Dror Bar-Natan: Talks: Kyoto-230727:

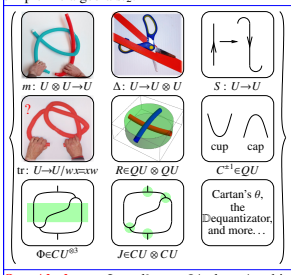
Thanks for inviting me to Kyoto! Continues Rozansky [Ro1, Ro2, Ro3], Ohtsuki [Oh], Overbay [Ov], joint with van der Veen [BV]. More at ωεβ:=http://drorbn.net/k23 ωεβ/talks

Everything around sl^ϵ_{2+} is DoPeGDO. So what?

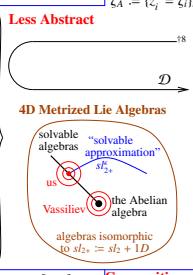
Abstract. I'll explain what "everything around" means: classical and quantum m, Δ, S, tr, R, C, and θ, as well as P, Φ, J, $\mathbb{D}$, and more, and all of their compositions. What **DoPeGDO** means: the category of Docile Perturbed Gaussian Differential Operators. And what sl^ϵ_{2+} means: a solvable approximation of the semi-simple Lie algebra sl_2.

Knot theorists should rejoice because all this leads to very powerful and well-behaved poly-time-computable knot invariants. Quantum algebraists should rejoice because it's a realistic playground for testing complicated equations and theories.

Conventions. 1. For a set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$.[†1] 2. Everything converges!



Less Abstract



DoPeGDO := The category with objects finite sets[†2] and $\mathrm{mor}(A\to B)$:

$$\{\mathcal{F} = \omega\exp(Q+P)\} \subset \mathbb{Q}[\![\zeta_A, z_B, \epsilon]\!]$$

Where: • ω is a scalar.[†3] • Q is a "small" ϵ-free quadratic in $\zeta_A \cup z_B$.[†4] • P is a "docile perturbation": $P = \sum_{k\ge1}\epsilon^k P^{(k)}$, where $\deg P^{(k)} \le 2k+2$.[†5] • Compositions:[†6]

$$\mathcal{F}/\!/\mathcal{G} := \mathcal{G}\circ\mathcal{F} := \left(\mathcal{G}|_{\zeta_i\to\partial_{z_i}}\mathcal{F}\right)_{z_i=0} = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

Cool! $(V^*)^{\otimes\Sigma}\otimes V^{\otimes S}$ explodes; the ranks of quadratics and bounded-degree polynomials grow slowly![†7] Representation theory is over-rated! **Cool!** How often do you see a computational toolbox so successful?

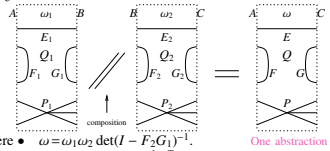
Our Algebras. Let $sl^\epsilon_{2+} := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t := \epsilon a - b$ is central and if $\exists\epsilon^{-1}$, $sl^\epsilon_{2+}/\langle t\rangle \cong sl_2$. ωεβ/ωa U is either $CU = \mathcal{U}(sl^\epsilon_{2+})[\![\hbar]\!]$ or $QU = \mathcal{U}_\hbar(sl^\epsilon_{2+}) = A\langle y,b,a,x\rangle[\![\hbar]\!]$ with $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $xy-qyx=(1-AB)/\hbar$, where $q=\Phi^{\hbar\epsilon}$, $A=\Phi^{-\hbar\epsilon a}$, and $B=\Phi^{-\hbar b}$. Set also $T=A^{-1}B=\Phi^{\hbar t}$.

The Quantum Leap. Also decree that in QU,

$$\Delta(y,b,a,x) = (y_1+B_1y_2, b_1+b_2, a_1+a_2, x_1+A_1x_2),$$
$$S(y,b,a,x) = (-B^{-1}y, -b, -a, -A^{-1}x),$$

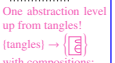
and $R = \sum\hbar^{j+k}y^kb^j\otimes a^jx^k/j!\,[k]_q!$.

Compositions (1). In $\mathrm{mor}(A\to B)$, $Q = \sum_{i\in A,j\in B}E_{ij}\zeta_i z_j + \frac12\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j + \frac12\sum_{i,j\in B}G_{ij}z_iz_j$



Where • $\omega=\omega_1\omega_2\det(I-F_2G_1)^{-1}$.
• $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$.
• $G = G_2 + E_2^TG_1(I-F_2G_1)^{-1}E_2$.
• $E = \sum_r E_1(F_2G_1)^rE_2 = E_1(I-F_2G_1)^{-1}E_2$.
• P is computed using "connected Feynman diagrams" or as the solution of a messy PDE (yet we're still in algebra!).

One abstraction level up from tangles! {tangles} → {Gs} with compositions:



Mid-Talk Debts. • What is this good for in quantum algebra?
• In knot theory?
• How does the "inclusion" $\mathcal{D}$: $\mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S}) \rightsquigarrow$ **DoPeGDO** work?
• Proofs that everything around sl^ϵ_{2+} really is **DoPeGDO**.
• Relations with prior art.
• The rest of the "compositions" story.

Theorem ([BG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\ge0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.

"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

DoPeGDO Footnotes. †1. Each variable has a "weight"$\in\{0,1,2\}$, and always $\mathrm{wt}\,z_i+\mathrm{wt}\,\zeta_i=2$.
†2. Really, "weight-graded finite sets" $A=A_0\sqcup A_1\sqcup A_2$.
†3. Really, a power series in the weight-0 variables[†9].
†4. The weight of Q must be 2, so it decomposes as $Q=Q_{20}+Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series[†9].
†5. Setting $\mathrm{wt}\,\epsilon=-2$, the weight of P is ≤2 (so the powers of the weight-0 variables are not constrained[†9]).
†6. There's also an obvious product
$\mathrm{mor}(A_1\to B_1)\times\mathrm{mor}(A_2\to B_2)\to\mathrm{mor}(A_1\sqcup A_2\to B_1\sqcup B_2)$.
†7. That is, if the weight-0 variables are ignored. Otherwise more care is needed yet the conclusion remains.
†8. $\mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S})\rightsquigarrow\mathrm{mor}(\{\eta_i,\beta_i,\tau_i,a_i,\xi_i\}_{i\in\Sigma}\to\{y_i,b_i,t_i,a_i,x_i\}_{i\in S})$, where $\mathrm{wt}(\eta_i,\xi_i,y_i,x_i)=1$ and $\mathrm{wt}(\beta_i,\tau_i,\alpha_i;b_i,t_i,a_i)=(2,2,0;0,0,2)$.
†9. For tangle invariants the wt-0 power series are always rational functions in the exponentials of the wt-0 variables (for knots: just one variable), with degrees bounded linearly by the crossing number.

$\mathcal{D}$: $\mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S})\to\mathbb{Q}[\![\eta_\Sigma,\beta_\Sigma,\alpha_\Sigma,\xi_\Sigma,y_S,b_S,a_S,x_S]\!]$. The PBW theorem for CU (always in the $ybax$ order), or its quantum analog for QU, say that if $U=CU$ or QU then $U^{\otimes S}$ is isomorphic as a vector space to $\mathbb{Q}[y_i,b_i,a_i,x_i]_{i\in S}[\![\hbar]\!]$; so it is enough to understand $\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$ for finite sets A and B.

Claim. $F\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])\xrightarrow[\mathcal{D}]{\cong}\mathbb{Q}[z_B][\![\zeta_A]\!]\ni\mathcal{F}$ via

$$\mathcal{D}(F) := \sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}F(z_A^n) = F\left(e^{\sum_{\omega\in A}\zeta_\omega z_\omega}\right) = \mathcal{F},$$
$$\mathcal{D}^{-1}(\mathcal{F})(p) = \left(p|_{z_\omega\to\partial_{\zeta_\omega}}\mathcal{F}\right)_{\zeta_\omega=0}\quad\text{for } p\in\mathbb{Q}[z_A].$$

Claim. Assuming convergence, if $F\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $G\in\mathrm{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, $\mathcal{F}=\mathcal{D}(F)$, and $\mathcal{G}=\mathcal{D}(G)$, then

$$\mathcal{D}(F/\!/G) = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

And so the title of the talk finally makes sense!

Example. $\mathcal{D}(id\colon U\to U) = e^{\eta y+\beta b+\alpha a+\xi x}$.

Example. Let $c\Delta^i_{jk}\colon CU^{\otimes\{i\}}\to CU^{\otimes\{j,k\}}$ be the standard coproduct, given by $c\Delta^i_{jk}(y_i,b_i,a_i,x_i) = (y_j+y_k, b_j+b_k, a_j+a_k, x_j+x_k)$. Then

$$\mathcal{D}(c\Delta^i_{jk}) = c\Delta^i_{jk}(e^{\eta_iy_i+\beta_ib_i+\alpha_ia_i+\xi_ix_i}) = e^{\eta_i(y_j+y_k)+\beta_i(b_j+b_k)+\alpha_i(a_j+a_k)+\xi_i(x_j+x_k)}.$$

Example. The standard commutative product m^{ij}_k of polynomials is given by $z_iz_j\to z_k$. Hence $\mathcal{D}(m^{ij}_k) = m^{ij}_k(e^{\zeta_iz_i+\zeta_jz_j}) = e^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{array}{ccc}\mathbb{Q}[z_i]\otimes\mathbb{Q}[z_j] & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z_k]\\ \| & & \|\\ \mathbb{Q}[z_i,z_j] & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z_k]\end{array}$$

A real DoPeGDO Example. Let $cm^{ij}_k\colon CU_i\otimes CU_j\to CU_k$ be "classical multiplication" for sl^ϵ_{2+}, and let $\mathbb{O}_i\colon\mathbb{Q}[y_i,b_i,a_i,x_i]\to CU_i$ be the PBW ordering map.

$$\begin{array}{ccc}CU_i\otimes CU_j & \xrightarrow{cm^{ij}_k} & CU_k\\ \uparrow{\scriptstyle\mathbb{O}_{i,j}} & & \uparrow{\scriptstyle\mathbb{O}_k}\\ \mathbb{Q}[y_i,b_i,a_i,x_i,y_j,b_j,a_j,x_j] & & \mathbb{Q}[y_k,b_k,a_k,x_k]\end{array}$$

Claim. Let (all brawn and no brains)

$$\Lambda = \left(\eta_i+\frac{e^{-\alpha_i}-e^{\beta_i}\eta_j}{1+e^{\beta_i}\xi_i}\right)y_k + \left(\beta_i+\beta_j+\frac{\log\left(1+e^{\beta_i}\xi_i\right)}{\epsilon}\right)b_k + \left(\alpha_i+\alpha_j+\log\left(1+e^{\beta_i}\xi_i\right)\right)a_k + \left(\frac{e^{-\alpha_j-e^{\beta_i}\xi_i}}{1+e^{\beta_i}\xi_i}+\xi_j\right)x_k$$

Then $e^{\eta_iy_i+\beta_ib_i+\alpha_ia_i+\xi_ix_i+\eta_jy_j+\beta_jb_j+\alpha_ja_j+\xi_jx_j}/\!/\mathbb{O}_{i,j}/\!/cm^{ij}_k = e^\Lambda/\!/\mathbb{O}_k$, and hence $\mathcal{D}(cm^{ij}_k) = e^\Lambda$ and cm^{ij}_k is DoPeGDO.

Proof. We compute in a faithful 2D representation $z\mapsto\hat z$ of CU: (ωεβ/cm)

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HL[𝒪_] := Style[𝒪, Background → If[TrueQ@𝒪, , ]];
{ŷ = ((0 0) (ε 0)), b̂ = ((0 0) (0 -ε)), â = ((1 0) (0 0)), x̂ = ((0 1) (0 0))};
HL/@{â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -ε ŷ,
  b̂.x̂ - x̂.b̂ == ε x̂, x̂.ŷ - ŷ.x̂ == b̂ + ε â}
{True, True, True, True, True}
HL@Simplify@With[{E = MatrixExp},
  E[η₁ ŷ].E[β₁ b̂].E[α₁ â].E[ξ₁ x̂].E[η₂ ŷ].E[β₂ b̂].
   E[α₂ â].E[ξ₂ x̂] == E[ŷ ∂_{y_k Λ}].E[b̂ ∂_{b_k Λ}].E[â ∂_{a_k Λ}].
   E[x̂ ∂_{x_k Λ}]]
True
Series[Λ, {ε, 0, 1}]
(a_k (α₁ + α₂) + y_k (η₁ + e^{-α₁} η₂) +
   b_k (β₁ + β₂ + η₂ ξ₁) + x_k (e^{-α₂} ξ₁ + ξ₂)) +
 (a_k η₂ ξ₁ - 1/2 b_k η₂² ξ₁² - e^{-α₁} y_k η₂ (β₁ + η₂ ξ₁) -
   e^{-α₂} x_k ξ₁ (β₂ + η₂ ξ₁)) ε + O[ε]²
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(Shame, but this technique fails for QU).

Claim. In QU, R is DoPeGDO.

Proof. Recall that with $q=\Phi^{\hbar\epsilon}$,

$$R = \sum\hbar^{j+k}y^kb^j\otimes a^jx^k/j!\,[k]_q! = \mathbb{O}\left(\Phi^{\hbar b_1a_2}\Phi_q^{\hbar y_1x_2}\right).$$

Now expand $\Phi_q^{\hbar y_1x_2}$ in powers of ϵ using:

Faddeev's Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n-1}{q-1}$, with $[n]_q! := [1]_q[2]_q\cdots[n]_q$ and with $\Phi_q^x := \sum_{n\ge0}\frac{x^n}{[n]_q!}$, we have

$$\log\Phi_q^x = \sum_{k\ge1}\frac{(1-q)^kx^k}{k(1-q^k)} = x+\frac{(1-q)^2x^2}{2(1-q^2)}+\dots.$$

Proof. We have that $\Phi_q^x = \frac{e_q^{qx}-e_q^x}{qx-x}$ ("the q-derivative of Φ_q^x is itself"), and hence $\Phi_q^{qx} = (1+(1-q)x)\Phi_q^x$, and

$$\log\Phi_q^{qx} = \log(1+(1-q)x)+\log\Phi_q^x.$$

Writing $\log\Phi_q^x = \sum_{k\ge1}a_kx^k$ and comparing powers of x, we get $q^ka_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

Compositions (2). Recall that with all indices i running in some set B,

$$\mathcal{F}/\!/\mathcal{G} = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0} \overset{(1)}{=} e^{\sum\partial_{\zeta_i}\partial_{z_i}(\mathcal{F}\mathcal{G})}\Big|_{z_i=\zeta_i=0},$$

(1) Strictly speaking, true only when $B\cap(A\cup C)=\emptyset$.

so in general we wish to understand

$$[F:\mathcal{E}]_B := e^{\frac12\sum_{i,j\in B}F_{ij}\partial_{\zeta_i}\partial_{z_j}}\mathcal{E}\quad\text{and}\quad\langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_{B|_{z_B\to0}},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\,e^{\frac12\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\,e^{\sum_{i\in B}y_iz_i}\right\rangle_B = e^{\frac12\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B$.

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P\quad\text{and}\quad\partial_\lambda Z_\lambda = \frac12\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda+(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

Complexity to ϵ^k, for an n-xing width w knot (by [LT], $w\in O(\sqrt n)$), is $O(n^2w^{2k+2}\log n) = O(n^{k+3}\log n)$ integer operations.