Pensieve header: For Luke Seaton.

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\KnotTheoryCongress-2502"];
```

In[•]:=
```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/ktc25/ap to compute rotation numbers.
Loading PolyPlot.m from http://drorbn.net/ktc25/ap to plot 2-variable polynomials.
```

CableComponent from https://katlas.org/wiki/CableComponent.m:

In[•]:=
```
CableComponent[BR[n_Integer, js_List], K_] :=
  Module[{BC, C0, C1, C2, CC1, CS1, CS2, L, S, a, e, h, i, i1, i2, j, j1, j2, jss, k, k1, kjs,
    out, out0, out1, p, p1, pos, q, r, s, ss, t, t0, t1, t2, tj, v, w, writhe}, L = PD[K];
   kjs = BR[L][[2]];
   For[i1 = 1;
    writhe = 0, i1 ≤ Length[kjs], i1++, writhe = writhe + Sign[kjs[[i1]]]];
   For[i2 = 1;
    jss[0] = js, i2 ≤ n Abs[writhe], i2++,
    jss[i2] = Flatten[{jss[i2 - 1], Table[-Sign[writhe] e, {e, n - 1}]}]];
   k1 = Length[jss[n Abs[writhe]]];
   For[i = 1, i ≤ n, i++, s[i] = a[i] = i];
   For[j = 1, j ≤ k1, j++, p = Select[Range[n], Abs[jss[n Abs[writhe]][[j]]] == a[#] &][[1]];
    q = Select[Range[n], a[#] == a[p] + 1 &][[1]];
    If[jss[n Abs[writhe]][[j]] > 0,
     K[j] = X[s[q], n + 2 j, n + 2 j - 1, s[p]], K[j] = X[s[p], s[q], n + 2 j, n + 2 j - 1]];
    s[p] = n + 2 j;
    s[q] = n + 2 j - 1;
    a[p]++;
    a[q]--];
   BC = Table[K[d], {d, k1}];
   If[Jones[L][q] === 1, For[j1 = 1, j1 ≤ Length[BC],
     j1++, For[i = 1, i ≤ n, i++, BC[[j1]] = BC[[j1]] /. s[i] :> a[i]]]];
    If[BC == {}, BC = {Loop[1]}];
    out1 = PD @@ BC, For[j2 = 1, j2 ≤ Length[BC],
     j2++, For[tj = 1, tj ≤ n, tj++, BC[[j2]] = BC[[j2]] /. tj :> 1[tj]]]];
    p1 = Select[Range[n], # ≠ s[#] &];
    S = Select[L, MemberQ[#, 1] && MemberQ[#, 2] &];
    pos = Position[S, 1][[1, 2]];
    r = Select[Table[i, {i, Length[L]}], L[[#]] == Flatten @@ S &][[1]];
    k = 0;
    out0 = L /. X[a_, b_, c_, d_] :> (++k;
        Table[X[h[i, j - 1, k], v[i, j, k], h[i, j, k], v[i - 1, j, k]], {i, n}, {j, n}] /.
          {h[i_, 0, _] :> a[i], h[i_, n, _] :> c[i]} /.
         If[d - b == 1 || b - d > 1, {v[0, j_, _] :> d[j], v[n, j_, _] :> b[j]},
```

```
        {v[0, j_, _] :> d[n + 1 - j], v[n, j_, _] :> b[n + 1 - j]}]);
  w = Flatten@out0[[r]];
  out = PD @@ Flatten[Join @@ out0];
  ss = Table[a[i], {i, n}][[p1]];
  CC1 = List @@ out;
  For[t0 = 1, t0 ≤ Length[ss], t0++, C0[t0] = Select[w, MemberQ[#, 1[ss[[t0]]]] &];
   C1[t0] = Select[C0[t0], Mod[Position[#, 1[ss[[t0]]]][[1, 1]], 2] == Mod[pos, 2] &];
   C2[t0] = C1[t0] /. 1[ss[[t0]]] :> s[Select[Range[n], a[#] == ss[[t0]] &][[1]]]];
  CS1 = Flatten[Table[C1[t1], {t1, Length[ss]}]];
  CS2 = Flatten[Table[C2[t2], {t2, Length[ss]}]];
  For[i = 1, i ≤ Length[CS1], i++, CC1 = DeleteCases[CC1, CS1[[i]]]];
  out1 = Union[BC, CC1, CS2];
  PD @@ out1;
  k = 0;
  out1 = PD @@ (out1 /. ((# → ++k) & /@ (List @@ Union @@ out1)))]];
```

In[•]:=
```
T3 = T1 T2;
```

In[•]:=
```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

In[•]:=
```
F1[{s_, i_, j_}] =
  CF[s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj - (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
      g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj + ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
        (T3^s - 1) g3ji (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji + (T2^s - 2) g2jj + g2ij)) / (T2^s - 1))];
```

In[•]:=
```
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] := CF[
  s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1 ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
```

In[•]:=
```
F3[φ_, k_] = -φ/2 + φ g3kk;
```

In[•]:=
```
Θ[K_] := Θ[K] = Module[{Cs, φ, n, A, Δ, G, ev, Θ},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += {{-T^s, T^s - 1}, {0, -1}})];
  Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
  G = Inverse[A];
  ev[ℰ_] := Factor[ℰ /. g_{ν_, α_, β_} :> (G[[α, β]] /. T → T_ν)];
  Θ = ev[∑_{k=1}^{n} F1[Cs[[k]]]];
  Θ += ev[∑_{k1=1}^{n} ∑_{k2=1}^{n} F2[Cs[[k1]], Cs[[k2]]]];

  Θ += ev[∑_{k=1}^{2 n} F3[φ[[k]], k]];
  Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) Θ}];
```

In[•]:=
```
Θ[Knot[3, 1]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=
$$\left\{\frac{1-T+T^2}{T},\ -\frac{1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4}{T_1^2 T_2^2}\right\}$$

In[•]:=
```
Knot[3, 1] // PD
```

Out[•]=
```
PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]]
```

In[•]:=
```
K = CableComponent[BR[4, {1, 2, 3}], Knot[3, 1]]
```

KnotTheory: Vogel's algorithm was implemented by Dan Carney in the summer of 2005 at the University of Toronto.

Out[•]=
```
PD[X[3, 8, 7, 1], X[5, 10, 9, 8], X[6, 12, 11, 10], X[9, 14, 13, 7], X[11, 16, 15, 14],
 X[12, 18, 17, 16], X[15, 20, 19, 13], X[17, 22, 21, 20], X[18, 24, 23, 22], X[21, 26, 25, 19],
 X[23, 28, 27, 26], X[24, 30, 29, 28], X[27, 32, 31, 25], X[29, 34, 33, 32], X[30, 36, 35, 34],
 X[33, 38, 37, 31], X[35, 40, 39, 38], X[36, 42, 41, 40], X[39, 44, 43, 37], X[41, 46, 45, 44],
 X[42, 48, 47, 46], X[45, 50, 49, 43], X[47, 52, 51, 50], X[48, 54, 53, 52], X[51, 56, 55, 49],
 X[53, 58, 57, 56], X[54, 60, 59, 58], X[57, 62, 61, 55], X[59, 64, 63, 62], X[60, 66, 65, 64],
 X[63, 68, 67, 61], X[65, 70, 69, 68], X[66, 72, 71, 70], X[69, 74, 73, 67],
 X[71, 76, 75, 74], X[72, 78, 77, 76], X[73, 139, 103, 95], X[75, 151, 112, 139],
 X[77, 163, 121, 151], X[78, 91, 130, 163], X[80, 2, 1, 79], X[81, 4, 3, 2], X[82, 6, 5, 4],
 X[87, 140, 104, 79], X[88, 152, 113, 140], X[89, 164, 122, 152], X[90, 99, 131, 164],
 X[95, 141, 105, 87], X[96, 153, 114, 141], X[97, 165, 123, 153], X[98, 83, 132, 165],
 X[103, 142, 106, 96], X[104, 143, 107, 80], X[105, 144, 108, 88], X[106, 145, 109, 97],
 X[107, 146, 110, 81], X[108, 147, 111, 89], X[109, 148, 83, 98], X[110, 149, 91, 82],
 X[111, 150, 99, 90], X[112, 154, 115, 142], X[113, 155, 116, 143], X[114, 156, 117, 144],
 X[115, 157, 118, 145], X[116, 158, 119, 146], X[117, 159, 120, 147], X[118, 160, 84, 148],
 X[119, 161, 92, 149], X[120, 162, 100, 150], X[121, 166, 124, 154], X[122, 167, 125, 155],
 X[123, 168, 126, 156], X[124, 169, 127, 157], X[125, 170, 128, 158], X[126, 171, 129, 159],
 X[127, 172, 85, 160], X[128, 173, 93, 161], X[129, 174, 101, 162], X[130, 92, 133, 166],
 X[131, 100, 134, 167], X[132, 84, 135, 168], X[133, 93, 136, 169], X[134, 101, 137, 170],
 X[135, 85, 138, 171], X[136, 94, 86, 172], X[137, 102, 94, 173], X[138, 86, 102, 174]]
```

In[•]:=
```
skel = Skeleton[K]
```

Out[•]=
```
{Loop[1, 8, 10, 12, 17, 21, 25, 32, 34, 36, 41, 45, 49, 56, 58, 60, 65, 69, 73, 103, 106,
  109, 83, 165, 153, 141, 87, 104, 107, 110, 91, 163, 151, 139, 95, 105, 108, 111, 99,
  164, 152, 140, 79, 2, 4, 6, 11, 15, 19, 26, 28, 30, 35, 39, 43, 50, 52, 54, 59, 63, 67,
  74, 76, 78, 130, 133, 136, 86, 174, 162, 150, 90, 131, 134, 137, 94, 172, 160, 148,
  98, 132, 135, 138, 102, 173, 161, 149, 82, 5, 9, 13, 20, 22, 24, 29, 33, 37, 44, 46,
  48, 53, 57, 61, 68, 70, 72, 77, 121, 124, 127, 85, 171, 159, 147, 89, 122, 125, 128,
  93, 169, 157, 145, 97, 123, 126, 129, 101, 170, 158, 146, 81, 3, 7, 14, 16, 18, 23,
  27, 31, 38, 40, 42, 47, 51, 55, 62, 64, 66, 71, 75, 112, 115, 118, 84, 168, 156, 144,
  88, 113, 116, 119, 92, 166, 154, 142, 96, 114, 117, 120, 100, 167, 155, 143, 80]}
```

In[•]:=
```
inds = Flatten[List @@@ skel]
```

Out[•]=
```
{1, 8, 10, 12, 17, 21, 25, 32, 34, 36, 41, 45, 49, 56, 58, 60, 65, 69, 73, 103, 106,
 109, 83, 165, 153, 141, 87, 104, 107, 110, 91, 163, 151, 139, 95, 105, 108, 111, 99,
 164, 152, 140, 79, 2, 4, 6, 11, 15, 19, 26, 28, 30, 35, 39, 43, 50, 52, 54, 59, 63,
 67, 74, 76, 78, 130, 133, 136, 86, 174, 162, 150, 90, 131, 134, 137, 94, 172, 160,
 148, 98, 132, 135, 138, 102, 173, 161, 149, 82, 5, 9, 13, 20, 22, 24, 29, 33, 37, 44,
 46, 48, 53, 57, 61, 68, 70, 72, 77, 121, 124, 127, 85, 171, 159, 147, 89, 122, 125,
 128, 93, 169, 157, 145, 97, 123, 126, 129, 101, 170, 158, 146, 81, 3, 7, 14, 16, 18,
 23, 27, 31, 38, 40, 42, 47, 51, 55, 62, 64, 66, 71, 75, 112, 115, 118, 84, 168, 156,
 144, 88, 113, 116, 119, 92, 166, 154, 142, 96, 114, 117, 120, 100, 167, 155, 143, 80}
```

```
In[•]:= Thread[inds → Range[Length[inds]]]
```

Out[•]=

$$\{1\to 1,\ 8\to 2,\ 10\to 3,\ 12\to 4,\ 17\to 5,\ 21\to 6,\ 25\to 7,\ 32\to 8,\ 34\to 9,\ 36\to 10,\ 41\to 11,\ 45\to 12,$$
$$49\to 13,\ 56\to 14,\ 58\to 15,\ 60\to 16,\ 65\to 17,\ 69\to 18,\ 73\to 19,\ 103\to 20,\ 106\to 21,\ 109\to 22,$$
$$83\to 23,\ 165\to 24,\ 153\to 25,\ 141\to 26,\ 87\to 27,\ 104\to 28,\ 107\to 29,\ 110\to 30,\ 91\to 31,$$
$$163\to 32,\ 151\to 33,\ 139\to 34,\ 95\to 35,\ 105\to 36,\ 108\to 37,\ 111\to 38,\ 99\to 39,\ 164\to 40,$$
$$152\to 41,\ 140\to 42,\ 79\to 43,\ 2\to 44,\ 4\to 45,\ 6\to 46,\ 11\to 47,\ 15\to 48,\ 19\to 49,\ 26\to 50,$$
$$28\to 51,\ 30\to 52,\ 35\to 53,\ 39\to 54,\ 43\to 55,\ 50\to 56,\ 52\to 57,\ 54\to 58,\ 59\to 59,\ 63\to 60,$$
$$67\to 61,\ 74\to 62,\ 76\to 63,\ 78\to 64,\ 130\to 65,\ 133\to 66,\ 136\to 67,\ 86\to 68,\ 174\to 69,\ 162\to 70,$$
$$150\to 71,\ 90\to 72,\ 131\to 73,\ 134\to 74,\ 137\to 75,\ 94\to 76,\ 172\to 77,\ 160\to 78,\ 148\to 79,$$
$$98\to 80,\ 132\to 81,\ 135\to 82,\ 138\to 83,\ 102\to 84,\ 173\to 85,\ 161\to 86,\ 149\to 87,\ 82\to 88,$$
$$5\to 89,\ 9\to 90,\ 13\to 91,\ 20\to 92,\ 22\to 93,\ 24\to 94,\ 29\to 95,\ 33\to 96,\ 37\to 97,\ 44\to 98,\ 46\to 99,$$
$$48\to 100,\ 53\to 101,\ 57\to 102,\ 61\to 103,\ 68\to 104,\ 70\to 105,\ 72\to 106,\ 77\to 107,\ 121\to 108,$$
$$124\to 109,\ 127\to 110,\ 85\to 111,\ 171\to 112,\ 159\to 113,\ 147\to 114,\ 89\to 115,\ 122\to 116,$$
$$125\to 117,\ 128\to 118,\ 93\to 119,\ 169\to 120,\ 157\to 121,\ 145\to 122,\ 97\to 123,\ 123\to 124,$$
$$126\to 125,\ 129\to 126,\ 101\to 127,\ 170\to 128,\ 158\to 129,\ 146\to 130,\ 81\to 131,\ 3\to 132,$$
$$7\to 133,\ 14\to 134,\ 16\to 135,\ 18\to 136,\ 23\to 137,\ 27\to 138,\ 31\to 139,\ 38\to 140,\ 40\to 141,$$
$$42\to 142,\ 47\to 143,\ 51\to 144,\ 55\to 145,\ 62\to 146,\ 64\to 147,\ 66\to 148,\ 71\to 149,\ 75\to 150,$$
$$112\to 151,\ 115\to 152,\ 118\to 153,\ 84\to 154,\ 168\to 155,\ 156\to 156,\ 144\to 157,\ 88\to 158,$$
$$113\to 159,\ 116\to 160,\ 119\to 161,\ 92\to 162,\ 166\to 163,\ 154\to 164,\ 142\to 165,\ 96\to 166,$$
$$114\to 167,\ 117\to 168,\ 120\to 169,\ 100\to 170,\ 167\to 171,\ 155\to 172,\ 143\to 173,\ 80\to 174\}$$

```
In[•]:= CF[pd_PD] := Module[{skel, inds},
          skel = Skeleton[K];
          inds = Flatten[List @@@ skel];
          pd /. Thread[inds → Range[Length[inds]]]
         ]
```

```
In[•]:= Θ[CF@K]
```

Out[•]=

$$\left\{\frac{1-T^4+T^8}{T^4},\right.$$
$$\frac{1}{T_1^8\,T_2^8}\Big(2-2\,T_1^4+2\,T_1^8-2\,T_1\,T_2+T_1^4\,T_2+T_1^5\,T_2-2\,T_1^8\,T_2-2\,T_1^2\,T_2^2+T_1^4\,T_2^2+T_1^6\,T_2^2-2\,T_1^8\,T_2^2-2\,T_1^3\,T_2^3+T_1^4\,T_2^3+$$
$$T_1^7\,T_2^3-2\,T_1^8\,T_2^3-2\,T_2^4+T_1\,T_2^4+T_1^2\,T_2^4+T_1^3\,T_2^4+3\,T_1^4\,T_2^4-T_1^5\,T_2^4-T_1^6\,T_2^4-T_1^7\,T_2^4+3\,T_1^8\,T_2^4+T_1^9\,T_2^4+$$
$$T_1^{10}\,T_2^4+T_1^{11}\,T_2^4-2\,T_1^{12}\,T_2^4+T_1\,T_2^5-T_1^4\,T_2^5-T_1^9\,T_2^5+T_1^{12}\,T_2^5+T_1^2\,T_2^6-T_1^4\,T_2^6-T_1^{10}\,T_2^6+T_1^{12}\,T_2^6+T_1^3\,T_2^7-$$
$$T_1^4\,T_2^7-T_1^{11}\,T_2^7+T_1^{12}\,T_2^7+2\,T_2^8-2\,T_1\,T_2^8-2\,T_1^2\,T_2^8-2\,T_1^3\,T_2^8+3\,T_1^4\,T_2^8+3\,T_1^{12}\,T_2^8-2\,T_1^{13}\,T_2^8-2\,T_1^{14}\,T_2^8-$$
$$2\,T_1^{15}\,T_2^8+2\,T_1^{16}\,T_2^8+T_1^4\,T_2^9-T_1^5\,T_2^9-T_1^{12}\,T_2^9+T_1^{13}\,T_2^9+T_1^4\,T_2^{10}-T_1^6\,T_2^{10}-T_1^{12}\,T_2^{10}+T_1^{14}\,T_2^{10}+T_1^4\,T_2^{11}-$$
$$T_1^7\,T_2^{11}-T_1^{12}\,T_2^{11}+T_1^{15}\,T_2^{11}-2\,T_1^4\,T_2^{12}+T_1^5\,T_2^{12}+T_1^6\,T_2^{12}+T_1^7\,T_2^{12}+3\,T_1^8\,T_2^{12}-T_1^9\,T_2^{12}-T_1^{10}\,T_2^{12}-T_1^{11}\,T_2^{12}+$$
$$3\,T_1^{12}\,T_2^{12}+T_1^{13}\,T_2^{12}+T_1^{14}\,T_2^{12}+T_1^{15}\,T_2^{12}-2\,T_1^{16}\,T_2^{12}-2\,T_1^8\,T_2^{13}+T_1^9\,T_2^{13}+T_1^{12}\,T_2^{13}-2\,T_1^{13}\,T_2^{13}-2\,T_1^8\,T_2^{14}+$$
$$\left.T_1^{10}\,T_2^{14}+T_1^{12}\,T_2^{14}-2\,T_1^{14}\,T_2^{14}-2\,T_1^8\,T_2^{15}+T_1^{11}\,T_2^{15}+T_1^{12}\,T_2^{15}-2\,T_1^{15}\,T_2^{15}+2\,T_1^8\,T_2^{16}-2\,T_1^{12}\,T_2^{16}+2\,T_1^{16}\,T_2^{16}\Big)\right\}$$

In[•]:=
```
Θ[CF@K] // PolyPlot
```

Out[•]=