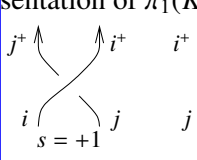


Homework 1-21

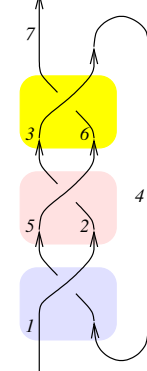
Abstract. I'll start with a review of my recent paper with van der Veen, "A Fast, Strong, Topologically Meaningful, and Fun Knot Invariant" [BV2], and then assign some homework. Much of what I'll say follows earlier work by Rozansky, Kriker, Garoufalidis, and Ohtsuki [Ro1, Ro2, Ro3, Kr, GR, Oh1].

Acknowledgement. This work was supported by NSERC grants RGPIN-2018-04350 and RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

A. With T an indeterminate, start from a presentation matrix A for the Alexander module of K , coming from the Wirtinger presentation of $\pi_1(K)$: $A := I_{2n+1} + \sum_c A_c$, where



$$A_c = \begin{vmatrix} i+1 & j+1 \\ i & j \end{vmatrix} \begin{vmatrix} -T^s & T^s-1 \\ 0 & -1 \end{vmatrix}$$

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$


$$\Delta \doteq \det(A)$$

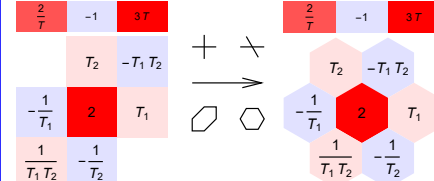
G. Let $G = (g_{\alpha\beta}) := A^{-1}$:

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let T_1 and T_2 be new indeterminates, let $T_3 = T_1 T_2$, and let $G_\nu = (g_{\nu\alpha\beta})$ be G with $T \rightarrow T_\nu$, for $\nu = 1, 2, 3$.

$$\theta \sim \Delta_1 \Delta_2 \Delta_3 \sum_{c_0, c_1} g_{1i_0 i_1} g_{2i_0 i_1} g_{3i_1 i_0} + \text{l.o.}$$

$$\Theta = (\Delta, \theta) \in \mathbb{Z}[T^{\pm 1}] \times \mathbb{Z}[T_1^{\pm 1}, T_2^{\pm 1}]$$



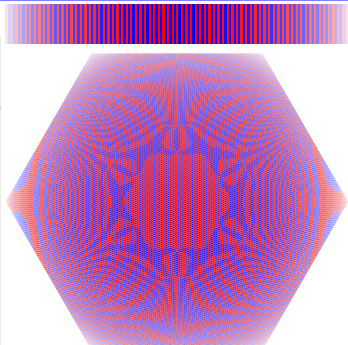
Fast.

Data (ouch)

```

F1([s_-, e_-, j_-]) := CF
S (1/2 - B11 + T1 B21 - B11 B21 - (T1-1) B21 B11 + 2 B21 B11 -
(1-T1) B21 B11 - B21 B11 - T1 B21 B11 + B11 B21 +
((T1-1) B21 (T1-1) B21 - T1 B21 + T1 B21) +
(T1-1) B21 (1-T1 B11 + B21 + (T1-2) B21 - (T1-1) (T1+1) B21)/
(T1-1))
F2([s0, i0, j0], [s1, i1, j1]) :=
CF[s1 (T1-1) (T1-1) (T1-1) (T1-1) B21 j1 i0 B21 j0 i1
((T1-1) B21 i1 i0 - B21 i1 j0) - (T1-1) B21 j1 i0 - B21 j1 j0]]
F3([phi, h_-]) = phi B21 - phi/2;
F4 = T1 T2;
CF[phi] := ExpandCollect[phi, g_-, F] /. F -> Factor;
Program
phi[phi_] := Module[{X, phi, n, A, delta, G, ev, phi, k1, k2},
{X, phi} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
Cases[X, {s_-, e_-, j_-} >> {A[[e[[j]], (i+1, j+1)]] += {
-T1 T1-1-1
0 -1-1
}}];
delta = (Total[phi]-Total[X[[All,1]]])/2 Det[A];
G = Inverse[A];
ev[phi_] := Factor[phi /. G_-, phi] /. {G[e_-, phi] /. F -> T_+};
ev[phi] := Sum[Sum[F2[X[[k1]], X[[k2]]], {k1, n}, {k2, n}]]];
ev[phi] := Sum[Sum[F3[X[[k1]], k1], {k1, Length[phi]}]]];
Factor[phi, (delta, F -> T1) (delta, F -> T2) (delta, F -> T3) phi];
];

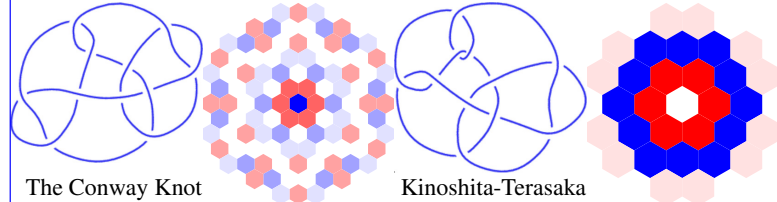
```



A random 300 xing knot from [DHOEBL]. For most invariants, 300 is science fiction.

Strong. Θ vs. a slew of other reasonably-computable invariants (deficits shown):

n	≤ 10	≤ 11	≤ 12	≤ 13	≤ 14	≤ 15
knots	249	801	2,977	12,965	59,937	313,230
Δ	(38)	(250)	(1,204)	(7,326)	(39,741)	(236,326)
σ_{LT}	(108)	(356)	(1,525)	(7,736)	(40,101)	(230,592)
J	(7)	(70)	(482)	(3,434)	(21,250)	(138,591)
Kh	(6)	(65)	(452)	(3,226)	(19,754)	(127,261)
H	(2)	(31)	(222)	(1,839)	(11,251)	(73,892)
Vol	(~6)	(~25)	(~113)	(~1,012)	(~6,353)	(~43,607)
(Kh, H, Vol)	(~0)	(~14)	(~84)	(~911)	(~5,917)	(~41,434)
(Δ, ρ_1)	(0)	(14)	(95)	(959)	(6,253)	(42,914)
(Δ, ρ_1, ρ_2)	(0)	(14)	(84)	(911)	(5,926)	(41,469)
$(\rho_1, \rho_2, Kh, H, Vol)$	(0)	(~14)	(~84)	(~911)	(~5,916)	(~41,432)
Θ	(0)	(3)	(19)	(194)	(1,118)	(6,758)
(Θ, ρ_2)	(0)	(3)	(10)	(169)	(982)	(6,341)
(Θ, σ_{LT})	(0)	(3)	(19)	(194)	(1,118)	(6,758)
(Θ, Kh)	(0)	(3)	(18)	(185)	(1,062)	(6,555)
(Θ, H)	(0)	(3)	(18)	(185)	(1,064)	(6,563)
(Θ, Vol)	(0)	(~3)	(~10)	(~169)	(~973)	(~6,308)
$(\Theta, \rho_2, Kh, H, Vol)$	(0)	(~3)	(~10)	(~169)	(~972)	(~6,304)



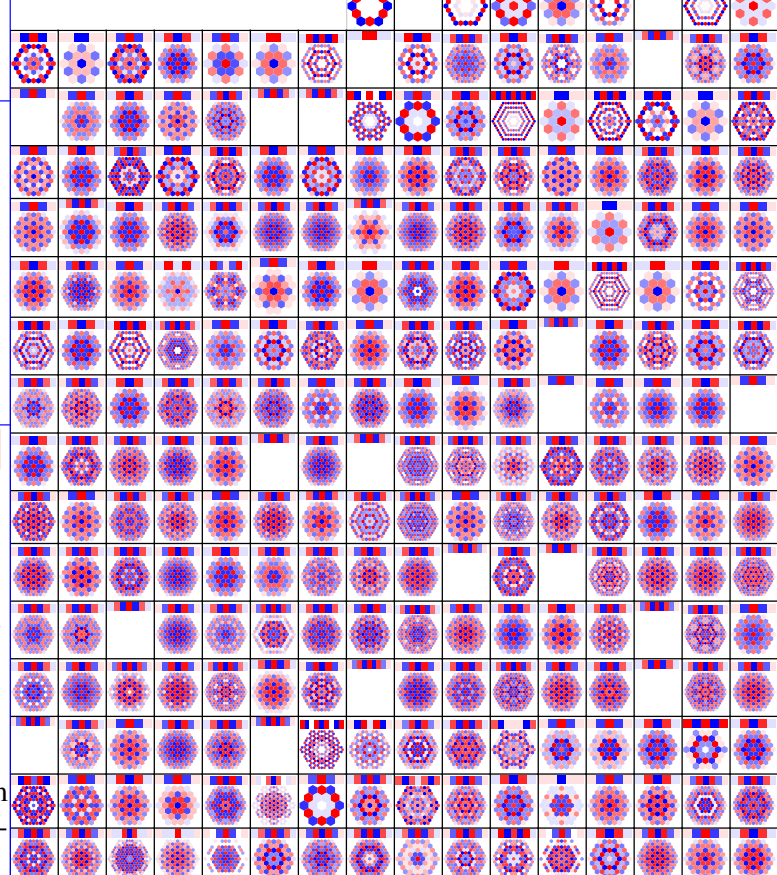
Topologically Meaningful. θ is near Δ and we dream that anything Δ can do, θ does too (sometimes better). The following two conjectures are verified for knots with ≤ 13 crossings:

Conjecture 1. $\deg_{T_1} \theta(K) \leq 2g(K)$.

Conjecture 2. If K is a fibered knot and d is the degree of $\Delta(K)$ (the highest power of T), then the coefficient of T_2^{2d} in $\theta(K)$, which is a polynomial in T_1 , is an integer multiple of $T_1^{2d} \Delta(K)|_{T \rightarrow T_1}$.

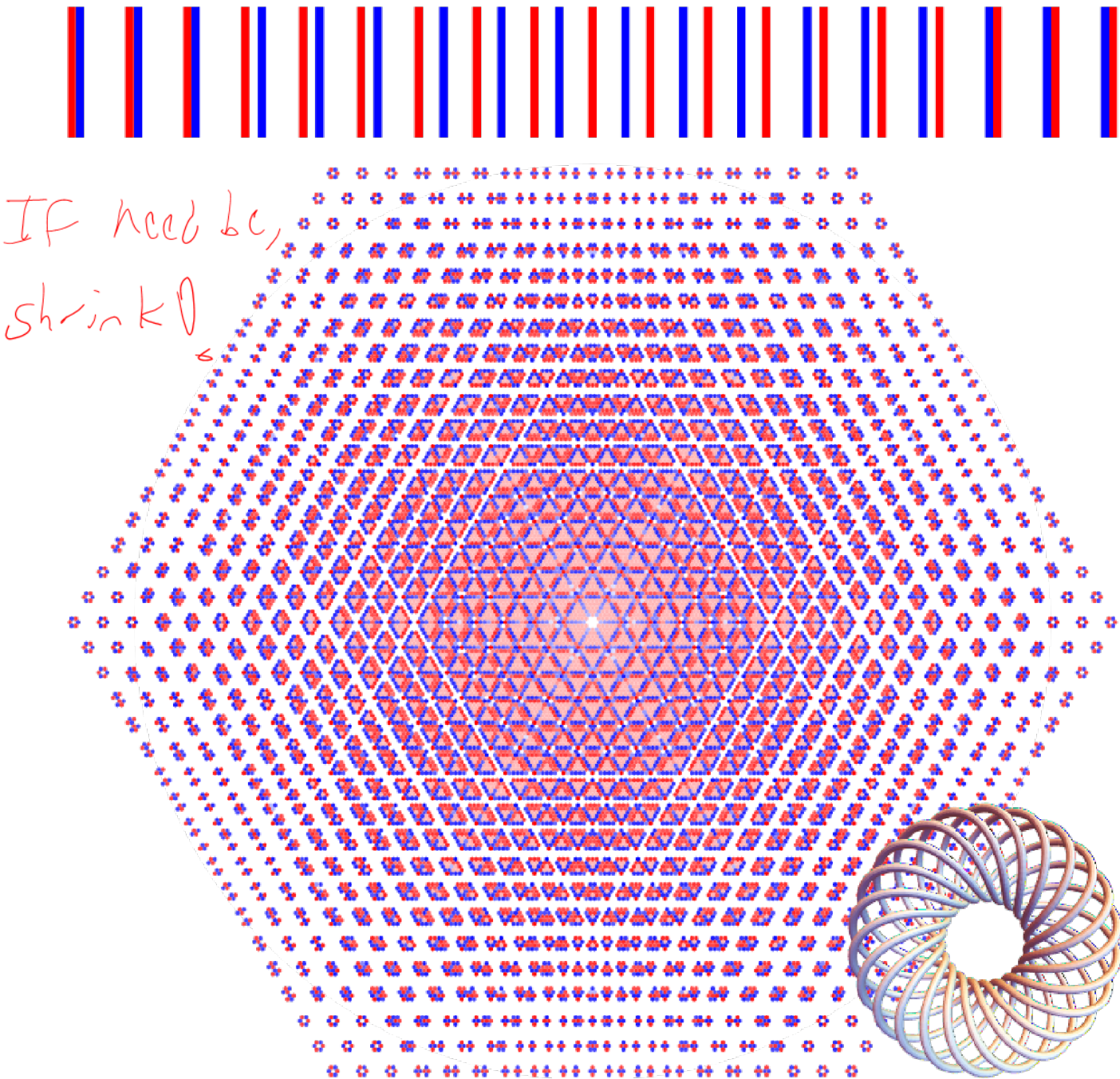
Dream. θ has something to say about ribbon knots.

Fun. Θ on Rolfsen's Table:



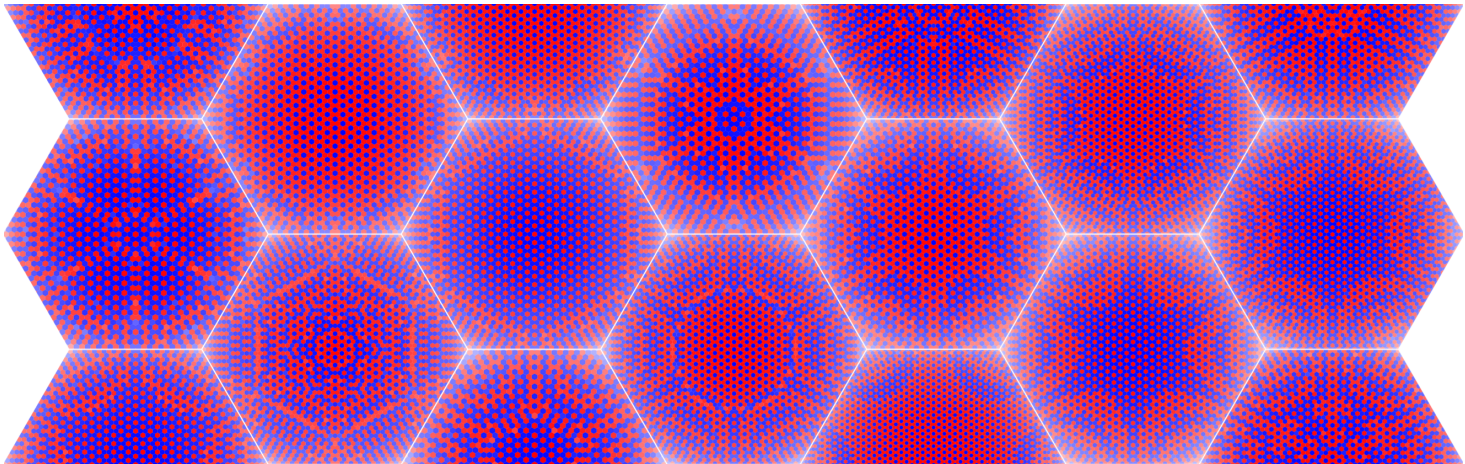
The 132-crossing torus knot $T_{22/7}$:

(many more at [ωεβ/TK](#))



Random knots from [DHOEBL], with 51 – 68 crossings:

(many more at [ωεβ/DK](#))



Homework Task 1. Make the “data” formulas human friendly.

That’s harder than it seems! The formulas don’t naively show any of that. Δ has a palindromic symmetry first conjectured in Alexander’s original paper [Al] — it is invariant under $T \rightarrow T^{-1}$. Proving this took a few years, and the proof starting from the Wirtinger presentation is quite involved (e.g. [CF, Chapter IX]).

This one should be easy with techniques from [BV2, Section 4.2].

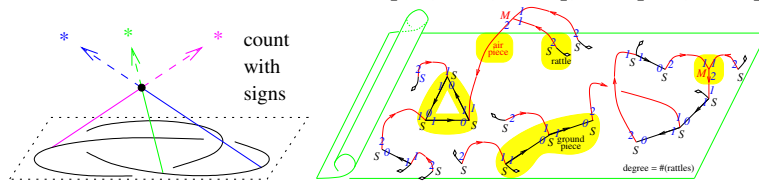
left: © Whipple Museum of the History of Science, University of Cambridge; right: CC-BY-SA 4.0 / Wikimedia / Matematica (IME USP) / Rodrigo Tetsuo Argenton



This is probably coming. One can bound the degree of $\Delta = \det(A)$ in terms of $g(K)$ using the Seifert presentation of the Alexander module. Pushing further, likely one can bound the degree of $(g_{\alpha\beta}) = A^{-1}$ in terms of $g(K)$, and that's probably enough.

They must be closely related to the equivariant linking numbers of [KY, GK, GT, Oh2, Le1].

See CS: [Th, Le2, BN1], BF: [CR, BN2]



My current best understanding of finite type invariants for Seifert surfaces goes through thick graphs.



If K is fibered, $\deg \Delta(K) = g(K)$ and $\Delta(K)$ is monic. Indeed, K is then the mapping cylinder of a diffeomorphism $f: \Sigma \rightarrow \Sigma$. The Alexander module of K is generated by $H_1(\Sigma)$ with relations $\{\gamma = Tf_*\gamma: \gamma \in H_1(\Sigma)\}$. Thus the highest monomial in Δ is $T^g \det(f_*)$ and $\det(f_*) = \pm 1$ as f_* preserves the intersection pairing. If only we had a formula for θ in terms of $f \dots$

Wirtinger is $2\{\text{xings}\} \rightarrow \{\text{edges}\}$. Dehn is $\{\text{xings}\} \rightarrow \{\text{faces}\}$. Co-Dehn is $\{\text{faces}\} \rightarrow \{\text{xings}\}$. Burau is $\{\text{braid strands}\} \rightarrow \{\text{braid strands}\}$. Seifert is $H_1(\Sigma) \rightarrow H_1(\Sigma)$, and so is the presentation from Task 9. Grid diagrams lead to $\{\text{grid number}\} \rightarrow \{\text{grid number}\}$ (may relate to HFK). There's more!

Claim (e.g., [BN3]). Cutting corners, with $\epsilon^2 = 0$,

$$\frac{1}{\Delta_1 \Delta_2 \Delta_3} \exp\left(\epsilon \cdot \frac{\theta}{\Delta_1 \Delta_2 \Delta_3}\right) \sim \oint_{\prod_e \mathbb{R}^6_{p_{1e}, p_{2e}, p_{3e}, x_{1e}, x_{2e}, x_{3e}}} \prod_c \mathbb{E}^{L_c},$$

$$\begin{aligned} & \mathbf{L}[\mathbf{x}_{i,j}, \mathbf{s}_{-}] := \text{Plus} [\\ & \sum_{i,v=1}^3 (\mathbf{x}_{vi} (\mathbf{p}_{vi}^+ - \mathbf{p}_{vi}) + \mathbf{x}_{vj} (\mathbf{p}_{vj}^+ - \mathbf{p}_{vj}) + (\mathbf{T}_v^s - 1) \mathbf{x}_{vi} (\mathbf{p}_{vi}^+ - \mathbf{p}_{vj}^+)), \\ & (\mathbf{T}_1^s - 1) \mathbf{p}_{1j} \mathbf{x}_{1i} (\mathbf{T}_2^s \mathbf{x}_{2i} - \mathbf{x}_{2j}), \\ & \in \mathbf{s} (\mathbf{T}_3^s - 1) \mathbf{p}_{1j} (\mathbf{p}_{2i} - \mathbf{p}_{2j}) \mathbf{x}_{3i} / (\mathbf{T}_2^s - 1), \\ & \in \mathbf{s} (1/2 + \mathbf{T}_2^s \mathbf{p}_{1i} \mathbf{p}_{2j} \mathbf{x}_{1i} \mathbf{x}_{2i} - \mathbf{p}_{1i} \mathbf{p}_{2j} \mathbf{x}_{1i} \mathbf{x}_{2j} - \mathbf{p}_{3i} \mathbf{x}_{3i} - (\mathbf{T}_2^s - 1) \mathbf{p}_{2j} \mathbf{p}_{3i} \mathbf{x}_{2i} \mathbf{x}_{3i} + \\ & (\mathbf{T}_3^s - 1) \mathbf{p}_{2j} \mathbf{p}_{3j} \mathbf{x}_{2i} \mathbf{x}_{3i} + 2 \mathbf{p}_{2j} \mathbf{p}_{3i} \mathbf{x}_{2j} \mathbf{x}_{3i} + \mathbf{p}_{1i} \mathbf{p}_{3j} \mathbf{x}_{1i} \mathbf{x}_{3j} - \mathbf{p}_{2i} \mathbf{p}_{3j} \mathbf{x}_{2i} \mathbf{x}_{3j} - \\ & \mathbf{T}_2^s \mathbf{p}_{2j} \mathbf{p}_{3j} \mathbf{x}_{2i} \mathbf{x}_{3j} + \\ & ((\mathbf{T}_1^s - 1) \mathbf{p}_{1j} \mathbf{x}_{1i} (\mathbf{T}_2^s \mathbf{p}_{2j} \mathbf{x}_{2i} - \mathbf{T}_2^s \mathbf{p}_{2j} \mathbf{x}_{2j} - (\mathbf{T}_2^s + 1) (\mathbf{T}_3^s - 1) \mathbf{p}_{3j} \mathbf{x}_{3i} + \\ & \mathbf{T}_2^s \mathbf{p}_{3j} \mathbf{x}_{3j}) + (\mathbf{T}_3^s - 1) \mathbf{p}_{3j} \mathbf{x}_{3i} \\ & (1 - \mathbf{T}_2^s \mathbf{p}_{1i} \mathbf{x}_{1i} + \mathbf{p}_{2i} \mathbf{x}_{2j} + (\mathbf{T}_2^s - 2) \mathbf{p}_{2j} \mathbf{x}_{2j})) / (\mathbf{T}_2^s - 1)] \end{aligned}$$

Homework Task 12. Find a similar perturbed Gaussian integral formula for θ , but with integration over $6H_1(\Sigma)$. The quadratic Q will be the same as in the Seifert-Alexander formula (but repeated 3 times, for each T_Y). The perturbation P_ϵ will be given by low-degree finite type invariants of curves on Σ (possibly also dependent on the intersection points of such curves, or on other information coming from Σ).

Composition with the inverse PBW isomorphism X^{-1} , $X^{-1}Z$ takes values in $\mathcal{D} = \{Z \in \mathfrak{g} \mid [Z, X] = 0\}$. Robinsky conjectured and Kricker proved [3] that

$$\log X^{-1}Z = \underbrace{\sum_n a_n \left[\text{rectangle} \right] + \sum_{nm} b_{nm} \left[\text{oval} \right]}_{Z(z)} + \text{hidden terms}$$

where $\sum a_n t^n = \Delta(t)$ and $\sum b_m t_1^n t_2^m = R(t_1, t_2)$
where Δ is Alexander and $R \in \mathbb{Z}[T_1^{\pm 1}, T_2^{\pm 1}]$ is
a two variable polynomial. Ohtsuki [3] studied R
extensively, and almost certainly, $R = \Theta$. Here that

Add a note about $\mathbb{Z}$

Homework Task 14. Complete and write up the ~~g~~ story.

13.6: Find a w-style Characterization

DO

Homework Task 15. Understand Chern-Simons theory with gauge group $\mathfrak{g}_\epsilon$.

Is there a gauge that leads to the formula $\mathcal{F}$ of Task 7?

Homework Task 16. What happens to representation theory as $\epsilon \rightarrow 0$? Is there any fun in continuous morphisms $\mathfrak{g}_\epsilon \rightarrow \mathfrak{gl}_{n,\epsilon}^+$?

Homework Task 17. Does Θ extend to knots in $\mathbb{ZHS}/\mathbb{QHS}$? $\mathbb{Z}$ and $\mathbb{Z}^{(2)}$ do.

Homework Task 18. Is there a surgery formula for Θ ? $\mathbb{Z}$ and $\mathbb{Z}^{(2)}$ have.

Homework Task 19. Extend Θ to tangles and figure out how it behaves under strand doubling.

$\mathbb{Z}$ and $\mathbb{Z}^{(2)}$ extend but their extensions depend on parenthesizations. From Task 14 we expect that Θ will extend without the need for parenthesizations, yet with an asymmetry built into the doubling operations.

Homework Task 20. Find a ribbon condition satisfied by Θ .

For a ribbon knot K , one may find a Seifert surface Σ half of whose homology is generated by the components of an unlink embedded in Σ . This makes for a presentation matrix A of the Alexander module of K that has big blocks of zeros, and this leads to the Fox-Milnor condition [FM], $\Delta \doteq \det(A) \doteq f(T)f(T^{-1})$ for some $f \in \mathbb{Z}[T^{\pm 1}]$. If $\det A$ is constrained for ribbon knots, perhaps so is A^{-1} and therefore Θ ?

Homework Task 21. Carthago delenda est and every knot polynomial must be categorized.

M. Khovanov & Cato the Elder



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*Add: Find the multi-variant Θ for 0. nics.

A FAST, STRONG, TOPOLOGICALLY MEANINGFUL, AND FUN KNOT INVARIANT

DROR BAR-NATAN AND ROLAND VAN DER VEEN

ABSTRACT. In this paper we discuss a pair of polynomial knot invariants $\Theta \sim (\Delta, \theta)$ which are:

- Theoretically and practically fast; Θ can be computed in polynomial time. We can compute it in full on random knots with over 300 crossings, and its evaluation at simple rational numbers on random knots with over 600 crossings.
- Strong. Its separation power is much greater than the hyperbolic volume, the HOMFLY-PT polynomial and Khovanov homology (taken together) on knots with up to 15 crossings while being computable on much larger knots.
- Topologically meaningful. It likely gives a gauge bound, and there are reasons to hope that it would be more.
- Fun. See also Figures 1.1–1.4, 3.1, and 6.2.

Δ is merely the Alexander polynomial. θ is almost certainly equal to an invariant that was studied extensively by Ohtsuki [Oht84], containing Rozansky, Kricker, and Garoufalidis [Roz01, Roz02, Roz03, Kr03, GH04]. Yet our formulas, proofs, and programs are much simpler and enable the computation even on very large knots.

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7. Acknowledgements
- References

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2020 Mathematics Subject Classification. Primary 57K14, secondary 16T99.
Key words and phrases. Alexander polynomial, loop expansion, solvable approximation, knot genus, filtered knots, ribbon knots, polynomial time computations, Feynman diagrams, perturbed Gaussian integration, Seifert surfaces.

This paper is available in electronic form, along with source files and a demo Mathematica notebook at [http://oeß.net/Theta](#) and at [arXiv:2509.18456](#).

1. FUN

The word “fun” rarely appears in the title of a math paper, so let us start with a brief justification.

Θ is a pair of polynomials. The first, Δ , is old news, the Alexander polynomial [Al]. It is a one-variable Laurent polynomial in a variable T . For example, $\Delta(\bigcirc) = T^{-1} - 1 + T$. We turn such a polynomial into a list of coefficients (for $\bigcirc$, it is $(1, -1, 1)$), and then to a chain of bars of varying colours: white for the zero coefficients, and red and blue for the positive and negative coefficients (with intensity proportional to the magnitude of the coefficients). The result is a “bar code”, and for the trefoil it is

Similarly, θ is a 2-variable Laurent polynomial, in variables T_1 and T_2 . We can turn such a polynomial into a 2D array of coefficients and then using the same rules, into a 2D array of colours, namely, into a picture. To highlight a certain conjectured hexagonal symmetry of the resulting pictures, we apply a shear transformation to the plane before printing. So is monomial $cT_1^{n_1}T_2^{n_2}$ gets printed at position $(n_1 - n_2/2, \sqrt{3}n_2/2)$ instead of the more straightforward (n_1, n_2) . On the right is the 2D picture corresponding to the polynomial $2 + T_1 - T_2T_2 + T_2 - T_1^{-1} + T_1T_2T_2 - T_2^{-1}$.

Thus Θ becomes a pair of pictures: a bar code, and a 2D picture that we call a “hexagonal QR code”. For the knots in the Rolfsen table (with the unknot prepended at the start), they are in Figure 1.1. For some alternating square wave knots, they are in Figure 1.2, and for a random square wave, in Figure 1.3. In addition, the hexagonal QR codes of 15 knots with ≥ 300 crossings are in Figure 1.4, and Θ of a 132-crossing torus knot is in Figure 3.1. Some further computations and figures, also highlighting the parity of coefficients rather than just their signs, are at [Al].

Clearly there are patterns in these figures. There is a hexagonal symmetry and the QR codes are nearly always hexagonal (these are independent properties). Much more can be seen in Figure 1.1. In Figure 1.4 there seem to be large-scale patterns perhaps reminiscent of the “Chladni figures” formed by powders atop vibrating plates (on right). We can’t prove any of these things, and the last one, we can’t even formulate properly. Yet they are clearly there, too clear to be the result of chance alone. We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.

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FIGURE 1.1. Θ as a bar code and a QR code, for all the knots in the Rolfsen table.

Two into a question?

A (2, 41, -41) pretzel for dessert

