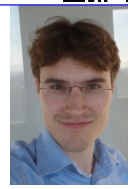
# Homework

OMG, thanks! ωεβ:=http://drorbn.net/ld26

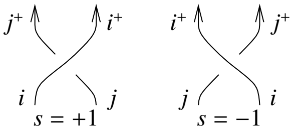
**Abstract.** I'll start with a review of my recent paper with van der Veen, "A Fast, Strong, Topologically Meaningful, and Fun Knot Invariant" [BV], and then assign some homework. Much of what I'll say follows earlier work by Rozansky, Kricker, Garoufalidis, and Ohtsuki [Ro1, Ro2, Ro3, Kr, GR, Oh].



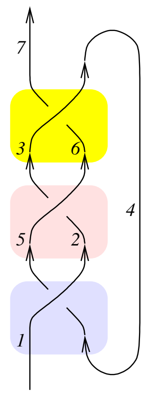
van der Veen

**Acknowledgement.** This work was supported by NSERC grants RGPIN-2018-04350 and RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

*A.* Let $T$, $T_1$, and $T_2$ be indeteminates, and let $T_3 = T_1T_2$. Start from a presentation matrix $A$ for the Alexander module of $K$, coming from the Wirtinger presentation of $\pi_1(K)$: $A := I_{2n+1} + \sum_c A_c$, where




$$\rightarrow \quad \begin{array}{c|cc} A_c & i+1 & j+1 \\ \hline i & -T^s & T^s-1 \\ j & 0 & -1 \end{array}$$




$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Delta \doteq \det(A)$$

*G.* Let $G = (g_{\alpha\beta}) := A^{-1}$:

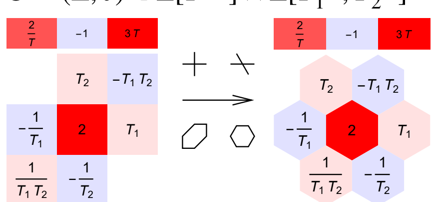
ωεβ/pi

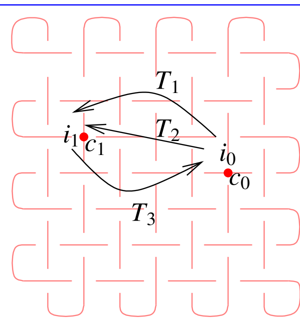
$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $G_\nu = (g_{\nu\alpha\beta})$ be $G$ with $T \to T_\nu$, for $\nu = 1, 2, 3$.

$$\theta \sim \Delta_1\Delta_2\Delta_3 \sum_{c_0,c_1} g_{1i_0i_1}g_{2i_0i_1}g_{3i_1i_0} + \text{l.o.}$$

$$\Theta = (\Delta, \theta) \in \mathbb{Z}[T^{\pm 1}] \times \mathbb{Z}[T_1^{\pm 1}, T_2^{\pm 1}]$$




## Fast.

```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
T3 = T1 T2;

F1[{s_, i_, j_}] := CF[
  s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj - (T2^s - 1) g2ji g3ii + 2 g2jj g3ii -
     (1 - T3^s) g2ji g3ji - g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
     ((T1^s - 1) g1ji (T2^(2s) g2ji - T2^s g2jj + T2^s g3jj) +
        (T3^s - 1) g3ji (1 - T2^s g1ii + g2ij + (T2^s - 2) g2jj - (T1^s - 1) (T2^s + 1) g1ji)) /
      (T2^s - 1))]
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
 CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
   ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
F3[φ_, k_] = φ g3kk - φ/2;

Θ[K_] := Θ[K] = Module[{X, φ, n, A, Δ, G, ev, θ, k, k1, k2},
   {X, φ} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
   Cases[X, {s_, i_, j_} :→ (A⟦{i, j}, {i + 1, j + 1}⟧ += ({{-T^s, T^s - 1}, {0, -1}}))];
   Δ = T^((-Total[φ] - Total[X⟦All,1⟧])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] := Factor[ℰ /. g_ν_,α_,β_ :→ (G⟦α, β⟧ /. T → T_ν)];
   θ = ev[Sum[F1[X⟦k⟧], {k, n}]];
   θ += ev[Sum[F2[X⟦k1⟧, X⟦k2⟧], {k1, n}, {k2, n}]];
   θ += ev[Sum[F3[φ⟦k⟧, k], {k, Length@φ}]];
   Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}
  ];
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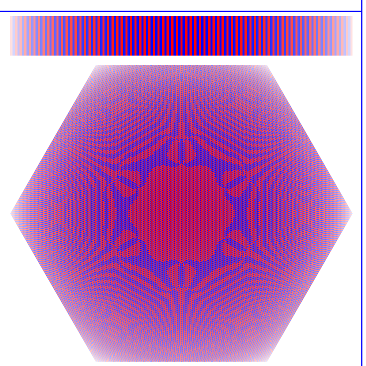
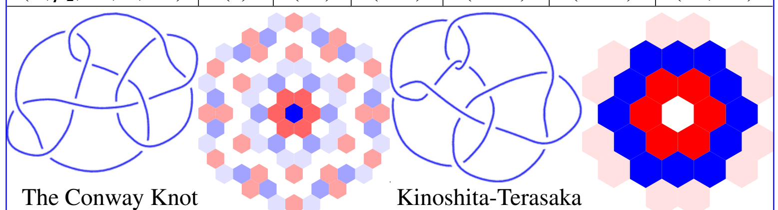


A random 300 xing knot from [DHOEBL]. For most invariants, 300 is science fiction.

**Strong.** Θ vs. a slew of other reasonably-computable invariants (deficits shown):

| $n$ | ≤ 10 | ≤ 11 | ≤ 12 | ≤ 13 | ≤ 14 | ≤ 15 |
|---|---|---|---|---|---|---|
| knots | 249 | 801 | 2,977 | 12,965 | 59,937 | 313,230 |
| $\Delta$ | (38) | (250) | (1,204) | (7,326) | (39,741) | (236,326) |
| $\sigma_{LT}$ | (108) | (356) | (1,525) | (7,736) | (40,101) | (230,592) |
| $J$ | (7) | (70) | (482) | (3,434) | (21,250) | (138,591) |
| $Kh$ | (6) | (65) | (452) | (3,226) | (19,754) | (127,261) |
| $H$ | (2) | (31) | (222) | (1,839) | (11,251) | (73,892) |
| $Vol$ | (~6) | (~25) | (~113) | (~1,012) | (~6,353) | (~43,607) |
| $(Kh, H, Vol)$ | (~0) | (~14) | (~84) | (~911) | (~5,917) | (~41,434) |
| $(\Delta, \rho_1)$ | (0) | (14) | (95) | (959) | (6,253) | (42,914) |
| $(\Delta, \rho_1, \rho_2)$ | (0) | (14) | (84) | (911) | (5,926) | (41,469) |
| $(\rho_1, \rho_2, Kh, H, Vol)$ | (0) | (~14) | (~84) | (~911) | (~5,916) | (~41,432) |
| $\Theta$ | (0) | (3) | (19) | (194) | (1,118) | (6,758) |
| $(\Theta, \rho_2)$ | (0) | (3) | (10) | (169) | (982) | (6,341) |
| $(\Theta, \sigma_{LT})$ | (0) | (3) | (19) | (194) | (1,118) | (6,758) |
| $(\Theta, Kh)$ | (0) | (3) | (18) | (185) | (1,062) | (6,555) |
| $(\Theta, H)$ | (0) | (3) | (18) | (185) | (1,064) | (6,563) |
| $(\Theta, Vol)$ | (0) | (~3) | (~10) | (~169) | (~973) | (~6,308) |
| $(\Theta, \rho_2, Kh, H, Vol)$ | (0) | (~3) | (~10) | (~169) | (~972) | (~6,304) |



The Conway Knot      Kinoshita-Terasaka

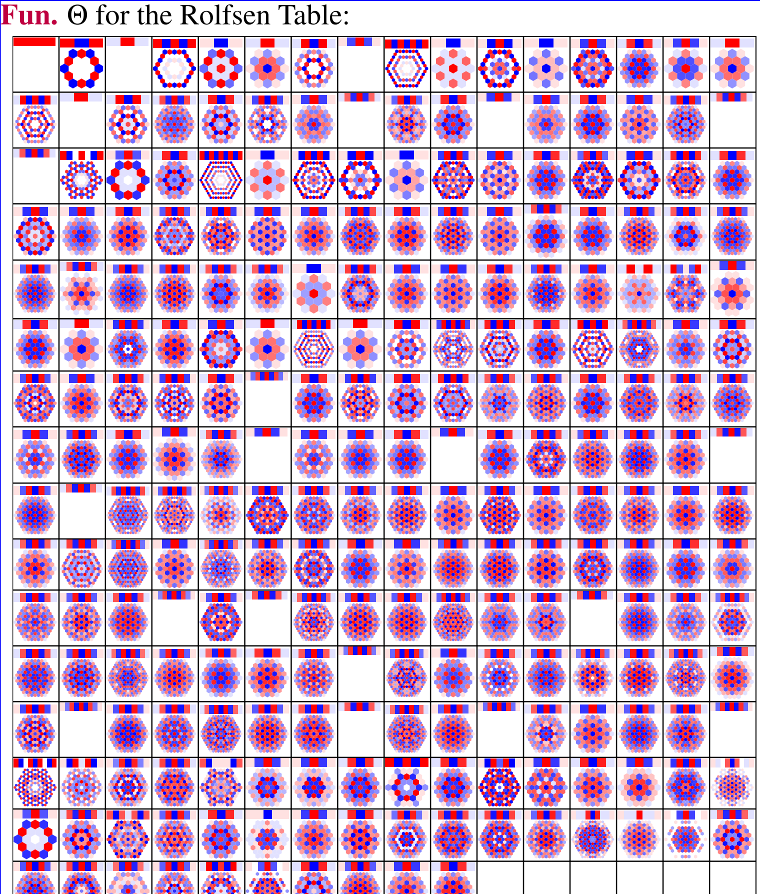
**Topologically Meaningful.** $\theta$ is near $\Delta$ and we dream that anything $\Delta$ can do, $\theta$ does too (sometimes better). The following are verified for knots with ≤ 13 crossings:

**Conjecture 1.** $\deg_{T_1} \theta(K) \leq 2g(K)$.

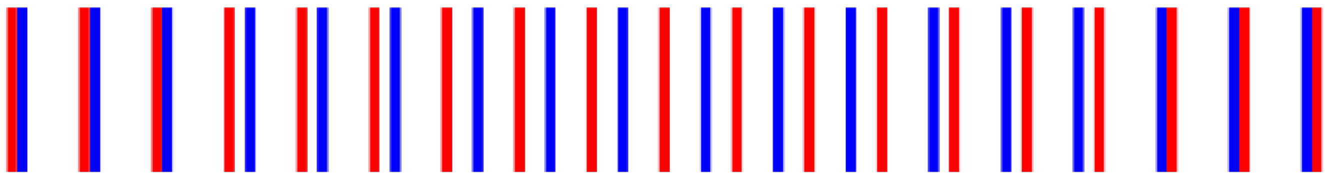
**Conjecture 2.** If $K$ is a fibered knot and $d$ is the degree of $\Delta(K)$ (the highest power of $T$), then the coefficient of $T_2^{2d}$ in $\theta(K)$, which is a polynomial in $T_1$, is an integer multiple of $T_1^d\Delta(K)|_{T\to T_1}$.

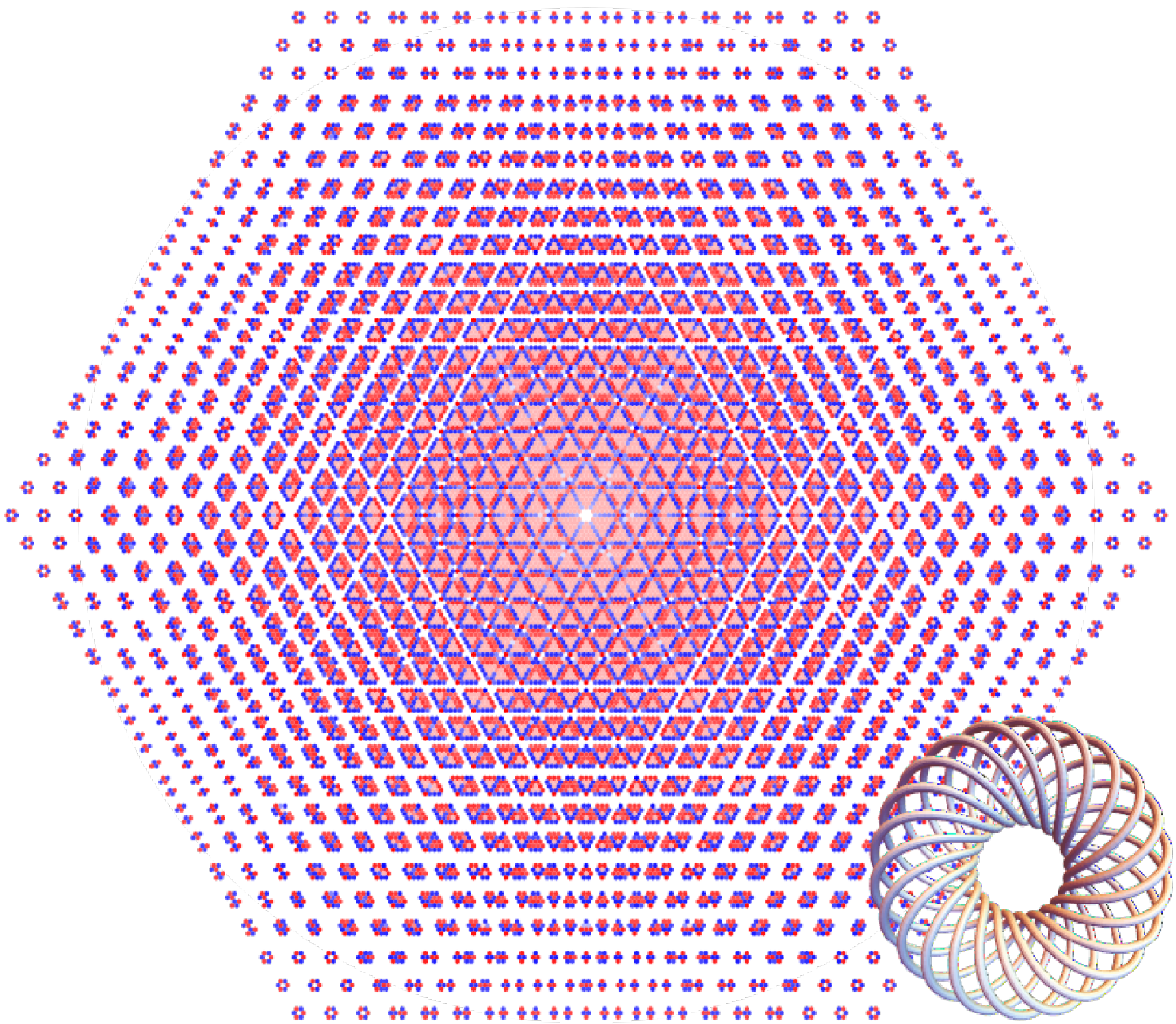
**Dream.** $\theta$ has something to say about ribbon knots.

**Fun.** Θ for the Rolfsen Table:

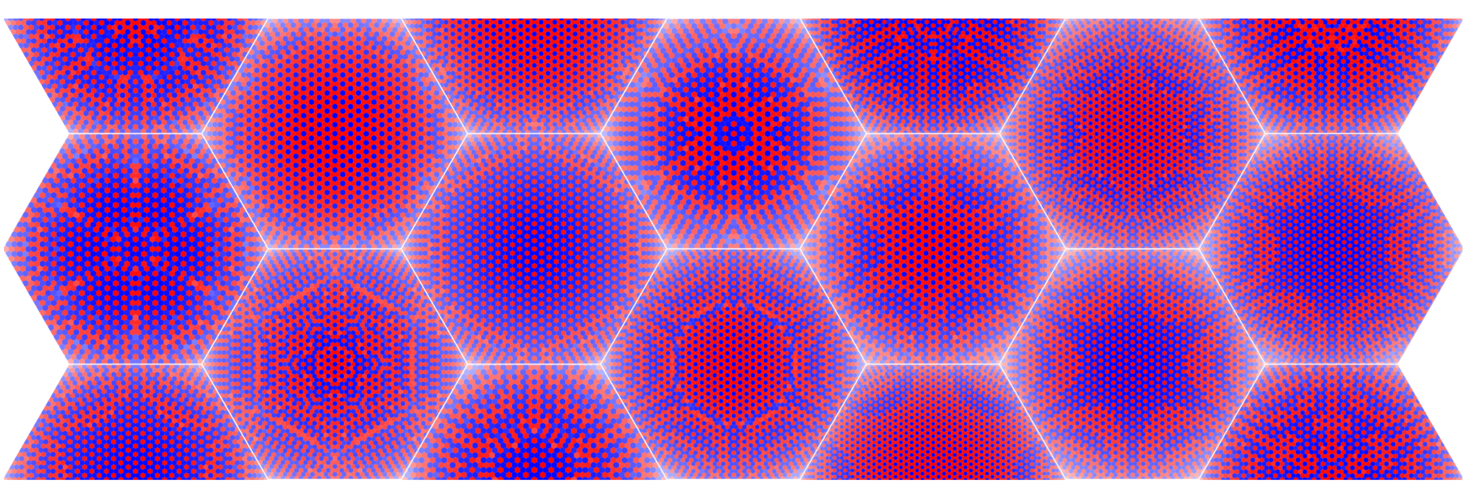


The 132-crossing torus knot $T_{22/7}$: (many more at ωεβ/TK)





Random knots from [DHOEBL], with 51 – 68 crossings: (many more at ωεβ/DK)



**Homework Task 1.** *Prove the hexagonal symmetry of $\theta(K)$, and that $\theta(K) = \theta(-K) = -\theta(\bar{K})$.*

That's harder than it seems! The formulas don't naively show any of that. $\Delta$ has a palindromic symmetry first conjectured in Alexander's original paper [Al] — it is invariant under $T \to T^{-1}$. Proving this took a few years, and the proof starting from the Wirtinger presentation is quite involved (e.g. [CF, Chapter IX]).

**Homework Task 2.** *Explain the "Chladni patterns". Are there "dominant parts" of $\theta$ that can be computed in isolation?*

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**Homework Task 3.** *Prove the genus bound of Conjecture 1.*

**Homework Task 4.** *Find a formula $\mathcal{F}$ for $\Theta(K)$ that starts from a Seifert surface $\Sigma$ of $K$. Better if $\mathcal{F}$ is completely 3D! Assuming Task 9, it is known that $\Theta$ depends only of invariants of type $\leq 3$ of $\Sigma$. Maybe $\mathcal{F}$ is about configuration space integrals / chopstick towers?*

**Homework Task 5.** *Is there an intrinsic theory of finite type invariants for Seifert surfaces? Does its gr map to functions on $H_1$?*

**Homework Task 6.** *Prove the the fibered condition of Conjecture 2.*

**Homework Task 7.** *In general, find a formula for $\Theta$ corresponding to each known formula for the Alexander polynomial.*

**Homework Task 8.** *Write up the integration story.*

**Homework Task 9.** *Prove that $\Theta$ is equal to the two-loop contribution to the Kontsevich integral.*

**Homework Task 10.** *Complete and write up the $\mathfrak{g}_\epsilon$ story.*

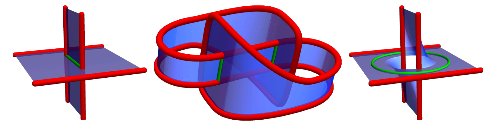
**Homework Task 11.** *Understand Chern-Simons theory with gauge group $\mathfrak{g}_\epsilon$.*

**Homework Task 12.** *What happens to representation theory as $\epsilon \to 0$? Is there any fun in continuous morphisms $\mathfrak{g}_\epsilon \to gl^+_{n,\epsilon}$?*

**Homework Task 13.** *Does $\Theta$ extend to knots in $\mathbb{Z}HS/\mathbb{Q}HS$?*

**Homework Task 14.** *Is there a surgery formula for $\Theta$?*

**Homework Task 15.** *Find a ribbon condition satisfied by $\Theta$.*



For a ribbon knot $K$, one may find a Seifert surface $\Sigma$ half of whose homology is generated by the components of an unlink embedded in $\Sigma$. This makes for a presentation matrix $A$ of the Alexander module of $K$ that has big blocks of zeros, and this leads to the Fox-Milnor condition [FM], $\Delta \doteq \det(A) \doteq f(T)f(T^{-1})$ for some $f \in \mathbb{Z}[T^{\pm 1}]$. If $\det A$ is constrained for ribbon knots, perhaps so is $A^{-1}$ and therefore $\Theta$?

**Homework Task 16.** *Carthago delenda est? Must $\Theta$ be categorified?*

## References.


[Al] J. W. Alexander, *Topological invariants of knots and link,* Trans. Amer. Math. Soc. **30** (1928) 275–306.

[BV] D. Bar-Natan and R. van der Veen, *A Fast, Strong, Topologically Meaningful, and Fun Knot Invariant,* ωεβ/Theta and arXiv:2509.18456.

[CF] R. H. Crowell and R. H. Fox, *Introduction to Knot Theory,* Springer-Verlag GTM **57** (1963).

[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharya, D. Lei, and others, *Random Knots: A Preliminary Report,* lecture notes at ωεβ/DHOEBL. Also a data file at ωεβ/DD.

[FM] R. H. Fox and J. W. Milnor, *Singularities of 2-Spheres in 4-Space and Cobordism of Knots,* Osaka J. Math. **3-2** (1966) 257–267.

[GR] S. Garoufalidis and L. Rozansky, *The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[Kr] A. Kricker, *The Lines of the Kontsevich Integral and Rozansky's Rationality Conjecture,* arXiv:math/0005284.

[Oh] T. Ohtsuki, *On the 2-Loop Polynomial of Knots,* Geometry & Topology **11** (2007) 1357–1475.

[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.


A $(2, 41, -41)$ pretzel for dessert: