Pensieve header: Implementing Θ - the main notebook accompanying Talks/KnotTheory-Congress-2502

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\begin{frame}

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Invariance under R3

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This is Theta.nb of http://drorbn.net/ktc25/ap.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\KnotTheoryCongress-2502"];
```

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In[∘]:=
```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/ktc25/ap to compute rotation numbers.
```

pdf

```
Loading PolyPlot.m from http://drorbn.net/ktc25/ap to plot 2-variable polynomials.
```

pdf

In[∘]:= $T_3 = T_1\, T_2;$

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In[∘]:=
```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

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In[∘]:=

$$F_1[\{s\_, i\_, j\_\}] = \mathrm{CF}\Big[s\,\Big(1/2 - g_{3ii} + T_2^s\, g_{1ii}\, g_{2ji} - g_{1ii}\, g_{2jj} - \left(T_2^s - 1\right) g_{2ji}\, g_{3ii} + 2\, g_{2jj}\, g_{3ii} - \left(1 - T_3^s\right) g_{2ji}\, g_{3ji} - g_{2ii}\, g_{3jj} - T_2^s\, g_{2ji}\, g_{3jj} + g_{1ii}\, g_{3jj} + \Big(\left(T_1^s - 1\right) g_{1ji}\, \left(T_2^{2\,s}\, g_{2ji} - T_2^s\, g_{2jj} + T_2^s\, g_{3jj}\right) + \left(T_3^s - 1\right) g_{3ji}\, \left(1 - T_2^s\, g_{1ii} - \left(T_1^s - 1\right)\left(T_2^s + 1\right) g_{1ji} + \left(T_2^s - 2\right) g_{2jj} + g_{2ij}\right)\Big) \Big/ \left(T_2^s - 1\right)\Big)\Big];$$

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In[∘]:=

$$F_2[\{s0\_, i0\_, j0\_\}, \{s1\_, i1\_, j1\_\}] := \mathrm{CF}\Big[s1\,\left(T_1^{s0} - 1\right)\left(T_2^{s1} - 1\right)^{-1}\left(T_3^{s1} - 1\right) g_{1,j1,i0}\, g_{3,j0,i1}\,\Big(\left(T_2^{s0}\, g_{2,i1,i0} - g_{2,i1,j0}\right) - \left(T_2^{s0}\, g_{2,j1,i0} - g_{2,j1,j0}\right)\Big)\Big]$$

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In[∘]:= $F_3[\varphi\_, k\_] = -\varphi/2 + \varphi\, g_{3kk};$

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\end{frame} \begin{frame}

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In[∘]:=

$$\delta_{i\_,j\_} := \mathrm{If}[i === j, 1, 0];$$

$$\mathrm{gR}_{s\_,i\_,j\_} := \{$$

$$g_{\nu\_ j\beta\_} :\!\to g_{\nu j^+\beta} + \delta_{j\beta},\ g_{\nu\_ i\beta\_} :\!\to T_\nu^s\, g_{\nu i^+\beta} + \left(1 - T_\nu^s\right) g_{\nu j^+\beta} + \delta_{i\beta},$$

$$g_{\nu\_\alpha\_ i^+} :\!\to T_\nu^s\, g_{\nu\alpha i} + \delta_{\alpha i^+},\ g_{\nu\_\alpha\_ j^+} :\!\to g_{\nu\alpha j} + \left(1 - T_\nu^s\right) g_{\nu\alpha i} + \delta_{\alpha j^+}$$

$$\}$$

Proof of Reidemeister 3:

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\[\resizebox{0.75\linewidth}{!}{\import{../Toronto-241030}{R3.pdftex_t}} \]

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$$\mathrm{DSum}[Cs\_\_\_] := \mathrm{Sum}[F_1[c], \{c, \{Cs\}\}] + \mathrm{Sum}[F_2[c0, c1], \{c0, \{Cs\}\}, \{c1, \{Cs\}\}]$$

$$\mathrm{lhs} = \mathrm{DSum}[\{1, j, k\}, \{1, i, k^+\}, \{1, i^+, j^+\}, \{s, m, n\}] \mathbin{//.} \mathrm{gR}_{1,j,k} \cup \mathrm{gR}_{1,i,k^+} \cup \mathrm{gR}_{1,i^+,j^+};$$

$$\mathrm{rhs} = \mathrm{DSum}[\{1, i, j\}, \{1, i^+, k\}, \{1, j^+, k^+\}, \{s, m, n\}] \mathbin{//.} \mathrm{gR}_{1,i,j} \cup \mathrm{gR}_{1,i^+,k} \cup \mathrm{gR}_{1,j^+,k^+};$$

$$\mathrm{Simplify}[\mathrm{lhs} == \mathrm{rhs}]$$

Out[∘]=

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True

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The Main Program

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In[∘]:=

$$\Theta[K\_] := \mathrm{Module}\Big[\{Cs, \varphi, n, A, \Delta, G, ev, \theta\},$$

$$\{Cs, \varphi\} = \mathrm{Rot}[K];\ n = \mathrm{Length}[Cs];$$

$$A = \mathrm{IdentityMatrix}[2n + 1];$$

$$\mathrm{Cases}\left[Cs, \{s\_, i\_, j\_\} :\!\to \left(A[\![\{i, j\}, \{i + 1, j + 1\}]\!] \mathrel{+}= \begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}\right)\right];$$

$$\Delta = T^{(-\mathrm{Total}[\varphi] - \mathrm{Total}[Cs[\![\mathrm{All},1]\!]])/2}\, \mathrm{Det}[A];$$

$$G = \mathrm{Inverse}[A];$$

$$ev[\mathcal{E}\_] := \mathrm{Factor}[\mathcal{E} \mathbin{/.} g_{\nu\_,\alpha\_,\beta\_} :\!\to (G[\![\alpha, \beta]\!] \mathbin{/.} T \to T_\nu)];$$

$$\theta = ev\left[\sum_{k=1}^{n} F_1[Cs[\![k]\!]]\right];$$

$$\theta \mathrel{+}= ev\left[\sum_{k1=1}^{n} \sum_{k2=1}^{n} F_2[Cs[\![k1]\!], Cs[\![k2]\!]]\right];$$

$$\theta \mathrel{+}= ev\left[\sum_{k=1}^{2n} F_3[\varphi[\![k]\!], k]\right];$$

$$\mathrm{Factor}@\{\Delta, (\Delta \mathbin{/.} T \to T_1)\, (\Delta \mathbin{/.} T \to T_2)\, (\Delta \mathbin{/.} T \to T_3)\, \theta\}\Big];$$

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\end{frame}

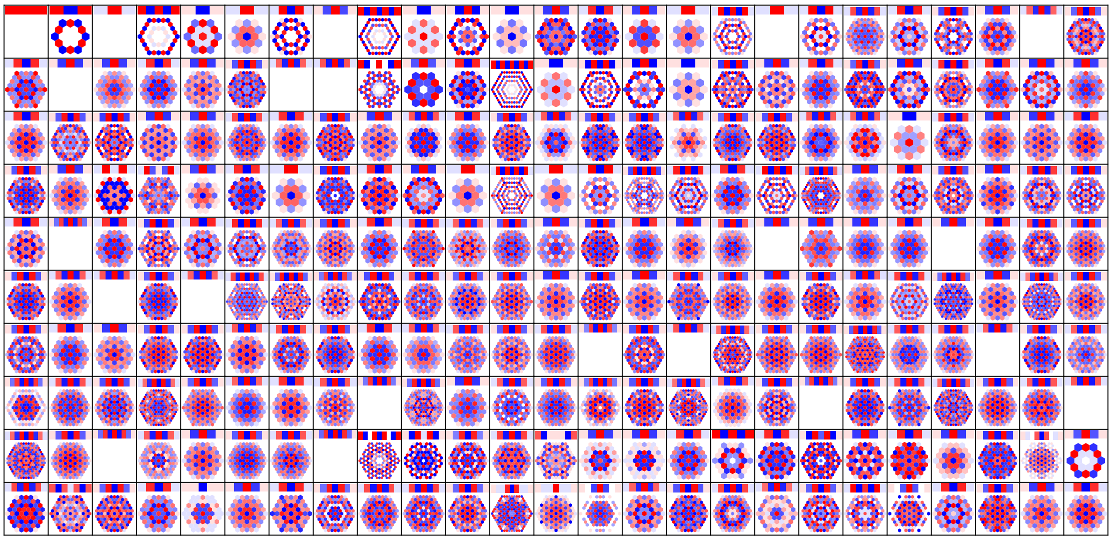
The Rolfsen Table

In[∘]:=
```
tab250 = {{1, 0}}~Join~Table[Θ[K], {K, AllKnots[{3, 10}]}];
```

KnotTheory: Loading precomputed data in PD4Knots`.

In[∘]:=
```
g250 = GraphicsGrid[Partition[PolyPlot /@ tab250, 25], Spacings → 0, Dividers → All]
```

Out[∘]=



In[∘]:=
```
Export["g250.png", g250]
```

Out[∘]=

g250.png

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The Trefoil Knot

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In[∘]:=
```
Θ[Knot[3, 1]] // Expand
```

Out[∘]=
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$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2\,T_2^2}+\frac{1}{T_1\,T_2^2}+\frac{1}{T_1^2\,T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2\,T_2-T_2^2+T_1\,T_2^2-T_1^2\,T_2^2\right\}$$

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In[◦]:= **PolyPlot[Θ[Knot[3, 1]], ImageSize → Tiny]**

Out[◦]=
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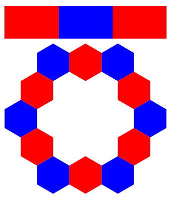

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\end{frame} \begin{frame}

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The Conway and Kinoshita-Terasaka Knots

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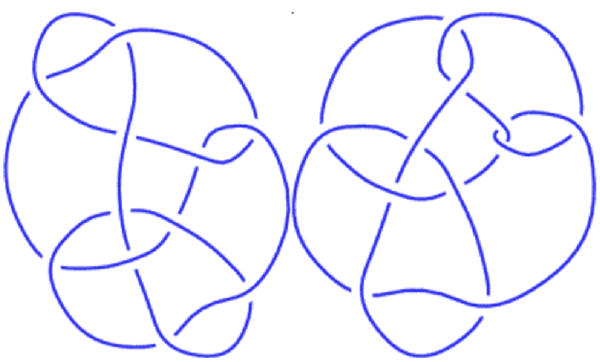
\[\includegraphics[width=15mm]{../Toronto-241030/K11n34.png}
 \qquad
 \includegraphics[width=15mm]{../Toronto-241030/K11n42.png}
\]



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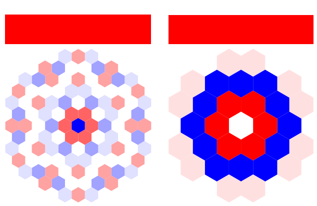
GraphicsRow[PolyPlot[Θ[Knot[#]], ImageSize → Tiny] & /@ {"K11n34", "K11n42"}]

Out[◦]=
pdf



tex

(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)
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The Torus Knots $TK_{13/2}$, $TK_{17/3}$, $TK_{13,5}$, and $TK_{7,6}$

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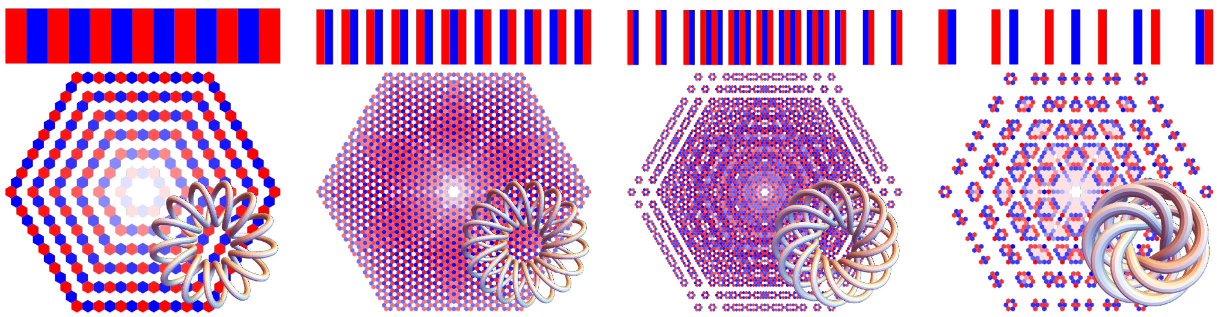
In[∘]:=
```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
   ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```

Out[∘]=
pdf



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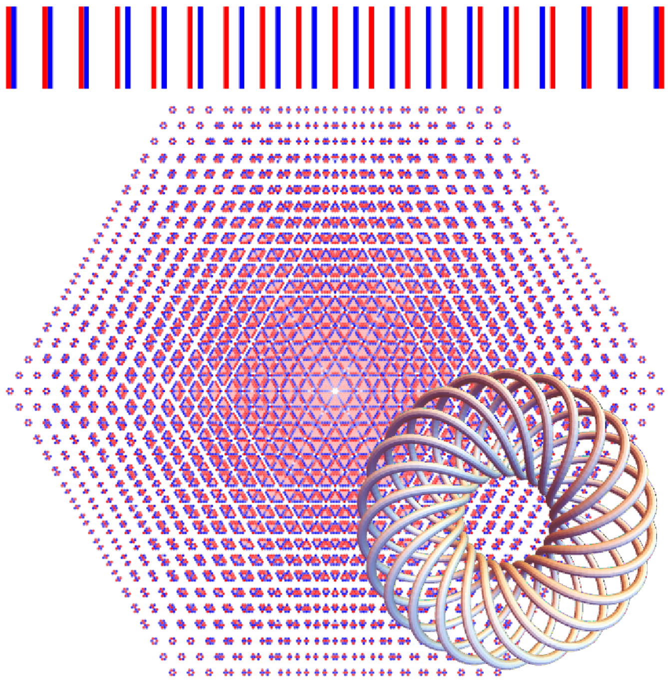
\end{frame}

The 132-crossing Torus Knot $TK_{22/7}$

In[∘]:=
```
AbsoluteTiming[T227 = Θ[TorusKnot[22, 7]];]
```

Out[∘]=
```
{924.759, Null}
```

In[∘]:=
```
T227Plot = ImageCompose[
  PolyPlot[T227, ImageSize → 720],
  TubePlot[TorusKnot[22, 7], ImageSize → 360],
  {Right, Bottom}, {Right, Bottom}
 ]
```
Out[∘]=



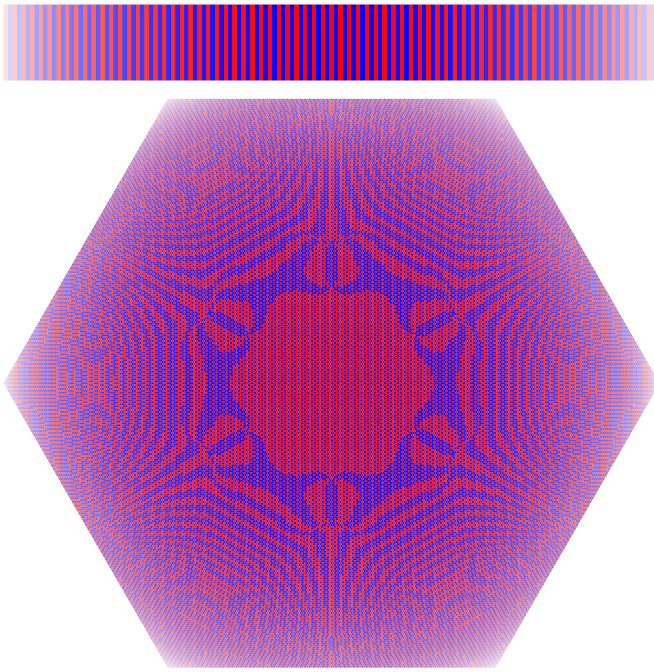
In[∘]:=
```
Export["T227Plot.pdf", T227Plot]
```
Out[∘]=
```
T227Plot.pdf
```
In[∘]:=
```
Export["T227Plot.png", RemoveBackground[Rasterize[T227Plot], {White, 1 / 250}]]
```
Out[∘]=
```
T227Plot.png
```

Large Random Knots

See also https://drorbn.net/AcademicPensieve/Projects/HigherRank/DunfieldKnots/.

In[ ]:=
```
DK300 = PolyPlot@Expand[
   Get["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank\\DunfieldKnots\\D300.m"][[2]] /.
    {T1 → T₁, T2 → T₂}]
```

Out[ ]=



In[ ]:=
```
Export["DK300.png", DK300, ImagePadding → None, PlotRangePadding → None]
```

Out[ ]=

DK300.png

In[ ]:=
```
Export["DK300.pdf", DK300]
```

Out[ ]=

DK300.pdf

Some Virtual Knots

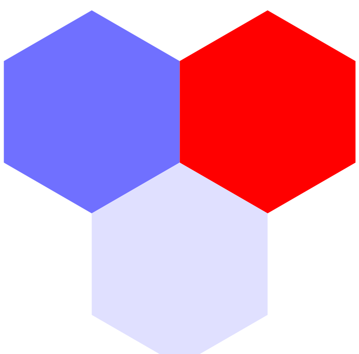
In[ ]:=
```
pd = PD[X[2, 4, 3, 1], X[3, 1, 4, 2]];
Rot[pd]
th = PowerExpand[Θ[pd]]
PolyPlot[th]
```

Out[ ]=
```
{{{-1, 4, 2}, {-1, 1, 3}}, {0, 0, 1, 0}}
```

Out[ ]=

$$\left\{\frac{1}{\sqrt{T}}, \frac{-1 - 2 T_2 + 4 T_1 T_2}{2 T_1 T_2}\right\}$$

Out[ ]=



In[ ]:=
```
pd = PD[X[3, 8, 4, 1], X[1, 6, 2, 7], X[5, 2, 6, 3], X[7, 5, 8, 4]];
Rot[pd]
th = PowerExpand[Θ[pd]]
PolyPlot[th]
```

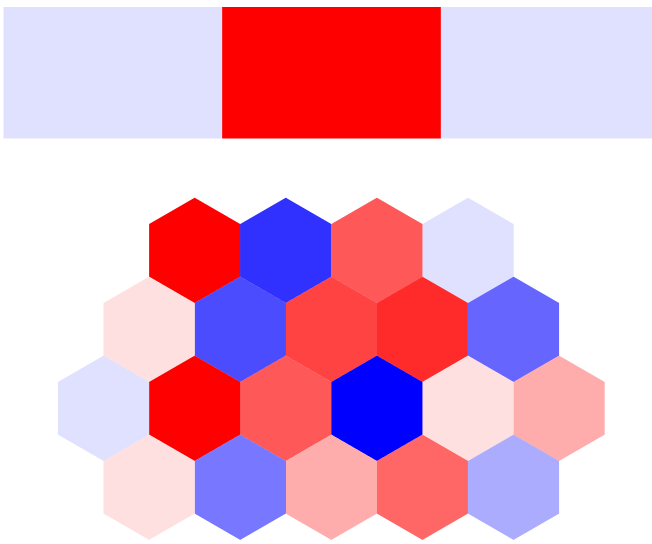
Out[ ]=

{{{-1, 8, 3}, {-1, 6, 1}, {-1, 2, 5}, {1, 4, 7}}, {0, 0, 0, 0, 1, -1, -1, -1}}

Out[ ]=

$$\left\{-\frac{1-3\,T+T^2}{T},\ -\frac{\left(1-3\,T_1+T_1^2\right)\left(-1+T_1+2\,T_1^2+T_2-7\,T_1^2\,T_2-2\,T_1^3\,T_2-T_1\,T_2^2+4\,T_1^2\,T_2^2+5\,T_1^3\,T_2^2-3\,T_1^2\,T_2^3+T_1^3\,T_2^3\right)}{T_1^2}\right\}$$

Out[ ]=



In[ ]:=
```
pd = PD[X[2, 8, 3, 1], X[3, 1, 4, 2], X[6, 4, 7, 5], X[7, 5, 8, 6]];
Rot[pd]
th = PowerExpand[Θ[pd]]
PolyPlot[th]
```

Out[ ]=

{{{-1, 8, 2}, {-1, 1, 3}, {-1, 4, 6}, {-1, 5, 7}}, {0, 0, 1, 0, 0, 1, 1, 0}}

Out[ ]=

$$\left\{\frac{1}{\sqrt{T}}, \frac{1}{2\,T_1^3\,T_2^4}\left(4 - 4\,T_1 - 6\,T_2 + 2\,T_1\,T_2 + 6\,T_1^2\,T_2 + 4\,T_1\,T_2^2 - 8\,T_1^2\,T_2^2 + 2\,T_2^3 + 2\,T_1\,T_2^3 + T_1^2\,T_2^3 - 4\,T_1^3\,T_2^3 - 2\,T_1\,T_2^4 - 6\,T_1^2\,T_2^4 + 10\,T_1^3\,T_2^4\right)\right\}$$

Out[ ]=