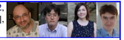
Everything around sl_{2+}^ϵ is **DoPeGDO**. So what?

Thanks for inviting me to Kyoto!
ωεβ:=http://drorbn.net/k23
More at ωεβ/talks

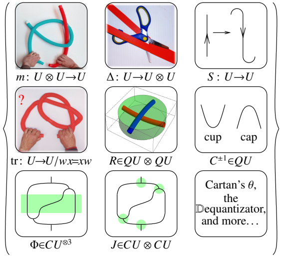
Continues Rozansky [Ro1, Ro2, Ro3], Ohtsuki [Oh], Overbay [Ov], joint with van der Veen [BV].



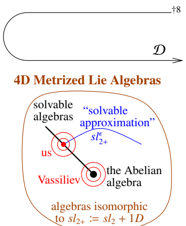
Abstract. I'll explain what "everything around" means: classical and quantum m, Δ, S, tr, R, C, and θ, as well as P, Φ, J, $\mathbb{D}$, and more, and all of their compositions. What **DoPeGDO** means: the category of Docile Perturbed Gaussian Differential Operators. And what sl_{2+}^ϵ means: a solvable approximation of the semi-simple Lie algebra sl_2.

Knot theorists should rejoice because all this leads to very powerful and well-behaved poly-time-computable knot invariants. Quantum algebraists should rejoice because it's a realistic playground for testing complicated equations and theories.

Conventions. 1. For a set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$.†1 2. Everything converges!



Less Abstract



DoPeGDO := The category with objects finite sets†2 and mor($A \to B$):

$$\{\mathcal{F} = \omega\exp(Q+P)\} \subset \mathbb{Q}[\![\zeta_A, z_B, \epsilon]\!]$$

Where: • ω is a scalar.†3 • Q is a "small" ϵ-free quadratic in $\zeta_A \cup z_B$.†4 • P is a "docile perturbation": $P = \sum_{k\geq 1}\epsilon^k P^{(k)}$, where $\deg P^{(k)} \leq 2k+2$.†5 • Compositions:†6

$$\mathcal{F}/\!\!/\mathcal{G} = \mathcal{G}\circ\mathcal{F} := \left(\mathcal{G}|_{\zeta_i\to\partial_{z_i}}\mathcal{F}\right)_{z_i=0} = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

Cool! $(V^*)^{\otimes\Sigma}\otimes V^{\otimes S}$ explodes; the ranks of quadratics and bounded-degree polynomials grow slowly!†7 Representation theory is over-rated!

Cool! How often do you see a computational toolbox so successful?

Our Algebras. Let $sl_{2+}^\epsilon := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t:=\epsilon a-b$ is central and if $\exists\epsilon^{-1}$, $sl_{2+}^\epsilon/\langle t\rangle \cong sl_2$. ωεβ/oa

U is either $CU = \mathcal{U}(sl_{2+}^\epsilon)[\![\hbar]\!]$ or $QU = \mathcal{U}_\hbar(sl_{2+}^\epsilon) = A\langle y,b,a,x\rangle[\![\hbar]\!]$ with $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $xy-qyx=(1-AB)/\hbar$, where $q=\mathbb{e}^{\hbar\epsilon}$, $A=\mathbb{e}^{-\hbar\epsilon a}$, and $B=\mathbb{e}^{-\hbar b}$. Set also $T=A^{-1}B=\mathbb{e}^{\hbar t}$.

The Quantum Leap. Also decree that in QU,

$$\Delta(y,b,a,x) = (y_1+B_1y_2, b_1+b_2, a_1+a_2, x_1+A_1x_2),$$
$$S(y,b,a,x) = (-B^{-1}y, -b, -a, -A^{-1}x),$$

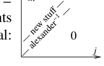
and $R=\sum\hbar^{j+k}y^kb^j\otimes a^jx^k/j![k]_q!$.

Mid-Talk Debts.
- What is this good for in quantum algebra?
- In knot theory?
- How does the "inclusion" $\mathcal{D}\colon \mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S}) \rightsquigarrow$ **DoPeGDO** work?
- Proofs that everything around sl_{2+}^ϵ really is **DoPeGDO**.
- Relations with prior art.
- The rest of the "compositions" story.

Theorem ([BG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

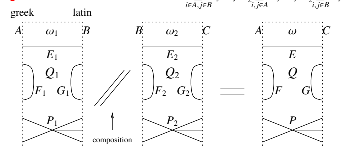
"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

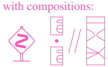
Compositions (1). In mor($A\to B$), $Q=\sum_{i\in A,j\in B}E_{ij}\zeta_iz_j+\frac12\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j+\frac12\sum_{i,j\in B}G_{ij}z_iz_j$



Where • $\omega=\omega_1\omega_2\det(I-F_2G_1)^{-1}$.
- $F=F_1+E_1F_2(I-G_1F_2)^{-1}E_1^T$.
- $G=G_2+E_2^TG_1(I-F_2G_1)^{-1}E_2$.
- $E=\sum_r E_1(F_2G_1)^rE_2=E_1(I-F_2G_1)^{-1}E_2$.
- P is computed using "connected Feynman diagrams" or as the solution of a messy PDE (yet we're still in algebra!).

Alexander lives here

One abstraction level up from tangles!
{tangles} → { }
with compositions:



DoPeGDO Footnotes. †1. Each variable has a "weight"$\in\{0,1,2\}$, and always $\mathrm{wt}\,z_i+\mathrm{wt}\,\zeta_i=2$.
†2. Really, "weight-graded finite sets" $A=A_0\sqcup A_1\sqcup A_2$.
†3. Really, a power series in the weight-0 variables†9.
†4. The weight of Q must be 2, so it decomposes as $Q=Q_{20}+Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series†9.
†5. Setting $\mathrm{wt}\,\epsilon=-2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained†9).
†6. There's also an obvious product
$$\mathrm{mor}(A_1\to B_1)\times\mathrm{mor}(A_2\to B_2)\to\mathrm{mor}(A_1\sqcup A_2\to B_1\sqcup B_2).$$
†7. That is, if the weight-0 variables are ignored. Otherwise more care is needed yet the conclusion remains.
†8. $\mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S})\rightsquigarrow\mathrm{mor}(\{\eta_i,\beta_i,\tau_i,\alpha_i,\xi_i\}_{i\in\Sigma}\to\{y_i,b_i,t_i,a_i,x_i\}_{i\in S})$, where $\mathrm{wt}(\eta_i,\xi_i,y_i,x_i)=1$ and $\mathrm{wt}(\beta_i,\tau_i,\alpha_i;b_i,t_i,a_i)=(2,2,0;0,0,2)$.
†9. For tangle invariants the wt-0 power series are always rational functions in the exponentials of the wt-0 variables (for knots: just one variable), with degrees bounded linearly by the crossing number.

$\mathcal{D}\colon \mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S})\to\mathbb{Q}[\![\eta_\Sigma,\beta_\Sigma,\alpha_\Sigma,\xi_\Sigma,y_S,b_S,a_S,x_S]\!]$. The PBW theorem for CU (always in the $ybax$ order), or its quantum analog for QU, say that if $U=CU$ or QU then $U^{\otimes S}$ is isomorphic as a vector space to $\mathbb{Q}[y_i,b_i,a_i,x_i]_{i\in S}[\![\hbar]\!]$; so it is enough to understand $\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$ for finite sets A and B.

Claim. $F\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])\xrightarrow[\mathcal{D}]{\sim}\mathbb{Q}[z_B][\![\zeta_A]\!]\ni\mathcal{F}$ via

$$\mathcal{D}(F) := \sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}F(z_A^n) = F\left(\mathbb{e}^{\sum_{a\in A}\zeta_az_a}\right)=\mathcal{F},$$
$$\mathcal{D}^{-1}(\mathcal{F})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{F}\right)_{\zeta_a=0}\quad\text{for } p\in\mathbb{Q}[z_A].$$

Claim. Assuming convergence, if $F\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $G\in\mathrm{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, $\mathcal{F}=\mathcal{D}(F)$, and $\mathcal{G}=\mathcal{D}(G)$, then

$$\mathcal{D}(F/\!\!/G) = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

And so the title of the talk finally makes sense!

Example. $\mathcal{D}(id\colon U\to U)=\mathbb{e}^{\eta y+\beta b+\alpha a+\xi x}$.

Example. Let $c\Delta^i_{jk}\colon CU^{\otimes\{i\}}\to CU^{\otimes\{j,k\}}$ be the standard co-product, given by $c\Delta^i_{jk}(y_i,b_i,a_i,x_i)=(y_j+y_k,b_j+b_k,a_j+a_k,x_j+x_k)$. Then

$$\mathcal{D}(c\Delta^i_{jk}) = c\Delta^i_{jk}(\mathbb{e}^{\eta_iy_i+\beta_ib_i+\alpha_ia_i+\xi_ix_i})$$
$$= \mathbb{e}^{\eta_i(y_j+y_k)+\beta_i(b_j+b_k)+\alpha_i(a_j+a_k)+\xi_i(x_j+x_k)}.$$

Example. The standard commutative product m^{ij}_k of polynomials is given by $z_i,z_j\to z_k$. Hence $\mathcal{D}(m^{ij}_k)=m^{ij}_k(\mathbb{e}^{\zeta_iz_i+\zeta_jz_j})=\mathbb{e}^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{array}{ccc}\mathbb{Q}[z]_i\otimes\mathbb{Q}[z]_j & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z]_k\\ \| & & \|\\ \mathbb{Q}[z_i,z_j] & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z_k]\end{array}$$

A real DoPeGDO Example. Let $cm^{ij}_k\colon CU_i\otimes CU_j\to CU_k$ be "classical multiplication" for sl_{2+}^ϵ, and let $\mathbb{O}_i\colon\mathbb{Q}[y_i,b_i,a_i,x_i]\to CU_i$ be the PBW ordering map.

$$\begin{array}{ccc}CU_i\otimes CU_j & \xrightarrow{cm^{ij}_k} & CU_k\\ \uparrow\mathbb{O}_{i,j} & & \uparrow\mathbb{O}_k\\ \mathbb{Q}[y_i,b_i,a_i,x_i,y_j,b_j,a_j,x_j] & & \mathbb{Q}[y_k,b_k,a_k,x_k]\end{array}$$

Claim. Let (all brawn and no brains)

$$\Lambda=\left(\eta_i+\frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k+\left(\beta_i+\beta_j+\frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k+$$
$$\left(\alpha_i+\alpha_j+\log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k+\left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i}+\xi_j\right)x_k$$

Then $\mathbb{e}^{\eta_iy_i+\beta_ib_i+\alpha_ia_i+\xi_ix_i+\eta_jy_j+\beta_jb_j+\alpha_ja_j+\xi_jx_j}/\!\!/\mathbb{O}_{i,j}/\!\!/cm^{ij}_k=\mathbb{e}^\Lambda/\!\!/\mathbb{O}_k$, and hence $\mathcal{D}(cm^{ij}_k)=\mathbb{e}^\Lambda$ and cm^{ij}_k is DoPeGDO.

Proof. We compute in a faithful 2D representation $z\mapsto\hat z$ of CU: (ωεβ/cm)

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HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
{ŷ = (0 0; ϵ 0), b̂ = (0 0; 0 -ϵ), â = (1 0; 0 0), x̂ = (0 1; 0 0)};
HL /@ {â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -ϵ ŷ,
  b̂.x̂ - x̂.b̂ == ϵ x̂, x̂.ŷ - ŷ.x̂ == b̂ + ϵ â}

{True, True, True, True, True}
HL@Simplify@With[{𝔼 = MatrixExp},
  𝔼[ηi ŷ].𝔼[βi b̂].𝔼[αi â].𝔼[ξi x̂].𝔼[ηj ŷ].𝔼[βj b̂].
   𝔼[αj â].𝔼[ξj x̂] == 𝔼[ŷ ∂yk Λ].𝔼[b̂ ∂bk Λ].𝔼[â ∂ak Λ].
   𝔼[x̂ ∂xk Λ]]

True
Series[Λ, {ϵ, 0, 1}]
(ak (αi + αj) + yk (ηi + e^(-αi) ηj) +
   bk (βi + βj + ηj ξi) + xk (e^(-αj) ξi + ξj)) +
 (ak ηj ξi - 1/2 bk ηj^2 ξi^2 - e^(-αi) yk ηj (βi + ηj ξi) -
   e^(-αj) xk ξi (βj + ηj ξi)) ϵ + O[ϵ]^2
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(Shame, but this technique fails for QU).

Claim. In QU, R is DoPeGDO.

Proof. Recall that with $q=\mathbb{e}^{\hbar\epsilon}$,

$$R=\sum\hbar^{j+k}y^kb^j\otimes a^jx^k/j![k]_q! = \mathbb{O}\left(\mathbb{e}^{\hbar b_1a_2}\mathbb{e}_q^{\hbar y_1x_2}\right).$$

Now expand $\mathbb{e}_q^{\hbar y_1x_2}$ in powers of ϵ using:

Faddeev's Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q:=\frac{q^n-1}{q-1}$, with $[n]_q!:=[1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x:=\sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have

$$\log\mathbb{e}_q^x=\sum_{k\geq 1}\frac{(1-q)^kx^k}{k(1-q^k)}=x+\frac{(1-q)^2x^2}{2(1-q^2)}+\dots.$$

Proof. We have that $\mathbb{e}_q^x=\frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx}=(1+(1-q)x)\mathbb{e}_q^x$, and

$$\log\mathbb{e}_q^{qx}=\log(1+(1-q)x)+\log\mathbb{e}_q^x.$$

Writing $\log\mathbb{e}_q^x=\sum_{k\geq 1}a_kx^k$ and comparing powers of x, we get $q^ka_k=-(1-q)^k/k+a_k$, or $a_k=\frac{(1-q)^k}{k(1-q^k)}$. □

Compositions (2). Recall that with all indices i running in some set B,

$$\mathcal{F}/\!\!/\mathcal{G}=\left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}\overset{(1)}{=}\mathbb{e}^{\sum\partial_{z_i}\partial_{\zeta_i}}(\mathcal{F}\mathcal{G})\Big|_{z_i=\zeta_i=0},$$

(1) Strictly speaking, true only when $B\cap(A\cup C)=\emptyset$.

so in general we wish to understand

$$[F:\mathcal{E}]_B:=\mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E}\quad\text{and}\quad\langle F:\mathcal{E}\rangle_B:=[F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\,\mathbb{e}^{\frac12\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B=\det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

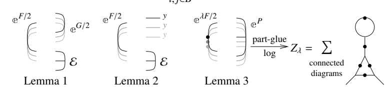
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B=\mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B$.

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda:=\log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0=P\quad\text{and}\quad\partial_\lambda Z_\lambda=\frac12\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda+(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Complexity to ϵ^k, for an n-xing width w knot (by [LT], $w\in O(\sqrt{n})$), is $O(n^2w^{2k+2}\log n)=O(n^{k+3}\log n)$ integer operations.