Dror Bar-Natan: Talks: Les Diablerets-1708:

Efficient PBW-like Computations

Goal.

In $\mathfrak{g}^{\epsilon} = \langle t, y, a, x\rangle / ([a, x] = x,\ [a, y] = -y,\ [x, y] = t - 2\epsilon a,\ [t, *] = 0)$, compute

Representing $\mathfrak{g}^{\epsilon}$.

```
ρt = (1 0
      0 1); ρy = ( 0  0
                  -ϵ  0); ρa = ((1 + 1/ϵ)/2        0
                                    0       -(1 - 1/ϵ)/2); ρx = (0 1
                                                                 0 0);
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy, ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}
```

{True, True, True}

```
ME = MatrixExp;
eqn = (ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
    ME[τ ρt].ME[η ρy].ME[α ρa].ME[ξ ρx])
```

$$\Big\{\Big\{e^{\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\Big(e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} - e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}}\epsilon\,\eta_2\,\xi_1\Big),\ e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}-\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\xi_1 + e^{\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\Big(e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} - e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}}\epsilon\,\eta_2\,\xi_1\Big)\xi_2\Big\},$$
$$\Big\{e^{\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\Big(-e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}}\epsilon\,\eta_1 - \epsilon\,\eta_2\Big(e^{-\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} - e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}}\epsilon\,\eta_1\,\xi_1\Big)\Big),$$
$$e^{-\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\Big(e^{-\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} - e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}}\epsilon\,\eta_1\,\xi_1\Big) + e^{\frac{\alpha_2}{2}+\frac{\alpha_2}{2\epsilon}}\Big(-e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}}\epsilon\,\eta_1 - \epsilon\,\eta_2\Big(e^{-\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}} - e^{\frac{\alpha_1}{2}+\frac{\alpha_1}{2\epsilon}}\epsilon\,\eta_1\,\xi_1\Big)\Big)\xi_2\Big\}\Big\} ==$$
$$\Big\{\Big\{e^{\frac{\alpha}{2}+\frac{\alpha}{2\epsilon}+\tau},\ e^{\frac{\alpha}{2}+\frac{\alpha}{2\epsilon}+\tau}\xi\Big\},\ \Big\{-e^{\frac{\alpha}{2}+\frac{\alpha}{2\epsilon}+\tau}\epsilon\,\eta,\ e^{-\frac{\alpha}{2}+\frac{\alpha}{2\epsilon}+\tau} - e^{\frac{\alpha}{2}+\frac{\alpha}{2\epsilon}+\tau}\epsilon\,\eta\,\xi\Big\}\Big\}$$

```
sol = Solve[Thread[Flatten /@ eqn], {τ, η, α, ξ}]〚1〛
```

$Aborted

```
ME = MatrixExp;
eqn = Simplify[ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
    τ0 * ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx]];
MatrixForm /@
 eqn
```

$$\begin{pmatrix} e^{\frac{(\alpha1+\alpha2)\,(1+\epsilon)}{2\epsilon}}\big(1-\epsilon\,\eta2\,\xi1\big) & e^{\frac{\alpha1+\alpha2+\alpha1\,\epsilon-\alpha2\,\epsilon}{2\epsilon}}\big(\xi1+e^{\alpha2}\,\xi2-e^{\alpha2}\,\epsilon\,\eta2\,\xi1\,\xi2\big) \\ e^{\frac{\alpha1+\alpha2-\alpha1\,\epsilon+\alpha2\,\epsilon}{2\epsilon}}\,\epsilon\,\big(-\eta2+e^{\alpha1}\,\eta1\,\big(-1+\epsilon\,\eta2\,\xi1\big)\big) & e^{-\frac{(\alpha1+\alpha2)\,(-1+\epsilon)}{2\epsilon}}\big(1-e^{\alpha1}\,\epsilon\,\eta1\,\xi1-e^{\alpha2}\,\epsilon\,\eta2\,\xi2+e^{\alpha1+\alpha2}\,\epsilon\,\eta1\,(-1+\epsilon \end{pmatrix}$$

```
sol = Solve[Thread[Flatten /@ eqn], {τ0, η0, α0, ξ0}]〚1〛 /.
  (var_ → val_) :> (var → Simplify[PowerExpand@val])
```

$Aborted

```
Solve[Simplify[#[[1, 1]] & /@ eqn], τ0]
```

$$\left\{\left\{\tau 0 \to -e^{-\frac{\alpha 0\,(1+\epsilon)}{2\epsilon}+\frac{(\alpha 1+\alpha 2)\,(1+\epsilon)}{2\epsilon}}\left(-1+\epsilon\,\eta 2\,\xi 1\right)\right\}\right\}$$

```
FullSimplify[-e^(-α0 (1+ϵ)/(2ϵ) + (α1+α2) (1+ϵ)/(2ϵ)) (-1 + ϵ η2 ξ1)]
```

$$e^{-\frac{(\alpha 0-\alpha 1-\alpha 2)\,(1+\epsilon)}{2\epsilon}}\left(1-\epsilon\,\eta 2\,\xi 1\right)$$

```
sol = Solve[Thread[Flatten /@ eqn] /. τ0 → e^(-(α0-α1-α2) (1+ϵ)/(2ϵ)) (1 - ϵ η2 ξ1), {η0, α0, ξ0}][[1]] /.
  (var_ → val_) :→ (var → FullSimplify[PowerExpand@val])
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete soluti

information.

$$\left\{\eta 0 \to \eta 1 + \frac{e^{-\alpha 1}\,\eta 2}{1-\epsilon\,\eta 2\,\xi 1},\ \alpha 0 \to \alpha 1+\alpha 2+2\,\mathrm{Log}[-1+\epsilon\,\eta 2\,\xi 1],\ \xi 0 \to \frac{e^{-\alpha 2}\,\xi 1}{1-\epsilon\,\eta 2\,\xi 1}+\xi 2\right\}$$

```
FullSimplify@PowerExpand[e^(-(α0-α1-α2) (1+ϵ)/(2ϵ)) (1 - ϵ η2 ξ1) /. α0 → α1 + α2 + 2 Log[-1 + ϵ η2 ξ1]]
```

$$-\left(-1+\epsilon\,\eta 2\,\xi 1\right)^{-1/\epsilon}$$

Lemma 3 at $\delta = 0$. $\mathbb{O}\left(e^{\alpha f+\beta e}\mid f\,e\right) = \mathbb{O}\left(e^{c h+a e-2\epsilon c l+b f}\mid e\,l\,f\right)$, with $\left\{a \to -\frac{\beta}{-1+\alpha\beta\epsilon},\ b \to -\frac{\alpha}{-1+\alpha\beta\epsilon},\ c \to \frac{\mathrm{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$.

Derivation.

```
ME[α ρf].ME[β ρe] // Simplify // MF

eqn = ME[α ρf].ME[β ρe] == ME[a ρe].ME[c (ρh - 2 ϵ ρl)].ME[b ρf]

sol = Solve[Thread[Flatten /@ eqn], {a, b, c}][[1]]

sol = sol /. C[1] → 0
```