Dror Bar-Natan: Talks: Les Diablerets-1708:

Pensieve header: Formulas for “The Dogma is Wrong” talk, Les Diablerets, August 31, 2017.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\LesDiablerets-1708"];
```

Efficient PBW-like Computations

Goal.

In $\mathfrak{g}^\epsilon = \langle t, y, a, x\rangle/([a,x]=x,\ [a,y]=-y,\ [x,y]=t-2\epsilon a,\ [t,*]=0)$, compute $\mathbb{O}\left(y_1^3 a_1^2 x_1^2 y_2^2 a_2^2 x_2 \;\middle|\; y_1 a_1 x_1 y_2 a_2 x_2 \to y a x\right)$.

Representing $\mathfrak{g}^\epsilon$.

```
ME = MatrixExp;
Simp[sol_] :=
  Flatten[sol] /. C[_] → 0 /. (var_ → val_) :> (var → Simplify[PowerExpand@val]);
```

2DRep

$$\rho t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\quad \rho y = \begin{pmatrix} 0 & 0 \\ -\epsilon & 0 \end{pmatrix};\quad \rho a = \begin{pmatrix} (1+1/\epsilon)/2 & 0 \\ 0 & -(1-1/\epsilon)/2 \end{pmatrix};\quad \rho x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

```
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy, ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}
```

2DRep

```
{True, True, True}
```

```
eqn = (ME[ξ1 ρx].ME[η1 ρy] == ME[η0 ρy].ME[λ0 (ρt - 2 ϵ ρa)].ME[ξ0 ρx]);
sol = Simp@Solve[Thread[Flatten /@ eqn], {λ0, η0, ξ0}]
```

$$\left\{\eta 0 \to \frac{\eta_1}{1-\epsilon\,\eta_1\,\xi_1},\ \xi 0 \to \frac{\xi_1}{1-\epsilon\,\eta_1\,\xi_1},\ \lambda 0 \to -\frac{\text{Log}[1-\epsilon\,\eta_1\,\xi_1]}{\epsilon}\right\}$$

The yaxyax formula.

Via $t_1\,y_1\,a_1[x_1\,t_2\,y_2]_{\to 3}\,a_2\,x_2 \to t_1\,t_2\,t_3\,y_1[a_1\,y_3]_{\to 4}\,a_3[x_3\,a_2]_{\to 5}\,x_2 \to t_1\,t_2\,t_3\,y_1\,y_4\,a_4\,a_3\,a_5\,x_5\,x_2 \to t_0\,y_0\,a_0\,x_0$:

```
eqn3 = (ME[ξ1 ρx].ME[η2 ρy] == ME[η0 ρy].ME[λ0 (ρt - 2 ϵ ρa)].ME[ξ0 ρx]);
sol3 = Echo@Simp@Solve[Thread[Flatten /@ eqn3], {λ0, η0, ξ0}];
rule3 = Simp[{τ3 → λ0, η3 → η0, α3 → -2 ϵ λ0, ξ3 → ξ0} /. sol3]
```

»

$$\left\{\eta 0 \to \frac{\eta_2}{1-\epsilon\,\eta_2\,\xi_1},\ \xi 0 \to \frac{\xi_1}{1-\epsilon\,\eta_2\,\xi_1},\ \lambda 0 \to -\frac{\text{Log}[1-\epsilon\,\eta_2\,\xi_1]}{\epsilon}\right\}$$

$$\left\{\tau_3 \to -\frac{\text{Log}[1-\epsilon\,\eta_2\,\xi_1]}{\epsilon},\ \eta_3 \to \frac{\eta_2}{1-\epsilon\,\eta_2\,\xi_1},\ \alpha_3 \to 2\,\text{Log}[1-\epsilon\,\eta_2\,\xi_1],\ \xi_3 \to \frac{\xi_1}{1-\epsilon\,\eta_2\,\xi_1}\right\}$$

```
eqn4 = (ME[α1 ρa].ME[η3 ρy] == ME[η0 ρy].ME[α0 ρa]);
sol4 = Echo@Simp@Solve[Thread[Flatten /@ eqn4], {α0, η0}];
rule4 = Simp[{η4 → η0, α4 → α0} /. sol4]
```

″ $\{\}$

$\{\eta_4 \to \eta 0,\ \alpha_4 \to \alpha 0\}$

```
MatrixForm /@ eqn4
```

$$\begin{pmatrix} e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} & 0 \\ -e^{-\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} \in \eta_3 & e^{-\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} \end{pmatrix} == \begin{pmatrix} e^{\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}} & 0 \\ -e^{\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}} \in \eta 0 & e^{-\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}} \end{pmatrix}$$

```
Thread[Flatten /@ eqn4]
```

$$\left\{e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} == e^{\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}},\ \text{True},\ -e^{-\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} \in \eta_3 == -e^{\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}} \in \eta 0,\ e^{-\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} == e^{-\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}}\right\}$$

```
Simp@Solve[Thread[Flatten /@ eqn4][[1]], {α0}]
```

$\{\alpha 0 \to \alpha_1\}$

```
Solve[Thread[Flatten /@ eqn4] /. Simp@Solve[Thread[Flatten /@ eqn4][[1]], {α0}], η0]
```

$\left\{\left\{\eta 0 \to e^{-\alpha_1}\ \eta_3\right\}\right\}$

```
rule4 = Simp[{η4 → η0, α4 → α0} /. {α0 → α1, η0 → e^-α1 η3}]
```

$\left\{\eta_4 \to e^{-\alpha_1}\ \eta_3,\ \alpha_4 \to \alpha_1\right\}$

```
eqn5 = (ME[ξ3 ρx].ME[α2 ρa] == ME[α0 ρa].ME[ξ0 ρx]);
sol5 = Echo@Simp@Solve[Thread[Flatten /@ eqn5], {α0, η0}];
rule5 = Simp[{η5 → η0, α5 → α0} /. sol5]
```

″ $\{\}$

$\{\eta_5 \to \eta 0,\ \alpha_5 \to \alpha 0\}$

```
MatrixForm /@ eqn5
```

$$\begin{pmatrix} e^{\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}} & e^{-\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\ \xi_3 \\ 0 & e^{-\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}} \end{pmatrix} == \begin{pmatrix} e^{\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}} & e^{\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}}\ \xi 0 \\ 0 & e^{-\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}} \end{pmatrix}$$

```
Thread[Flatten /@ eqn5]
```

$$\left\{e^{\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}} == e^{\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}},\ e^{-\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\ \xi_3 == e^{\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}}\ \xi 0,\ \text{True},\ e^{-\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}} == e^{-\frac{\alpha 0}{2}+\frac{\alpha 0}{2\epsilon}}\right\}$$

```
Simp@Solve[Thread[Flatten /@ eqn5][[1]], {α0}]
```

$\{\alpha 0 \to \alpha_2\}$

```
Thread[Flatten /@ eqn5] /. Simp@Solve[Thread[Flatten /@ eqn5][[1]], α0]
```

$\left\{\text{True},\ e^{-\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\ \xi_3 == e^{\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\ \xi 0,\ \text{True},\ \text{True}\right\}$

```
Solve[Thread[Flatten /@ eqn5] /. Simp@Solve[Thread[Flatten /@ eqn5][[1]], {α0}], ξ0]
```

$\left\{\left\{\xi 0 \to e^{-\alpha_2}\ \xi_3\right\}\right\}$

```
rule5 = Simp[{ξ5 → ξ0, α5 → α0} /. {α0 → α2, ξ0 → e^-α2 ξ3}]
```

$$\{\xi_5 \to e^{-\alpha_2}\,\xi_3,\ \alpha_5 \to \alpha_2\}$$

```
yaxyaxrule =
 Simp[{τ0 → τ1 + τ2 + τ3, η0 → η1 + η4, α0 → α4 + α3 + α5, ξ0 → ξ5 + ξ2} /. rule5 /. rule4 /. rule3]
```

$$\left\{\tau_0 \to -\frac{\text{Log}[1-\epsilon\,\eta_2\,\xi_1]}{\epsilon} + \tau_1 + \tau_2,\ \eta_0 \to \eta_1 + \frac{e^{-\alpha_1}\,\eta_2}{1-\epsilon\,\eta_2\,\xi_1},\right.$$
$$\left.\alpha_0 \to 2\,\text{Log}[1-\epsilon\,\eta_2\,\xi_1] + \alpha_1 + \alpha_2,\ \xi_0 \to \frac{e^{-\alpha_2}\,\xi_1}{1-\epsilon\,\eta_2\,\xi_1} + \xi_2\right\}$$

```
Series[{τ0, η0, α0, ξ0} /. yaxyaxrule, {ϵ, 0, 1}]
```

$$\left\{(\eta_2\,\xi_1 + \tau_1 + \tau_2) + \frac{1}{2}\,\eta_2^2\,\xi_1^2\,\epsilon + O[\epsilon]^2,\ \left(\eta_1 + e^{-\alpha_1}\,\eta_2\right) + e^{-\alpha_1}\,\eta_2^2\,\xi_1\,\epsilon + O[\epsilon]^2,\right.$$
$$\left.(\alpha_1 + \alpha_2) - 2\,(\eta_2\,\xi_1)\,\epsilon + O[\epsilon]^2,\ \left(e^{-\alpha_2}\,\xi_1 + \xi_2\right) + e^{-\alpha_2}\,\eta_2\,\xi_1^2\,\epsilon + O[\epsilon]^2\right\}$$

yaxyax

```
ME = MatrixExp;
Simplify[
 ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
   ME[τ0 ρt].ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx] /. {τ0 → -Log[1 - ϵ η2 ξ1]/ϵ + τ1 + τ2,
   η0 → η1 + e^-α1 η2/(1 - ϵ η2 ξ1), α0 → 2 Log[1 - ϵ η2 ξ1] + α1 + α2, ξ0 → e^-α2 ξ1/(1 - ϵ η2 ξ1) + ξ2}]
```

yaxyax

```
True
```

```
D[e^τ0 t+η0 y+α0 a+ξ0 x /. yaxyaxrule, {η1, 3}] /. (τ | η | α | ξ)1|2 → 0
```

$$y^3$$

```
D[e^τ0 t+η0 y+α0 a+ξ0 x /. yaxyaxrule, {η1, 3}, {α1, 2}, {ξ1, 2}] /. (τ | η | α | ξ)1|2 → 0
```

$$a^2\,x^2\,y^3$$

```
Expand[
 D[e^τ0 t+η0 y+α0 a+ξ0 x /. yaxyaxrule, {η1, 3}, {α1, 2}, {ξ1, 2}, {η2, 4}] /. (τ | η | α | ξ)1|2 → 0]
```

$$48\,t^2\,y^5 - 48\,a\,t^2\,y^5 + 12\,a^2\,t^2\,y^5 + 72\,t\,x\,y^6 - 48\,a\,t\,x\,y^6 + 8\,a^2\,t\,x\,y^6 + 16\,x^2\,y^7 - 8\,a\,x^2\,y^7 +$$
$$a^2\,x^2\,y^7 + 240\,t\,y^5\,\epsilon - 432\,a\,t\,y^5\,\epsilon + 252\,a^2\,t\,y^5\,\epsilon - 48\,a^3\,t\,y^5\,\epsilon + 288\,x\,y^6\,\epsilon - 336\,a\,x\,y^6\,\epsilon +$$
$$128\,a^2\,x\,y^6\,\epsilon - 16\,a^3\,x\,y^6\,\epsilon + 288\,y^5\,\epsilon^2 - 768\,a\,y^5\,\epsilon^2 + 744\,a^2\,y^5\,\epsilon^2 - 312\,a^3\,y^5\,\epsilon^2 + 48\,a^4\,y^5\,\epsilon^2$$

```
Expand[D[e^τ0 t+η0 y+α0 a+ξ0 x /. yaxyaxrule, {η1, 3},
   {α1, 2}, {ξ1, 2}, {η2, 2}, {α2, 2}, {ξ2, 1}] /. (τ | η | α | ξ)1|2 → 0]
```

$$2\,a^4\,t^2\,x\,y^3 + 4\,t\,x^2\,y^4 - 16\,a\,t\,x^2\,y^4 + 24\,a^2\,t\,x^2\,y^4 - 16\,a^3\,t\,x^2\,y^4 + 4\,a^4\,t\,x^2\,y^4 + 16\,x^3\,y^5 -$$
$$32\,a\,x^3\,y^5 + 24\,a^2\,x^3\,y^5 - 8\,a^3\,x^3\,y^5 + a^4\,x^3\,y^5 + 2\,a^4\,t\,x\,y^3\,\epsilon - 8\,a^5\,t\,x\,y^3\,\epsilon + 8\,x^2\,y^4\,\epsilon -$$
$$40\,a\,x^2\,y^4\,\epsilon + 80\,a^2\,x^2\,y^4\,\epsilon - 80\,a^3\,x^2\,y^4\,\epsilon + 40\,a^4\,x^2\,y^4\,\epsilon - 8\,a^5\,x^2\,y^4\,\epsilon - 4\,a^5\,x\,y^3\,\epsilon^2 + 8\,a^6\,x\,y^3\,\epsilon^2$$

```
Expand[
 ∂{η1,3} ∂{α1,2} ∂{ξ1,2} ∂{η2,2} ∂{α2,2} ∂{ξ2,1} (e^(τ0 t+η0 y+α0 a+ξ0 x) /. yaxyaxrule) /. (τ | η | α | ξ)1|2 → 0]
```

$$2\,a^4\,t^2\,x\,y^3 + 4\,t\,x^2\,y^4 - 16\,a\,t\,x^2\,y^4 + 24\,a^2\,t\,x^2\,y^4 - 16\,a^3\,t\,x^2\,y^4 + 4\,a^4\,t\,x^2\,y^4 + 16\,x^3\,y^5 - 32\,a\,x^3\,y^5 + 24\,a^2\,x^3\,y^5 - 8\,a^3\,x^3\,y^5 + a^4\,x^3\,y^5 + 2\,a^4\,t\,x\,y^3\,\epsilon - 8\,a^5\,t\,x\,y^3\,\epsilon + 8\,x^2\,y^4\,\epsilon - 40\,a\,x^2\,y^4\,\epsilon + 80\,a^2\,x^2\,y^4\,\epsilon - 80\,a^3\,x^2\,y^4\,\epsilon + 40\,a^4\,x^2\,y^4\,\epsilon - 8\,a^5\,x^2\,y^4\,\epsilon - 4\,a^5\,x\,y^3\,\epsilon^2 + 8\,a^6\,x\,y^3\,\epsilon^2$$

322221

```
Expand[∂{η1,3} ∂{α1,2} ∂{ξ1,2} ∂{η2,2} ∂{α2,2} ∂{ξ2,1} Exp[
          (-(Log[1 - ϵ η2 ξ1]/ϵ) + τ1 + τ2) t +
           (η1 + e^(-α1) η2/(1 - ϵ η2 ξ1)) y + (2 Log[1 - ϵ η2 ξ1] + α1 + α2) a + (e^(-α2) ξ1/(1 - ϵ η2 ξ1) + ξ2) x
         ] /. (τ | η | α | ξ)1|2 →
   0]
```

322221

$$2\,a^4\,t^2\,x\,y^3 + 4\,t\,x^2\,y^4 - 16\,a\,t\,x^2\,y^4 + 24\,a^2\,t\,x^2\,y^4 - 16\,a^3\,t\,x^2\,y^4 + 4\,a^4\,t\,x^2\,y^4 + 16\,x^3\,y^5 - 32\,a\,x^3\,y^5 + 24\,a^2\,x^3\,y^5 - 8\,a^3\,x^3\,y^5 + a^4\,x^3\,y^5 + 2\,a^4\,t\,x\,y^3\,\epsilon - 8\,a^5\,t\,x\,y^3\,\epsilon + 8\,x^2\,y^4\,\epsilon - 40\,a\,x^2\,y^4\,\epsilon + 80\,a^2\,x^2\,y^4\,\epsilon - 80\,a^3\,x^2\,y^4\,\epsilon + 40\,a^4\,x^2\,y^4\,\epsilon - 8\,a^5\,x^2\,y^4\,\epsilon - 4\,a^5\,x\,y^3\,\epsilon^2 + 8\,a^6\,x\,y^3\,\epsilon^2$$

Exporting the above as PDF files

The below is adapted from pensieve://Talks/NCSU-1604/GaussGassnerDemo.nb.

```
ConditionalExport[fname_String, rest___] := Module[{temp, exists},
  temp = "ConditionalExportTemporary" <> "." <> FileExtension[fname];
  exists = FileExistsQ[fname];
  Export[temp, rest];
  If[exists && FileByteCount[fname] === FileByteCount[temp],
   DeleteFile[temp],
   (* else *) Print["Exporting " <> fname <> "..."];
   If[exists, DeleteFile[fname]];
   RenameFile[temp, fname]
  ];
  fname
 ]

SetOptions[$FrontEndSession, PrintingStyleEnvironment → "Working"];
TagProperties[_] := {};
TagProperties["131"] = {PageWidth → 3.2/0.66};
Options[CellExport] = {
    PageWidth → 4/0.66, CellFilter → Identity,
    ExportDirectory → "Snips", ExportBaseFilename → Automatic,
    ExportFormat → ".pdf", ExportOptions → {}, Split → False
  };
CellExport[tag_String, opts___Rule] := CellExport[
    NotebookGet[EvaluationNotebook[]],
    tag, opts
  ];
CellExport[nb_Notebook, tag_String] := CellExport[nb, tag, TagProperties[tag]];
```

```
CellExport[nb_Notebook, tag_String, OptionsPattern[]] := Module[
   {cells, cell, filename, format},
   filename = FileNameJoin[{
      OptionValue[ExportDirectory] /. Automatic → Directory[],
      OptionValue[ExportBaseFilename] /. Automatic → tag
     }];
   format = OptionValue[ExportFormat];
   cells = OptionValue[CellFilter][Cases[
      nb, c_Cell /; FreeQ[List @@ c, Cell] && ! FreeQ[c, CellTags → tag],
      Infinity
     ]];
   If[! OptionValue[Split],
    If[Length[cells] ≥ 1,
     If[Length[cells] == 1,
      cells = Join[First[cells], Cell[
         PageWidth → 1.2 × 72 OptionValue[PageWidth], Background → {White, Opacity[0]}]]],
      cells = Cell[CellGroup[cells], PageWidth → 72 OptionValue[PageWidth]]
     ];
     ConditionalExport[
      filename <> format, cells,
      ImageResolution → 300,
      OptionValue[ExportOptions]
     ]
    ],
    k = 0;
    Table[
     ++k;
     ConditionalExport[
      filename <> "-" <> ToString[k] <> format,
      Append[cell, PageWidth → 72 OptionValue[PageWidth]],
      ImageResolution → 300,
      OptionValue[ExportOptions]
     ],
     {cell, cells}
    ]
   ]
  ];

ExportCells := (
   nb = NotebookGet[EvaluationNotebook[]];
   tags = Cases[nb, (CellTags → tag_String) :↦ tag, Infinity] // Union;
   Print[tags];
   CellExport /@ tags;
   Print["Done."]
  );

ExportCells
```

```
{2DRep, 322221, yaxyax}

Exporting Snips\322221.pdf...

Done.
```