

Homework 1-23

ωεβ:=<http://drorbn.net/ld26>



Strong. Θ vs. a slew of other reasonably-computable invariants (deficits shown):

Abstract. I'll start with a review of my recent paper with van der Veen, "A **Fast, Strong, Topologically Meaningful**, and **Fun** Knot Invariant" [BV3], and then assign some homework. Much of what I'll say follows earlier work by Rozansky, Kriker, Garoufalidis, and Ohtsuki [Ro1, Ro2, Ro4, Kr, GR, Oh2].



van der Veen

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A. With T an indeterminate, start from a presentation matrix A for the Alexander module of K , coming from the Wirtinger presentation of $\pi_1(K)$: $A := I_{2n+1} + \sum_c A_c$, where

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Delta \doteq \det(A)$

G. Let $G = (g_{\alpha\beta}) := A^{-1}$:

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{T^2-T+1}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



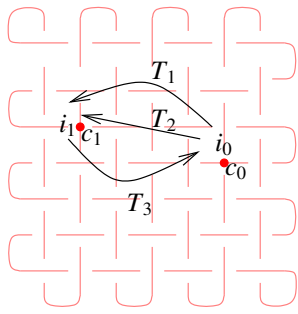
ωεβ/pi

Let T_1 and T_2 be new indeterminates, let $T_3 = T_1 T_2$, and let $G_\nu = (g_{\nu\alpha\beta})$ be G with $T \rightarrow T_\nu$, for $\nu = 1, 2, 3$.

$$\theta \sim \Delta_1 \Delta_2 \Delta_3 \sum_{C_0, C_1} g_{1i_0 i_1} g_{2i_0 i_1} g_{3i_1 i_0} + \text{l.o.}$$

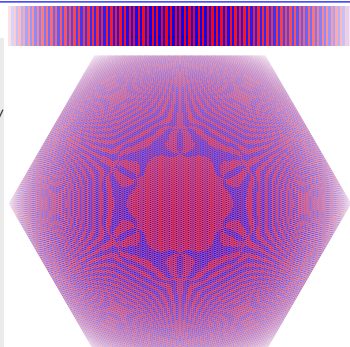
$$\Theta = (\Delta, \theta) \in \mathbb{Z}[T^{\pm 1}] \times \mathbb{Z}[T_1^{\pm 1}, T_2^{\pm 1}]$$

$$\begin{pmatrix} \frac{2}{T} & -1 & 3T \\ T_2 & -T_1 T_2 & \\ -\frac{1}{T_1} & 2 & T_1 \\ \frac{1}{T_1 T_2} & -\frac{1}{T_2} & \end{pmatrix} \rightarrow \begin{pmatrix} \frac{2}{T} & -1 & 3T \\ T_2 & -T_1 T_2 & \\ -\frac{1}{T_1} & 2 & T_1 \\ \frac{1}{T_1 T_2} & -\frac{1}{T_2} & \end{pmatrix}$$



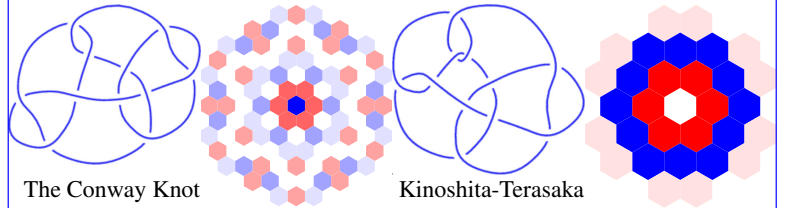
Fast.

```
F1([S_-, i_-, j_-]) := CF[
  S [1/2 - B11 + T1 B21 - B21 B22 - (T1-1) B21 B21 + 2 B21 B21 -
    (1-T1) B21 B21 - B21 B21 - T1 B21 B21 + B21 B21 +
    ((T1-1) B21 (T1-1) B21 - T1 B21 + T1 B21) +
    (T1-1) B21 (1-T1 B21 + B21 + (T1-2) B21 - (T1-1) (T1+1) B21)] /
    (T1-1)]
F2([S0, i0, j0], [S1, i1, j1]) :=
  CF[S1 (T1-1) (T1-1) (T1-1) (T1-1) B21 B21 B21 B21]
  ((T1-1) B21 B21 B21 B21 - (T1-1) B21 B21 B21 B21)]
F3([S_-, i_-, j_-]) = 1/2 B21 - 1/2;
F4 := T1 T2;
CF[0] := ExpandCollect[0, g_-, F] / F + Factor;
Program
0[K_] := 0[K] = Module[{X, phi, n, A, Delta, G, ev, phi, k1, k2},
  {X, phi} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
  Cases[X, {S_-, i_-, j_-} >> {0[K], {i, j}, {i+1, j+1}} + {
    -T1 T1-1, 1}];
  Delta = (-Total[phi] - Total[phi[All, 1]])/2 Det[A];
  G = Inverse[A];
  ev[0_] := Factor[0 / (G[phi, phi] / (G[phi, phi] / T - T - 1));
  phi = ev[Sum[F2[X[K], {k1, n}], {k1, n}];
  phi = ev[Sum[F2[X[K], {k1, n}], {k1, n}];
  phi = ev[Sum[F2[X[K], {k1, n}], {k1, n}];
  Factor[0, (Delta / T - T1) (Delta / T - T2) (Delta / T - T3) 0];
];
```



A random 300 xing knot from [DHOEBL]. For most invariants, 300 is science fiction.

n	≤ 10	≤ 11	≤ 12	≤ 13	≤ 14	≤ 15
knots	249	801	2,977	12,965	59,937	313,230
Δ	(38)	(250)	(1,204)	(7,326)	(39,741)	(236,326)
σ_{LT}	(108)	(356)	(1,525)	(7,736)	(40,101)	(230,592)
J	(7)	(70)	(482)	(3,434)	(21,250)	(138,591)
Kh	(6)	(65)	(452)	(3,226)	(19,754)	(127,261)
H	(2)	(31)	(222)	(1,839)	(11,251)	(73,892)
Vol	(~6)	(~25)	(~113)	(~1,012)	(~6,353)	(~43,607)
(Kh, H, Vol)	(~0)	(~14)	(~84)	(~911)	(~5,917)	(~41,434)
(Δ, ρ_1)	(0)	(14)	(95)	(959)	(6,253)	(42,914)
(Δ, ρ_1, ρ_2)	(0)	(14)	(84)	(911)	(5,926)	(41,469)
$(\rho_1, \rho_2, Kh, H, Vol)$	(0)	(~14)	(~84)	(~911)	(~5,916)	(~41,432)
Θ	(0)	(3)	(19)	(194)	(1,118)	(6,758)
(Θ, ρ_2)	(0)	(3)	(10)	(169)	(982)	(6,341)
(Θ, σ_{LT})	(0)	(3)	(19)	(194)	(1,118)	(6,758)
(Θ, Kh)	(0)	(3)	(18)	(185)	(1,062)	(6,555)
(Θ, H)	(0)	(3)	(18)	(185)	(1,064)	(6,563)
(Θ, Vol)	(0)	(~3)	(~10)	(~169)	(~973)	(~6,308)
$(\Theta, \rho_2, Kh, H, Vol)$	(0)	(~3)	(~10)	(~169)	(~972)	(~6,304)



Topologically Meaningful. θ is near Δ and we dream that anything Δ can do, θ does too (sometimes better). The following two conjectures are verified for knots with ≤ 13 crossings:

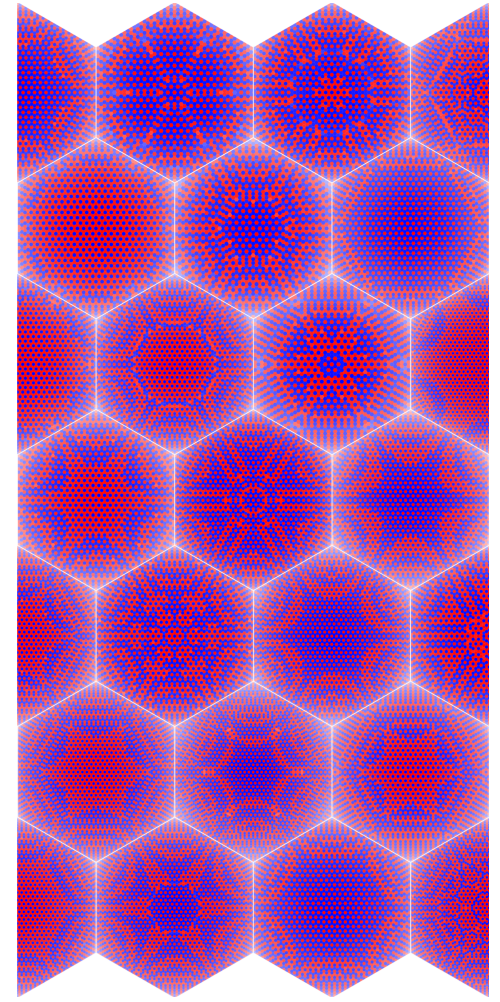
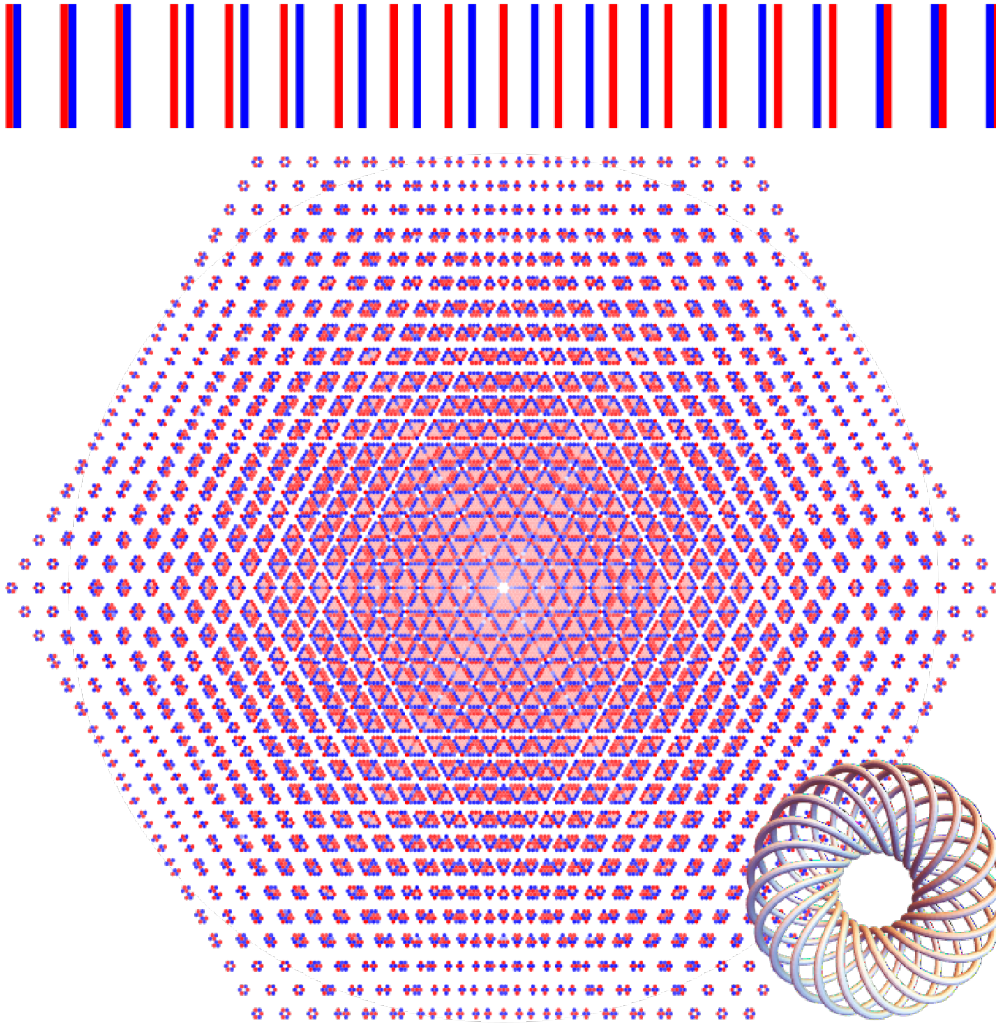
Conjecture 1. $\deg_{T_1} \theta(K) \leq 2g(K)$.
Conjecture 2. If K is a fibered knot and d is the degree of $\Delta(K)$ (the highest power of T), then the coefficient of T_1^{2d} in $\theta(K)$, which is a polynomial in T_1 , is an integer multiple of $T_1^{2d} \Delta(K)|_{T \rightarrow T_1}$.
Dream. θ has something to say about ribbon knots.

Fun. Θ on Rolfsen's Table:

The 132-crossing torus knot $T_{22/7}$:

(many more at $\omega\epsilon\beta/\text{TK}$)

Random knots from [DHOEBL] with 51 – 75 crossings: (many more at $\omega\epsilon\beta/\text{DK}$)



Moral. We must come to terms with Θ !

Task 1. Make the “data” formulas human friendly.

Task 2. Prove the hexagonal symmetry of $\theta(K)$, and that $\theta(K) = \theta(-K) = -\theta(\bar{K})$.

That’s harder than it seems! The formulas don’t naively show any of that. Δ has a palindromic symmetry first conjectured in Alexander’s original paper [A1] — it is invariant under $T \rightarrow T^{-1}$. Proving this took a few years, and the proof starting from the Wirtinger presentation is quite involved (e.g. [CF, Chapter IX]).

Task 3. Show that θ dominates the Rozansky-Overbay invariant ρ_1 [Ro1, Ro2, Ro4, Ov, BV1]. Precisely, show that $\rho_1 = -\theta|_{T_1 \rightarrow T, T_2 \rightarrow 1}$.

This one should be easy with techniques from [BV3, Section 4.2].

Task 4. Explain the “Chladni patterns”. Are there “dominant parts” of θ that can be computed in isolation?



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Task 5. Prove the genus bound of Conjecture 1.

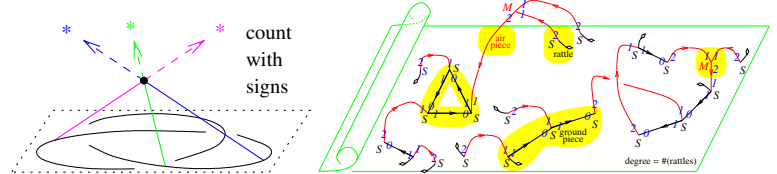
This is probably coming. One can bound the degree of $\Delta = \det(A)$ in terms of $g(K)$ using the Seifert presentation of the Alexander module. Pushing further, likely one can bound the degree of

$(g_{\alpha\beta}) = A^{-1}$ in terms of $g(K)$, and that’s probably enough.

Task 6. Find a 3D interpretation of the $g_{\alpha\beta}$ ’s.

They must be closely related to the equivariant linking numbers of [KY, GK, GT, Oh3, Le1].

Task 7. Find a formula $\mathcal{F}$ for $\Theta(K)$ that starts from a Seifert surface Σ of K . Better if $\mathcal{F}$ is completely 3D! Assuming Task 13, it is known that Θ depends only of invariants of type ≤ 3 of Σ . Maybe $\mathcal{F}$ is about configuration space integrals / chopstick towers? See CS: [Th, Le2, BN1], BF: [CR, BN2]



Task 8. Is there an intrinsic theory of finite type invariants for Seifert surfaces? For task 11, does its gr map to functions on H_1 ?

My current best understanding of finite type invariants for Seifert surfaces goes through thick graphs.



Task 9. Prove the the fibered condition of Conjecture 2.

If K is fibered, $\deg \Delta(K) = g(K)$ and $\Delta(K)$ is monic. Indeed, K is then the mapping cylinder of a diffeomorphism $f: \Sigma \rightarrow \Sigma$. The Alexander module of K is generated by $H_1(\Sigma)$ with relations

$\{\gamma = T f_* \gamma : \gamma \in H_1(\Sigma)\}$. Thus the highest monomial in Δ is $T^g \det(f_*)$ and $\det(f_*) = \pm 1$ as f_* preserves the intersection pairing. If only we had a formula for θ in terms of $f \dots$

Task 10. In general, find a formula for Θ corresponding to each known presentation of the Alexander module.

Wirtinger is $2\{\text{xings}\} \rightarrow \{\text{edges}\}$. Dehn is $\{\text{xings}\} \rightarrow \{\text{faces}\}$. Co-Dehn is $\{\text{faces}\} \rightarrow \{\text{xings}\}$. Burau is $\{\text{braid strands}\} \rightarrow \{\text{braid strands}\}$. Seifert is $H_1(\Sigma) \rightarrow H_1(\Sigma)$, and so is the presentation from Task 9. Grid diagrams lead to $\{\text{grid number}\} \rightarrow \{\text{grid number}\}$ (may relate to HFK). There's more!

Task 11. Write up the integration story.

Claim (e.g., [BN4]). Cutting corners, with $\epsilon^2 = 0$,

$$\frac{1}{\Delta_1 \Delta_2 \Delta_3} \exp\left(\epsilon \cdot \frac{\theta}{\Delta_1 \Delta_2 \Delta_3}\right) \sim \oint_{\prod_{e \in \mathbb{R}^6} p_{1e}, p_{2e}, p_{3e}, x_{1e}, x_{2e}, x_{3e}} \prod_C e^{L_C},$$

where $\oint$ denotes perturbed formal Gaussian integration (i.e., “Feynman Diagrams”) and L_C is

$$\begin{aligned} L[X_{i,j}, [S_-]] := & \text{Plus} [\\ & \sum_{v=1}^3 (x_{vi} (p_{vi}^+ - p_{vi}) + x_{vj} (p_{vj}^+ - p_{vj}) + (T_v^S - 1) x_{vi} (p_{vi}^+ - p_{vj}^+)), \\ & (T_1^S - 1) p_{3j} x_{1i} (T_2^S x_{2i} - x_{2j}), \\ & \epsilon \in (T_3^S - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T_2^S - 1), \\ & \epsilon \in (1/2 + T_2^S p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} - (T_2^S - 1) p_{2j} p_{3i} x_{2i} x_{3i} + \\ & (T_3^S - 1) p_{2j} p_{3j} x_{2i} x_{3i} + 2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} - \\ & T_2^S p_{2j} p_{3j} x_{2i} x_{3j} + \\ & ((T_1^S - 1) p_{1j} x_{1i} (T_2^S p_{2j} x_{2i} - T_2^S p_{2j} x_{2j} - (T_2^S + 1) (T_3^S - 1) p_{3j} x_{3i} + \\ & T_2^S p_{3j} x_{3j}) + (T_3^S - 1) p_{3j} x_{3i} \\ & (1 - T_2^S p_{1i} x_{1i} + p_{2i} x_{2j} + (T_2^S - 2) p_{2j} x_{2j})) / (T_2^S - 1))] \end{aligned}$$

In fact, we first found L_C using the method of undetermined coefficients, and then derived F_1 and F_2 from it.

Task 12. Find a similar perturbed Gaussian integral formula for θ , but with integration over $6H_1(\Sigma)$. The quadratic Q will be the same as in the Seifert-Alexander formula (but repeated 3 times, for each T_v). The perturbation P_ϵ will be given by low-degree finite type invariants of curves on Σ (possibly also dependent on the intersection points of such curves, or on other information coming from Σ).

Task 13. Prove that θ is equal to the two-loop contribution $Z^{(2)}$ to the Kontsevich integral Z .

Composed with the inverse PBW isomorphism χ^{-1} , $\chi^{-1} \circ Z$ takes values in univalent Jacobi diagrams, $\mathcal{B} = \{\oslash \circ \dots\} / IHX$. Rozansky conjectured [Ro3, GR] and Kricker proved [Kr] that

$$\log(\chi^{-1} \circ Z) = f_1 \left(\text{diagram with } t \right) + f_2 \left(\text{diagram with } t_1, t_2 \right) + \text{higher loops},$$

where $t^k \text{---} t := \text{---} | \cdots n \cdots | \text{---}$, $f_1 \in \mathbb{Q}[[t]]$, and $f_2 \in \mathbb{Q}[[t_1, t_2]]$ satisfy $f_1 = \frac{1}{2} \log \frac{\sinh(t/2)}{t \Delta(e^t/2)}$ and $f_2 = Z^{(2)}(e^{t_1}, e^{t_2}) / \Delta(e^{t_1}) \Delta(e^{t_1}) \Delta(e^{t_1+t_2})$ where $Z^{(2)} \in \mathbb{Z}[T_1^{\pm 1}, T_2^{\pm 1}]$ is the “two loop polynomial”. Ohtsuki [Oh2] studied $Z^{(2)}$ extensively, and almost certainly, $Z^{(2)} = \theta$. Prove that!

Task 14. Complete and write up the g_ϵ^+ story.

Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{b}^u$ and $\mathfrak{b}^l$ be its upper and lower Borel subalgebras. Then $\mathfrak{b}^u$ has a bracket β , and as the dual of $\mathfrak{b}^l$ it also has a cobracket δ , and in fact, $\mathfrak{g} \oplus \mathfrak{h} \equiv \text{Double}(\mathfrak{b}^u, \beta, \delta)$. Let $g_\epsilon^+ := \text{Double}(\mathfrak{b}^u, \beta, \epsilon \delta) \pmod{\epsilon^{d+1}}$ it is solvable for any d . We expect that Θ is the universal invariant (in the sense of Lawrence and Ohtsuki [La, Oh1]) corresponding to $sl_{3,\epsilon}^+$, computed modulo ϵ^2 (in fact, that's how we guessed it). See [BN3, BV2].

Task 15. Understand Chern-Simons theory with gauge group g_ϵ^+ .

Is there a gauge that leads to the formula $\mathcal{F}$ of Task 7?

Task 16. What happens to representation theory as $\epsilon \rightarrow 0$? Is there any fun in continuous morphisms $g_\epsilon^+ \rightarrow gl_{n,\epsilon}^+$?

Task 17. Does Θ extend to knots in $\mathbb{Z}HS/\mathbb{Q}HS$?

Z and $Z^{(2)}$ do.

Task 18. Is there a surgery formula for Θ ?

Z and $Z^{(2)}$ have.

Task 19. Extend Θ to tangles and figure out how it behaves under strand doubling.

Z and $Z^{(2)}$ extend but their extensions depend on parenthesizations. From Task 14 we expect that Θ will extend without the need for parenthesizations, yet with an asymmetry built into the doubling operations.

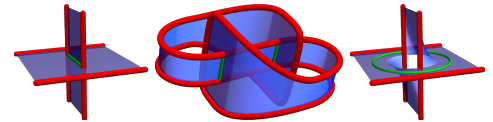
Task 20. Make Kricker / Ohtsuki [Kr, Oh2] more computable!

Task 21. Find a multi-variable version of θ for links, like there is a multi-variable Alexander for links (e.g. [Ka, Chapter 7]).

It is predicted g_ϵ^+ consideration, but not by the loop expansion.

Task 22. Find a ribbon condition satisfied by Θ .

For a ribbon knot K , one may find a Seifert surface Σ half of whose homology is



generated by the components of an unlink embedded in Σ . This makes for a presentation matrix A of the Alexander module of K that has big blocks of zeros, and this leads to the Fox-Milnor condition [FM], $\Delta \doteq \det(A) \doteq f(T)f(T^{-1})$ for some $f \in \mathbb{Z}[T^{\pm 1}]$. If $\det A$ is constrained for ribbon knots, perhaps so is A^{-1} and therefore Θ ?

Task 23. Carthago delenda est and every knot polynomial must be categorified.

M. Khovanov & Cato the Elder



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A: Task 14.3 Relate the q_1^+ story w/ (rotational) virtual knots, $\mathbb{A}^2$, and quantization of Lie bi-algebras $\sim \hbar$ Ek, Enriquez and Severn.

Task 14.4 Find $\sim$ w-style characterization.

A FAST, STRONG, TOPOLOGICALLY MEANINGFUL, AND FUN KNOT INVARIANT

DROR BAR-NATAN AND ROLAND VAN DER VEEN

ABSTRACT. In this paper we discuss a pair of polynomial knot invariants $\Theta \sim (\Delta, \theta)$ which is:

- Theoretically and practically fast; Θ can be computed in polynomial time. We can compute it in full on random knots with over 300 crossings, and its evaluation at simple rational numbers on random knots with over 600 crossings.
- Strong. Its separation power is much greater than the hyperbolic volume, the HOMFLY-PT polynomial and Khovanov homology (taken together) on knots with up to 15 crossings (while being computable on much larger knots).
- Topologically meaningful. It likely gives a genus bound, and there are reasons to hope that it would be more.
- Fun. See also Figures 1.1–1.4, 3.1, and 6.2.

Δ is merely the Alexander polynomial. θ is almost certainly equal to an invariant that was studied extensively by Ohtsuki [Ohts04], containing Rozansky, Kricker, and Garoufalidis [Roz01, Roz02, Roz03, Kr, GR]. Yet our formulas, proofs, and programs are much simpler and enable the computation even on very large knots.

CONTENTS

1. Fun
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This paper is available in electronic form, along with source files and a demo Mathematica notebook at [http://drcb.nyu.edu/Theta](#) and at [arXiv:2509.18456](#).

1. FUN

The word “fun” rarely appears in the title of a math paper, so let us start with a brief justification.

Θ is a pair of polynomials. The first, Δ , is old news, the Alexander polynomial [Al]. It is a one-variable Laurent polynomial in a variable T . For example, $\Delta(\bigcirc) = T^{-1} - 1 + T$. We turn such a polynomial into a list of coefficients (for Δ , it is $(1, -1, 1)$), and then to a chain of bars of varying colours: white for the zero coefficients, and red and blue for the positive and negative coefficients (with intensity proportional to the magnitude of the coefficients). The result is a “bar code”, and for the trefoil Δ it is

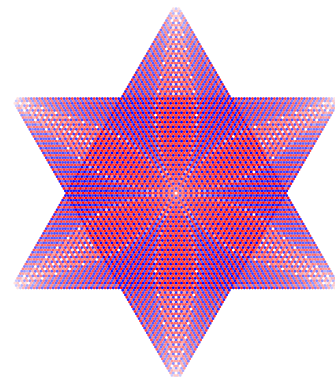
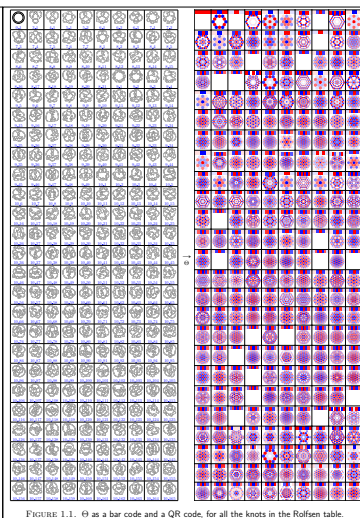
Similarly, θ is a 2-variable Laurent polynomial in variables T_1 and T_2 . We can turn such a polynomial into a 2D array of coefficients and then using the same rules, into a 2D array of colours, namely, into a picture.

To highlight a certain conjectured hexagonal symmetry of the resulting pictures, we apply a shear transformation to the plane before printing. So a monomial $cT_1^{n_1}T_2^{n_2}$ gets printed at position $(n_1 - n_2/2, \sqrt{3}n_2/2)$ instead of the more straightforward (n_1, n_2) . On the right is the 2D picture corresponding to the polynomial $2 + T_1 - T_1T_2 + T_2 - T_1^{-1} - T_1^{-1}T_2 + T_2^{-1}$.

Thus Θ becomes a pair of pictures: a bar code, and a 2D picture that we call a “hexagonal QR code”. For the knots in the Rolfsen table (with the unknot prepended at the start), they are in Figure 1.1. For some alternating square weave knots, they are in Figure 1.2, and for a random square weave, in Figure 1.3. In addition, the hexagonal QR codes of 15 knots with ≥ 300 crossings are in Figure 1.4, and Θ of a 132-crossing torus knot is in Figure 3.1. Some further computations and figures, also highlighting the parity of coefficients rather than just their signs, are at [La].

Clearly there are patterns in these figures. There is a hexagonal symmetry and the QR codes are nearly always hexagons (these are independent properties). Much more can be seen in Figure 1.1. In Figure 1.4 there seem to be large-scale patterns perhaps reminiscent of the “Chladni figures” formed by powders atop vibrating plates (on right). We can’t prove any of these things, and the last one, we can’t even formulate properly. Yet they are clearly there, too clear to be the result of chance alone. We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.

Left: © William Brewster of the History of Science, University of Cambridge; right: CC BY-SA 4.0 / Wikimedia / Monastera (DHR GSH) / Rodrigo Torres Arce.



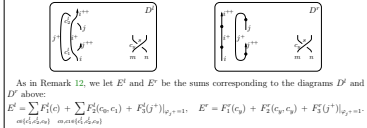
A (2, 41, -41) pretzel for dessert

Turn into a question?

$\mathbf{R1} = \{(1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)\}$
 $\mathbf{R2} = \{(1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)\}$
 $\mathbf{R3} = \{(1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)\}$
 $\mathbf{R4} = \{(1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)\}$

Proposition 13. The quantity θ_1 is invariant under the $\mathbf{R1}$ and $\mathbf{R2}$.

Proof. For $\mathbf{R1}$, we follow the same logic as in the proof of Proposition 11, as simplified by Remark 12. We start with the figure that replaces Figure 1.4 (note the null moves in D' and their minimal effect as in Lemma 7 and Remark 8):



As in Remark 12, let E^1 and E^2 be the sums corresponding to the diagrams D and D' above:

We used to say that $E^1 = E^2$ after all relevant $\mathbf{g}$ -rules are applied to both sides.

To compute these E sums we first have to extend the $\mathbf{E}$ sum notation to accept also a list of pairs $\{(i, j)\}$ of the form (rotation number, edge label):

$\mathbf{E} = \sum_{(i,j) \in \{(1,1), (1,1), (1,1), (1,1)\}} \mathbf{E}(\mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)})$
 $\mathbf{E} = \sum_{(i,j) \in \{(1,1), (1,1), (1,1), (1,1)\}} \mathbf{E}(\mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)})$
 $\mathbf{E} = \sum_{(i,j) \in \{(1,1), (1,1), (1,1), (1,1)\}} \mathbf{E}(\mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)})$
 $\mathbf{E} = \sum_{(i,j) \in \{(1,1), (1,1), (1,1), (1,1)\}} \mathbf{E}(\mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)})$

The computation of $\mathbf{E}$ is simpler, as we follow the same logic as in the proof of Proposition 11, as simplified by Remark 12. We start with the figure that replaces Figure 1.4 (note the null moves in D' and their minimal effect as in Lemma 7 and Remark 8):

$\mathbf{E} = \sum_{(i,j) \in \{(1,1), (1,1), (1,1), (1,1)\}} \mathbf{E}(\mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)})$
 $\mathbf{E} = \sum_{(i,j) \in \{(1,1), (1,1), (1,1), (1,1)\}} \mathbf{E}(\mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)})$
 $\mathbf{E} = \sum_{(i,j) \in \{(1,1), (1,1), (1,1), (1,1)\}} \mathbf{E}(\mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)})$
 $\mathbf{E} = \sum_{(i,j) \in \{(1,1), (1,1), (1,1), (1,1)\}} \mathbf{E}(\mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)} \circ \mathbf{g}_{(i,j)})$

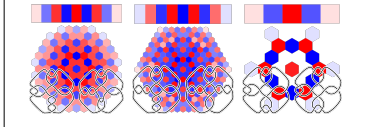
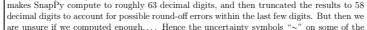
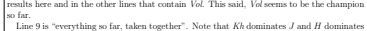


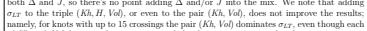
FIGURE 5.1. The three pairs responsible for the deficit δ in the column $n = 11$ of line 12 of Table 5.1. They are $(1, 1, 1, 1)$, $(1, 1, 1, 1)$, and $(1, 1, 1, 1)$, and each pair is a pair of Mutator links (defined by θ sometimes does separate mutant pairs, as was shown in Section 3.2).

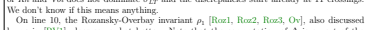


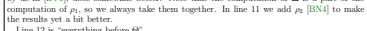
FIGURE 5.2. The 48-crossing Gompf-Scharlemann-Thompson (GST) knot [GST].

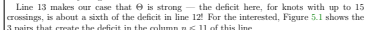


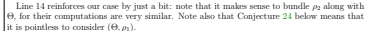


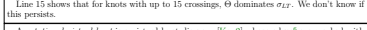


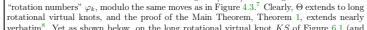


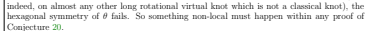


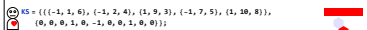






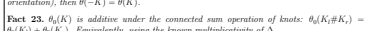


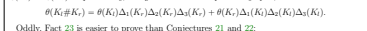


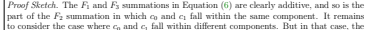


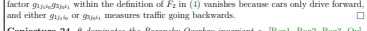


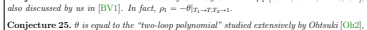


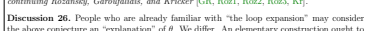


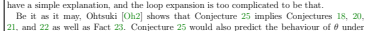


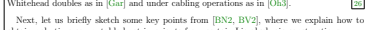


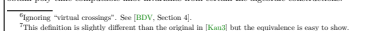










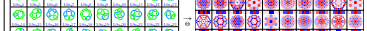










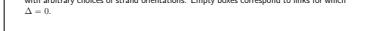


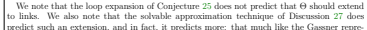


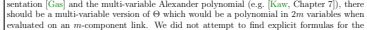




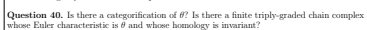


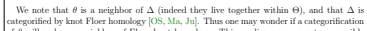


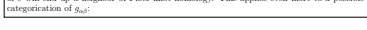














For $\mathbf{R2}$, we allow ourselves to even more concise:

$\mathbf{R1} = \{(1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)\}$
 $\mathbf{R2} = \{(1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)\}$
 $\mathbf{R3} = \{(1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)\}$
 $\mathbf{R4} = \{(1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)\}$

