Θ corresponding to each known presentation of the Alexander module.

Wirtinger is $2\{\text{xings}\} \rightarrow \{\text{edges}\}$. Dehn is $\{\text{xings}\} \rightarrow \{\text{faces}\}$. Co-Dehn is $\{\text{faces}\} \rightarrow \{\text{xings}\}$. Burau is $\{\text{braid strands}\} \rightarrow \{\text{braid strands}\}$. Seifert is $H_1(\Sigma) \rightarrow H_1(\Sigma)$, and so is the presentation from Task 9. Grid diagrams lead to $\{\text{grid number}\} \rightarrow \{\text{grid number}\}$ (may relate to HFK). There's more!

Task 11. Write up the integration story.

Claim (e.g., [BN4]). Cutting corners, with $\epsilon^2 = 0$,

$$\frac{1}{\Delta_1 \Delta_2 \Delta_3} \exp\left(\epsilon \cdot \frac{\theta}{\Delta_1 \Delta_2 \Delta_3}\right) \sim \oint_{\prod_e \mathbb{R}^6_{p_{1e}, p_{2e}, p_{3e}, x_{1e}, x_{2e}, x_{3e}}} \prod_C e^{L_C},$$

where $\oint$ denotes perturbed formal Gaussian integration (i.e., “Feynman Diagrams”) and L_C is

$$\begin{aligned} L[\mathbf{x}_{i,j}, [\mathbf{s}_-]] := & \text{Plus} [\\ & \sum_{v=1}^3 (\mathbf{x}_{vi} (\mathbf{p}_{vi}^+ - \mathbf{p}_{vi}) + \mathbf{x}_{vj} (\mathbf{p}_{vj}^+ - \mathbf{p}_{vj}) + (\mathbf{T}_v^s - 1) \mathbf{x}_{vi} (\mathbf{p}_{vi}^+ - \mathbf{p}_{vj}^+)), \\ & (\mathbf{T}_1^s - 1) \mathbf{p}_{3j} \mathbf{x}_{1i} (\mathbf{T}_2^s \mathbf{x}_{2i} - \mathbf{x}_{2j}), \\ & \epsilon \in (\mathbf{T}_3^s - 1) \mathbf{p}_{1j} (\mathbf{p}_{2i} - \mathbf{p}_{2j}) \mathbf{x}_{3i} / (\mathbf{T}_2^s - 1), \\ & \epsilon \in (1/2 + \mathbf{T}_2^s \mathbf{p}_{1i} \mathbf{p}_{2j} \mathbf{x}_{1i} \mathbf{x}_{2i} - \mathbf{p}_{1i} \mathbf{p}_{2j} \mathbf{x}_{1i} \mathbf{x}_{2j} - \mathbf{p}_{3i} \mathbf{x}_{3i} - (\mathbf{T}_2^s - 1) \mathbf{p}_{2j} \mathbf{p}_{3i} \mathbf{x}_{2i} \mathbf{x}_{3i} + \\ & (\mathbf{T}_3^s - 1) \mathbf{p}_{2j} \mathbf{p}_{3j} \mathbf{x}_{2i} \mathbf{x}_{3i} + 2 \mathbf{p}_{2j} \mathbf{p}_{3i} \mathbf{x}_{2j} \mathbf{x}_{3i} + \mathbf{p}_{1i} \mathbf{p}_{3j} \mathbf{x}_{1i} \mathbf{x}_{3j} - \mathbf{p}_{2i} \mathbf{p}_{3j} \mathbf{x}_{2i} \mathbf{x}_{3j} - \\ & \mathbf{T}_2^s \mathbf{p}_{2j} \mathbf{p}_{3j} \mathbf{x}_{2i} \mathbf{x}_{3j} + \\ & ((\mathbf{T}_1^s - 1) \mathbf{p}_{1j} \mathbf{x}_{1i} (\mathbf{T}_2^s \mathbf{p}_{2j} \mathbf{x}_{2i} - \mathbf{T}_2^s \mathbf{p}_{2j} \mathbf{x}_{2j} - (\mathbf{T}_2^s + 1) (\mathbf{T}_3^s - 1) \mathbf{p}_{3j} \mathbf{x}_{3i} + \\ & \mathbf{T}_2^s \mathbf{p}_{3j} \mathbf{x}_{3j}) + (\mathbf{T}_3^s - 1) \mathbf{p}_{3j} \mathbf{x}_{3i} \\ & (1 - \mathbf{T}_2^s \mathbf{p}_{1i} \mathbf{x}_{1i} + \mathbf{p}_{2i} \mathbf{x}_{2j} + (\mathbf{T}_2^s - 2) \mathbf{p}_{2j} \mathbf{x}_{2j})) / (\mathbf{T}_2^s - 1))] \end{aligned}$$

In fact, we first found L_C using the method of undetermined coefficients, and then derived F_1 and F_2 from it.

Task 12. Find a similar perturbed Gaussian integral formula for θ , but with integration over $6H_1(\Sigma)$. The quadratic Q will be the same as in the Seifert-Alexander formula (but repeated 3 times, for each T_v). The perturbation P_ϵ will be given by low-degree finite type invariants of curves on Σ (possibly also dependent on the intersection points of such curves, or on other information coming from Σ).

Task 13. Prove that θ is equal to the two-loop contribution $Z^{(2)}$ to the Kontsevich integral Z .

Composed with the inverse PBW isomorphism χ^{-1} , $\chi^{-1} \circ Z$ takes values in univalent Jacobi diagrams, $\mathcal{B} = \{\boxtimes \circ \dots\} / IHX$. Rozansky conjectured [Ro3, GR] and Kricker proved [Kr] that

$$\log(\chi^{-1} \circ Z) = f_1 \left(\begin{array}{|c|} \hline t \\ \hline \end{array} \right) + f_2 \left(\begin{array}{|c|} \hline t_1 \\ \hline t_2 \\ \hline \end{array} \right) + \text{higher loops},$$

where $t^k \begin{array}{|c|} \hline t \\ \hline \end{array} := \begin{array}{|c|} \hline | \\ \hline \end{array} \dots n \dots \begin{array}{|c|} \hline | \\ \hline \end{array}$, $f_1 \in \mathbb{Q}[[t]]$, and $f_2 \in \mathbb{Q}[[t_1, t_2]]$ satisfy $f_1 = \frac{1}{2} \log \frac{\sinh(t/2)}{t \Delta(e^t/2)}$ and $f_2 = Z^{(2)}(e^{t_1}, e^{t_2}) / \Delta(e^{t_1}) \Delta(e^{t_1}) \Delta(e^{t_1+t_2})$ where $Z^{(2)} \in \mathbb{Z}[T_1^{\pm 1}, T_2^{\pm 1}]$ is the “two loop polynomial”. Ohtsuki [Oh2] studied $Z^{(2)}$ extensively, and almost certainly, $Z^{(2)} = \theta$. Prove that!

Task 14. Complete and write up the $\mathfrak{g}_\epsilon^+$ story.

Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{b}^u$ and $\mathfrak{b}^l$ be its upper and lower Borel subalgebras. Then $\mathfrak{b}^u$ has a bracket β , and as the dual of $\mathfrak{b}^l$ it also has a cobracket δ , and in fact, $\mathfrak{g} \oplus \mathfrak{h} \equiv \text{Double}(\mathfrak{b}^u, \beta, \delta)$. Let $\mathfrak{g}_\epsilon^+ := \text{Double}(\mathfrak{b}^u, \beta, \epsilon \delta) \pmod{\epsilon^{d+1}}$ it is solvable for any d . We expect that Θ is the universal invariant (in the sense of Lawrence and Ohtsuki [La, Oh1]) corresponding to $sl_{3,\epsilon}^+$, computed modulo ϵ^2 (in fact, that's how we guessed it). See [BN3, BV2].

Task 15. Go beyond sl_3 and the first power of ϵ !

This sounds very appealing, and you will indeed get stronger and stronger invariants. But they will become less and less computable $\odot$.

Task 16. Relate the $\mathfrak{g}_\epsilon^+$ story with (rotational) virtual knots [Kau], with $\vec{\mathcal{A}}$ [Po], and with quantization of Lie bialgebras [EK1, EK2, En, Se]

We expect that there is a commutative diagram as

Re top on or the right which descends to the bottom one, with $\oplus$ Re case of $\mathfrak{g} = \mathfrak{sl}_3$ $\downarrow k=1$. But $\odot$

Task 17. Find a w-style characterization of Θ .

A word about [GR]. Compare with (say) [BD], where Δ is characterized on w-knots by the overcrossings / tails commute relation. Similarly it should be possible to characterize Θ on rotational virtual knots by some “overcrossings / tails nearly commute” relation.

Task 18. Understand Chern-Simons theory with gauge group $\mathfrak{g}_\epsilon^+$.

Is there a gauge that leads to the formula $\mathcal{F}$ of Task 7?

Task 19. What happens to representation theory as $\epsilon \rightarrow 0$? Is there any fun in continuous morphisms $\mathfrak{g}_\epsilon^+ \rightarrow \mathfrak{gl}_{n,\epsilon}^+$?

Task 20. Does Θ extend to knots in $\mathbb{Z}HS/\mathbb{Q}HS$?

Z and $Z^{(2)}$ do.

Task 21. Is there a surgery formula for Θ ?

Z and $Z^{(2)}$ have.

Task 22. Extend Θ to tangles and figure out how it behaves under strand doubling.

Z and $Z^{(2)}$ extend but their extensions depend on parenthesizations. From Task 14 we expect that Θ will extend without the need for parenthesizations, yet with an asymmetry built into the doubling operations. A

Task 23. Make Kricker / Ohtsuki [Kr, Oh2] more computable!

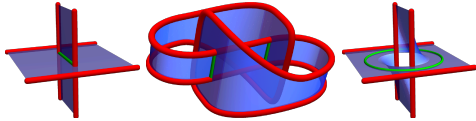
Task 24. Find a multi-variable version of θ for links, like there is a multi-variable Alexander for links (e.g. [Kaw, Chapter 7]). It is predicted g_e^+ consideration, but not by the loop expansion.

Task 25. Find a ribbon condition satisfied by Θ .

For a ribbon knot K , one may find a Seifert surface Σ half of whose homology is generated by the components of an unlink embedded in Σ . This makes for a presentation matrix A of the Alexander module of K that has big blocks of zeros, and this leads to the Fox-Milnor condition [FM], $\Delta \doteq \det(A) \doteq f(T)f(T^{-1})$ for some $f \in \mathbb{Z}[T^{\pm 1}]$. If $\det A$ is constrained for ribbon knots, perhaps so is A^{-1} and therefore Θ ?

Bonus Task. Carthago delenda est and every knot polynomial must be categorified.

M. Khovanov & Cato the Elder



[AJ] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928) 275–306.
[BN1] D. Bar-Natan, *Cosmic Coincidences and Several Other Stories*, talk given in Tennessee, March 2011. Handout and video: [oeß/Ten](#).
[BN2] D. Bar-Natan, *A Partial Reduction of BF Theory to Combinatorics*, talk given in Vienna, February 2014. Handout and video: [oeß/Vie](#).

References.

[BN3] D. Bar-Natan, *Everything around sl_5^+ is DoPeGDO*. So what?, talk given in “Quantum Topology and Hyperbolic Geometry Conference”, Da Nang, Vietnam, May 2019. Handout and video at [oeß/DPG](#).
[BN4] D. Bar-Natan, *Knot Invariants from Zero-Dimensional QFT*, talk given in Bonn, May 2025. Handout and video: [oeß/Bonn](#).
[BD] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial*, Alg. and Geom. Top. **16-2** (2016) 1063–1133, [arXiv:1405.1956](#).
[BV1] D. Bar-Natan and R. van der Veen, *A Perturbed-Alexander Invariant*, Quantum Topology **15** (2024) 449–472, [arXiv:2206.12298](#).
[BV2] D. Bar-Natan and R. van der Veen, *Perturbed Gaussian Generating Functions for Universal Knot Invariants*, [arXiv:2109.02057](#).
[BV3] D. Bar-Natan and R. van der Veen, *A Fast, Strong, Topologically Meaningful, and Fun Knot Invariant*, [oeß/Theta](#) and [arXiv:2509.18456](#).
[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Comm. Math. Phys. **256** (2005) 513–537, [arXiv:math-ph/0210037](#).
[CF] R. H. Crowell and R. H. Fox, *Introduction to Knot Theory*, Springer-Verlag GTM **57** (1963).
[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharyya, D. Lei, and others, *Random Knots: A Preliminary Report*, lecture notes at [oeß/DHOEBL](#). Also a data file at [oeß/DD](#).
[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras*, Adv. in Math. **197-2** (2005) 430–479, [arXiv:math/0212325](#).
[EK1] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Sel. Math., NS **2** (1996) 1–41, [arXiv:q-alg/9506005](#).
[EK2] P. Etingof and D. Kazhdan, *Quantization of Lie bialgebras, II*, Sel. Math., NS **4** (1998) 213–231, [arXiv:q-alg/9701038](#).
[FM] R. H. Fox and J. W. Milnor, *Singularities of 2-Spheres in 4-Space and Cobordism of Knots*, Osaka J. Math. **3-2** (1966) 257–267.
[GK] S. Garoufalidis and A. Kricker, *A Rational Noncommutative Invariant of Boundary Links*, Geom. & Top. **8** (2004) 115–204, [arXiv:math/0105028](#).
[GR] S. Garoufalidis and L. Rozansky, *The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence*, [arXiv:math.GT/0003187](#).
[GT] S. Garoufalidis and P. Teichner, *On Knots with Trivial Alexander Polynomial*, J. Diff. Geom. **67** (2004) 165–191, [arXiv:math/0206023](#).
[Kau] L. H. Kauffman, *Rotational Virtual Knots and Quantum Link Invariants*, J. of Knot Theory and its Ramifications **24-13** (2015), [arXiv:1509.00578](#).
[Kaw] A. Kawachi, *A Survey of Knot Theory*, Birkhauser Verlag, 1996.
[KY] S. Kojima and M. Yamasaki, *Some New Invariants of Links*, Invent. Math. **54** (1979) 213–228.
[Kr] A. Kricker, *The Lines of the Kontsevich Integral and Rozansky’s Rationality Conjecture*, [arXiv:math/0005284](#).
[La] R. J. Lawrence, *Universal Link Invariants using Quantum Groups*, Proc. XVII Int. Conf. on Diff. Geom. Methods in Theor. Phys., Chester, England, August 1988. World Scientific (1989) 55–63.
[Le1] C. Lescop, *Knot Invariants Derived from the Equivariant Linking Pairing*, AMS/IP Stud. in Adv. Math. **50** (2011) 217–242, [arXiv:1001.4474](#).
[Le2] C. Lescop, *Invariants of Links and 3-Manifolds from Graph Configurations*, EMS Monographs, 2024, [arXiv:2001.09929](#).
[Oh1] T. Ohtsuki, *Quantum Invariants*, Series on Knots and Everything **29**, World Scientific 2002.
[Oh2] T. Ohtsuki, *On the 2-Loop Polynomial of Knots*, Geometry & Topology **11** (2007) 1357–1475.
[Oh3] T. Ohtsuki, *Invariants of Knots Derived from Equivariant Linking Matrices of their Surgery Presentations*, Int. J. Math. **20-7** (2009) 883–913.
[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, Ph.D. thesis, University of North Carolina, August 2013, [oeß/Ov](#).
[Po] M. Polyak, *On the Algebra of Arrow Diagrams*, Let. Math. Phys. **51** (2000) 275–291.
[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten’s Invariant of 3D Manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, [arXiv:hep-th/9401061](#).
[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. Math. **134-1** (1998) 1–31, [arXiv:q-alg/9604005](#).
[Ro3] L. Rozansky, *A Rational Structure of Generating Functions for Vassiliev Invariants*, Yale University preprint, July 1999.
[Ro4] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, [arXiv:math/0201139](#).
[Se] P. Severa, *Quantization of Lie Bialgebras Revisited*, Sel. Math., NS, to appear, [arXiv:1401.6164](#).
[Th] D. Thurston, *Integral expressions for the Vassiliev knot invariants*, Harvard University senior thesis, April 1995, [arXiv:math.QA/9901110](#).

A: Note that tangles & strand doubling are keys to “Algebraic knot Theory” [AKTJ].

better if burger.

A FAST, STRONG, TOPOLOGICALLY MEANINGFUL, AND FUN KNOT INVARIANT

DROR BAR-NATAN AND ROLAND VAN DER VEEN

ABSTRACT. In this paper we discuss a pair of polynomial knot invariants $\Theta \sim (\Delta, \theta)$ which

- Theoretically and practically fast, Θ can be computed in polynomial time. We can compute it in full on random knots with over 300 crossings, and its evaluation at simple rational numbers on random knots with over 600 crossings.
- Strong. Its separation power is much greater than the hyperbolic volume, the HOMFLY-PT polynomial and Khovanov homology (taken together) on knots with up to 15 crossings while being computable on much larger knots.
- Topologically meaningful. It likely gives a genus bound, and there are reasons to hope that it would be more.
- Fun. See below to Figures 1.1–1.4, 3.1, and 6.2.

Δ is merely the Alexander polynomial. θ is almost certainly equal to an invariant that was studied extensively by Ohtsuki [Oh2], containing Rozansky, Kricker, and Garoufalidis [Roz1, Roz2, Roz3, Kr, GH]. Yet our formulas, proofs, and programs are much simpler and enable the computation even on very large knots.

CONTENTS

1. Fun
2. The Main Theorem
3. Implementation and Examples
- 3.1. Implementation
- 3.2. Examples
4. Proof of the Main Theorem, Theorem 1
- 4.1. Proof of Invariance
- 4.2. Proof of Polynomiality
5. Strong and Meaningful
- 5.1. Strong
- 5.2. Meaningful
- 5.2.1. The Knot Genus
- 5.2.2. Filtered Knots
6. Stories, Conjectures, and Dreams
7. Acknowledgement
- References

Date: First edition September 22, 2025. This edition December 17, 2025.
2020 Mathematics Subject Classification. Primary 57K14, secondary 16T99.
Key words and phrases. Alexander polynomial, loop expansion, solvable approximation, knot genus, filtered knots, ribbon knots, polynomial time computations, Feynman diagrams, perturbed Gaussian integration, Seifert surfaces.
This paper is available in electronic form, along with source files and a demo Mathematica notebook at [http://oeß.net/Theta](#) and at [arXiv:2509.18456](#).

1. FUN
The word “fun” rarely appears in the title of a math paper, so let us start with a brief justification.

Θ is a pair of polynomials. The first, Δ , is old news, the Alexander polynomial [AJ]. It is a one-variable Laurent polynomial in a variable T . For example, $\Delta(\bigcirc) = T^{-1} - 1 + T$. We turn such a polynomial into a list of coefficients (for $\bigcirc$, it is $(1, -1, 1)$), and then to a chain of bars of varying colours: white for the zero coefficients, and red and blue for the positive and negative coefficients (with intensity proportional to the magnitude of the coefficients).

Similarly, θ is a 2-variable Laurent polynomial, in variables T_1 and T_2 . We can turn such a polynomial into a 2D array of coefficients and then using the same rules, into a 2D array of colours, namely, into a picture. To highlight a certain conjectured hexagonal symmetry of the resulting pictures, we apply a shear transformation to the plane before printing. So is monomial $cT_1^a T_2^b$ gets printed at position $(a - b/2, \sqrt{3}b/2)$ instead of the more straightforward (a, b) . On the right is the 2D picture corresponding to the polynomial $2 + T_1 - T_2 + T_1 T_2 + T_2 - T_1^{-1} + T_1^{-1} T_2^{-1} - T_2^{-1}$.

Thus Θ becomes a pair of pictures: a bar code, and a 2D picture that we call a “hexagonal QR code”. For the knots in the Rolfsen table (with the unknot prepended at the start), they are in Figure 1.1. For some alternating square wave knots, they are in Figure 1.2, and for a random square wave, in Figure 1.3. In addition, the hexagonal QR codes of 15 knots with ≥ 300 crossings are in Figure 1.4, and Θ of a 132-crossing torus knot is in Figure 3.1. Some further computations and figures, also highlighting the parity of coefficients rather than just their signs, are at [La].

Clearly there are patterns in these figures. There is a hexagonal symmetry and the QR codes are nearly always hexagons (these are independent properties). Much more can be seen in Figure 1.1. In Figure 1.4 there seem to be large-scale patterns perhaps reminiscent of the “Chladni figures” formed by powders atop vibrating plates (on right). We can’t prove any of these things, and the last one, we can’t even formulate properly. Yet they are clearly there, too clear to be the result of chance alone. We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.



Left: © Wolfgang Stoyan of the Institute of Science, University of Göttingen. Right: © J. H. Conway, J. Lagarias, and J. Lagarias. Wikimedia / Mathematics (DHL CC BY) / Bridgeway Images.

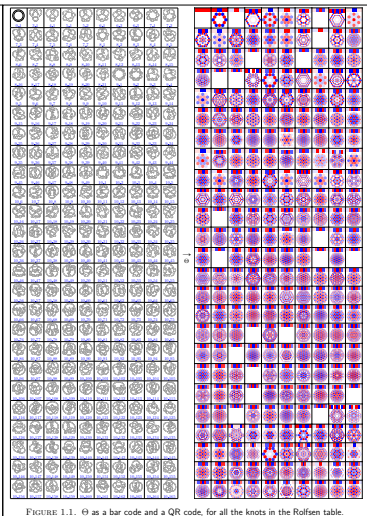
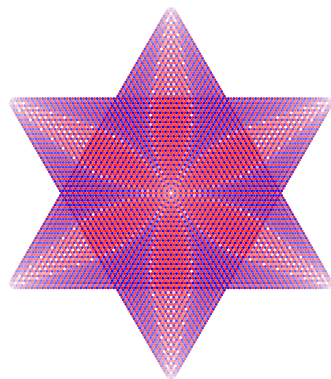


FIGURE 1.1. Θ as a bar code and a QR code, for all the knots in the Rolfsen table.



A (2, 41, -41) pretzel for dessert

