Watched Talks:
Khovanov: Webs and foams in historical perspective
Hirasawa: Construction and manipulation of Seifert surfaces in knot theory
Naoko Kamada: A Method for Constructing Welded Links from Existing Welded Links
Rasmussen: Satellite operators, curves, and the Fukaya category
Gukov: Knot Theory and AI
Silver: The Penrose Polynomial
Vassiliev: Varieties of chord diagrams and finite dimensional knot spaces
Zupan: Determining the ribbon numbers of ribbon knots
Kalfagianni: Colored Jones polynomials and crossing numbers of satellite knots
Morton: Framing and knot invariants
Kawauchi: Ribbon Surface-Link Overview
Walker: Higher-dimensional Temperly-Lieb categories via partition relations
Witten: From the Jones Polynomial to Khovanov Homology Via Gauge Theory

Missed Talks:
Seiichi Kamada: Generalized Alexander Quandles and Their Classification
Nelson: Quandle Cohomology Quiver Representations
Libgober: Geometric applications of Alexander invariants of the complements to algebraic curves
Qiuyu Ren: Lasagna s-invariant detect exotic 4-manifolds
Stoimenow: Strong quasipositivity, Thurston-Bennequin invariants, and arc index
Bryden: Computing Deloup's Invariant (possibly this talk was not given)
Manturov: The photography method: The state of the art and open problems
Baldridge: A new way to prove the Four Color Theorem using gauge theory
Gren: The classification of algebraic tangles
Lambropoulou: Celebrating Lou Kauffman: Working with Lou
Kauffman: Multiple Virtual Knot Theory
Bellingeri: Congruence subgroups of braid groups and crystallographic quotients
Traczyk: Burau braid representation for n=4. A problem in linear algebra?
Manolescu: Generalizations of Rasmussen's invariant
Melikhov: Is every knot isotopic to the unknot?
Schneider: Roto-Welded Knot Theory (revisited)
Turaev: Knots and links in 2-complexes
Lawrence: TBA
Piccirillo: New constructions and invariants of closed smooth 4-manifolds
Jin: Twist polynomial as a weight system
Menasco: A Seifert algorithm for integral homology spheres
Blanchet: Homology of configurations in ribbon graphs
Fiedler: Distinguishing knots without invariants
Gordon: The Unknotting Problem for Surfaces in the 4-sphere
Ozsvath: Bordered Floer homology
Gügümcü: Mock Alexander Polynomials
Bardakov: Multi-virtual braid groups and their representations
Millett: The HOMFLY-PT polynomials and the Conway and Kinoshita-Terasaka Knots
Szabo: Knot Floer homology and Kauffman states
Henrich: A Tour of Knotty Tricks, Games, and Puzzles
Przytycki: Homology of Yang-Baxter operators: from Lou Kauffman "Braids" paper to HOMFLYPT Yang-Baxter homology
Dynnikov: Rectangular diagrams of links, surfaces, and foliations
Reshetikhin: Invariants of tangles with a flat connection in the complement
Caprau: Graphical calculi and invariants for classical and colored links

Include ρ_2 ?

It is in the Beijing talk and only needs $\mathfrak{g}$-fication.

Put it in page 4 which would otherwise be shrunk.

Add a conjecture of the relation with Chern-Simons theory at \frakg_\eps^+.

The Strongest Genuinely Computable Knot Invariant in ~~2024~~ 2025

Abstract. "Genuinely computable" means we have computed it for random knots with over 300 crossings. "Strongest" means it separates prime knots with up to 15 crossings better than the less-computable HOMFLY-PT and Khovanov homology taken together. And hey, it's also meaningful and fun.

van der Veen

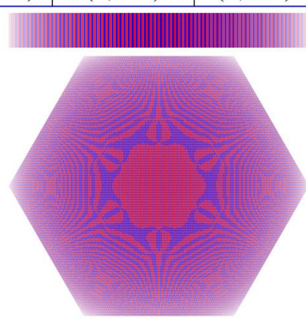
Continues Rozansky, Garoufalidis, Kricker, and Ohtsuki, joint with van der Veen.

Acknowledgement. This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Strongest. Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis): (ρ_1: [Ro1, Ro2, Ro3, Ov, BV1])

	knots	(H, Kh)	(Δ, ρ_1)	$\Theta = (\Delta, \theta)$	together
reign		2005-22	2022-24	2024-	
xing ≤ 10	249	248 (1)	249 (0)	249 (0)	249 (0)
xing ≤ 11	801	771 (30)	787 (14)	798 (3)	798 (3)
xing ≤ 12	2,977	(214)	(95)	(19)	(18)
xing ≤ 13	12,965	(1,771)	(959)	(194)	(185)
xing ≤ 14	59,937	(10,788)	(6,253)	(1,118)	(1,062)
xing ≤ 15	313,230	(70,245)	(42,914)	(6,758)	(6,555)

Genuinely Computable. Here's Θ on a random 300 crossing knot (from [DHOEBL]). For almost every other invariant, that's science fiction.

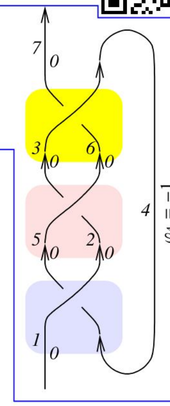


Fun. There's so much more to see in 2D pictures than in 1D ones! Yet almost nothing of the patterns you see we know how to prove. We'll have fun with that over the next few years. Would you join?

Meaningful. θ gives a genus bound (unproven yet with confidence). We hope (with reason) it says something about ribbon knots.

Conventions. T, T_1, and T_2 are indeterminates and $T_3 := T_1T_2$.

Preparation. Draw an n-crossing knot K as a diagram D as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \ldots, 2n+1\}$ and with rotation numbers φ_k.



Model T Traffic Rules. Cars always drive forward. When a car crosses over a sign-s bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0$. At the very end, cars fall off and disappear. On various edges *traffic counters* are placed. See also [Jo, LTW].

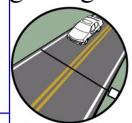


image credits: diamondtraffic.com

$p = 1 - T^s$

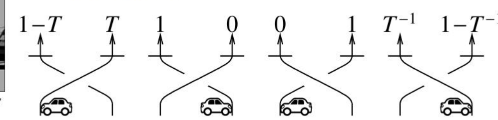
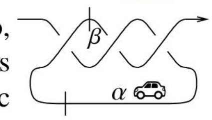


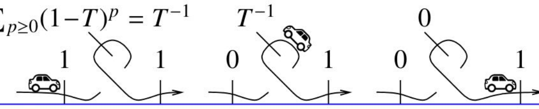
image credits: Dall-E

Definition. The *traffic function* $G = (g_{\alpha\beta})$ (also, the *Green function* or the *two-point function*) is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point). There are also model-T_ν traffic functions $G_\nu = (g_{\nu\alpha\beta})$ for $\nu = 1, 2, 3$.



Example.

$\sum_{p\geq 0}(1-T)^p = T^{-1}$



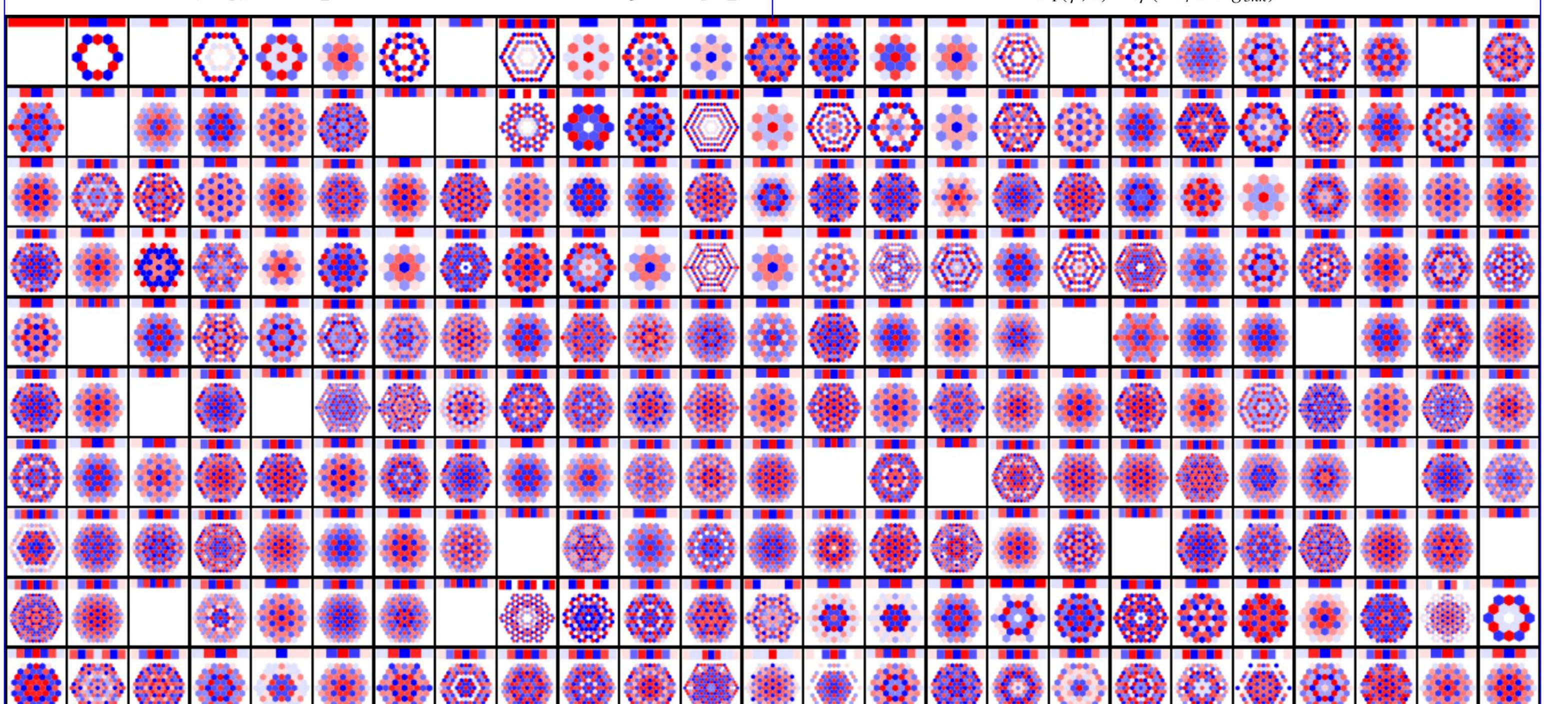
$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Don't Look.

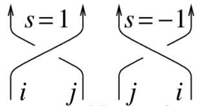
$$\begin{aligned} R_{11}(c) = s\Big[&1/2 - g_{3ii} + T_2^s g_{1ii} g_{2ji} - T_2^s g_{3jj} g_{2ji} - (T_2^s - 1) g_{3ii} g_{2ji} \\ &+ (T_3^s - 1) g_{2ji} g_{3ji} - g_{1ii} g_{2jj} + 2 g_{3ii} g_{2jj} + g_{1ii} g_{3jj} - g_{2ii} g_{3jj}\Big] \\ &+ \frac{s}{T_2^s - 1}\Big[(T_1^s - 1) T_2^s \left(g_{3jj} g_{1ji} - g_{2jj} g_{1ji} + T_2^s g_{1ji} g_{2ji}\right) \\ &+ (T_3^s - 1)\left(g_{3ji} - T_2^s g_{1ii} g_{3ji} + g_{2ij} g_{3ji} + (T_2^s - 2) g_{2jj} g_{3ji}\right) \\ &- (T_1^s - 1)(T_2^s + 1)(T_3^s - 1) g_{1ji} g_{3ji}\Big] \end{aligned}$$

$$R_{12}(c_0, c_1) = \frac{s_1 (T_1^{s_0} - 1)(T_3^{s_1} - 1) g_{1 j_1 i_0} g_{3 j_0 i_1}}{T_2^{s_1} - 1}\left(T_2^{s_0} g_{2 i_1 i_0} + g_{2 j_1 j_0} - T_2^{s_0} g_{2 j_1 i_0} - g_{2 i_1 j_0}\right)$$

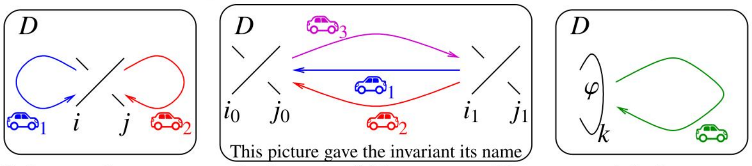
$$\Gamma_1(\varphi, k) = \varphi(-1/2 + g_{3kk})$$



Theorem. With $c=(s,i,j)$, $c_0=(s_0,i_0,j_0)$, and $c_1=(s_1,i_1,j_1)$ denoting crossings, there is a quadratic $R_{11}(c)\in\mathbb{Q}(T_\nu)[g_{\nu\alpha\beta}:\alpha,\beta\in\{i,j\}]$, a cubic $R_{12}(c_0,c_1)\in\mathbb{Q}(T_\nu)[g_{\nu\alpha\beta}:\alpha,\beta\in\{i_0,j_0,i_1,j_1\}]$, and a linear $\Gamma_1(\varphi,k)$ such that the following is a knot invariant:



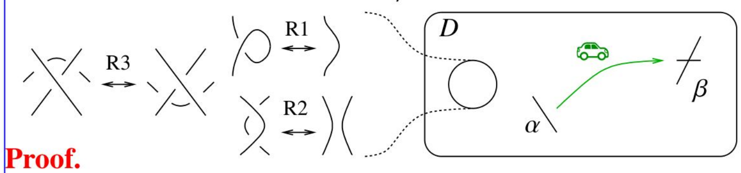
$$\theta(D) := \underbrace{\Delta_1\Delta_2\Delta_3}_{\substack{\text{normalization,}\\ \text{see later}}}\left(\sum_c R_{11}(c)+\sum_{c_0,c_1}R_{12}(c_0,c_1)+\sum_k\Gamma_1(\varphi_k,k)\right),$$



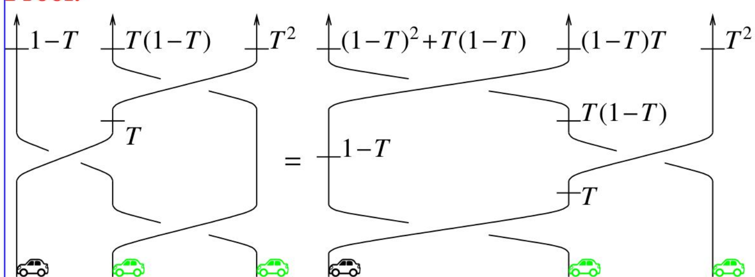
This picture gave the invariant its name

If these pictures remind you of Feynman diagrams, it's because they are Feynman diagrams [BN2].

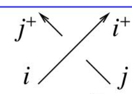
Lemma 1. The traffic function $g_{\alpha\beta}$ is a "relative invariant":



Proof.



Lemma 2. With $k^+ := k+1$, the "g-rules" hold near a crossing $c=(s,i,j)$:



$g_{j\beta}=g_{j^+\beta}+\delta_{j\beta}\quad g_{i\beta}=T^s g_{i^+\beta}+(1-T^s)g_{j^+\beta}+\delta_{i\beta}\quad g_{2n^+,\beta}=\delta_{2n^+,\beta}$

$g_{\alpha i^+}=T^s g_{\alpha i}+\delta_{\alpha i^+}\quad g_{\alpha j^+}=g_{\alpha j}+(1-T^s)g_{\alpha i}+\delta_{\alpha j^+}\quad g_{\alpha,1}=\delta_{\alpha,1}$

Corollary 1. G is easily computable, for $AG=I\ (=GA)$, with A the $(2n+1)\times(2n+1)$ identity matrix with additional contributions:

$$c=(s,i,j)\mapsto\begin{array}{c|cc} A & \text{col } i^+ & \text{col } j^+ \\ \hline \text{row } i & -T^s & T^s-1 \\ \text{row } j & 0 & -1\end{array}$$

For the trefoil example, we have:

$$A=\begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -T & 0 & 0 & T-1\\ 0 & 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & T-1 & 0 & 1 & -T & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

$$G=\begin{pmatrix} 1 & T & 1 & T & 1 & T & 1\\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1\\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1\\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1\\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1\\ 0 & 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

Note. The Alexander polynomial Δ is given by

$$\Delta=T^{(-\varphi-w)/2}\det(A),\qquad \text{with } \varphi=\textstyle\sum_k\varphi_k,\ w=\sum_c s.$$

We also set $\Delta_\nu := \Delta(T_\nu)$ for $\nu=1,2,3$.

Questions, Conjectures, Expectations, Dreams.

Question 1. What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

Conjecture 2. On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

Conjecture 3. θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinery on the double $\mathcal{D}(\mathfrak{b},b,\epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

Conjecture 4. θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $g(K)<\lceil\deg_{T_1}\theta(K)\rceil$.

Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

Expectation 8. There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

Dream 9. These invariants can be explained by something less foreign than semisimple Lie algebras.

Dream 10. θ will have something to say about ribbon knots.

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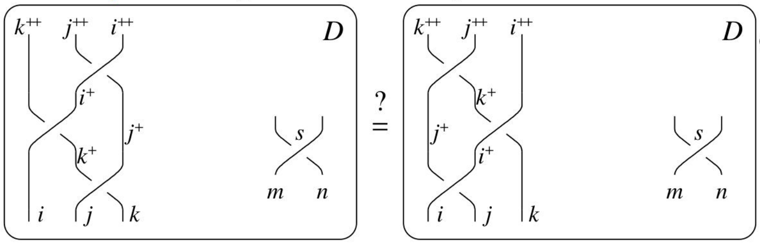
[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Corollary 2. Proving invariance is easy:



Invariance under R3

This is Theta.nb of http://drorbn.net/to24/ap.

```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

```
T3 = T1 T2;
```

```
CF[ℰ_] :=
  Module[{vs = Union@Cases[ℰ, g__, ∞], ps, c},
   Total[CoefficientRules[Expand[ℰ], vs] /.
      (ps_ → c_) ⧴ Factor[c] (Times @@ vs^ps)]];
```

```
R11[{s_, i_, j_}] := R11[s, i_, j_] =
  CF[
   s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj -
      (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
      g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
      ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
        (T3^s - 1) g3ji
         (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji +
           (T2^s - 2) g2jj + g2ij)) / (T2^s - 1))];
```

```
R12[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=      as above.
  CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
    ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
```

```
Γ1[φ_, k_] = -φ/2 + φ g3kk;
```

```
δi_,j_ := If[i === j, 1, 0];
gRs_,i_,j_ := {
   gν_jβ_ ⧴ gνj+β + δjβ,
   gν_iβ_ ⧴ Tν^s gνi+β + (1 - Tν^s) gνj+β + δiβ,
   gν_α_i+ ⧴ Tν^s gναi + δαi+,
   gν_α_j+ ⧴ gναj + (1 - Tν^s) gναi + δαj+
  }
```

```
DSum[Cs___] := Sum[R11[c], {c, {Cs}}] +
  Sum[R12[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k+}, {1, i+, j+},
    {s, m, n}] //. gR1,j,k ⋃ gR1,i,k+ ⋃ gR1,i+,j+;
rhs = DSum[{1, i, j}, {1, i+, k}, {1, j+, k+},
    {s, m, n}] //. gR1,i,j ⋃ gR1,i+,k ⋃ gR1,j+,k+;
Simplify[lhs == rhs]
```

```
True
```

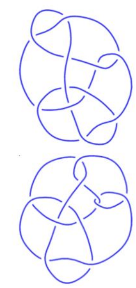
The Main Program

```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴
     (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                        0     -1    ))];
   Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] :=
    Factor[ℰ /. gν_,α_,β_ ⧴ (G[[α, β]] /. T → Tν)];
   θ = ev[Sum_{k1=1}^n Sum_{k2=1}^n R12[Cs[[k1]], Cs[[k2]]]];
   θ += ev[Sum_{k=1}^n R11[Cs[[k]]]];
   θ += ev[Sum_{k=1}^{2 n} Γ1[φ[[k]], k]];
   Factor@
    {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}];
```

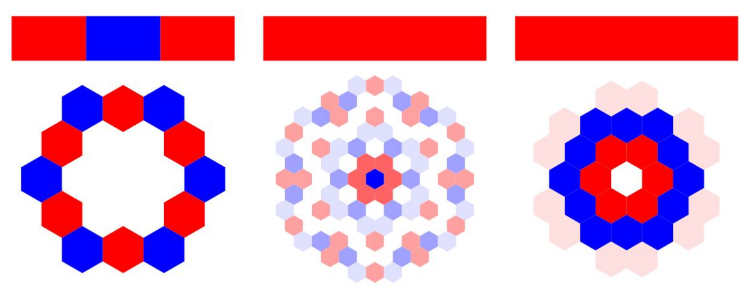
The Trefoil, Conway, and Kinoshita-Terasaka

```
Θ[Knot[3, 1]] // Expand
```

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$



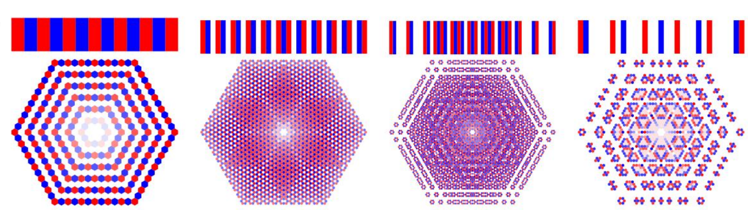
```
GraphicsRow[PolyPlot[Θ[Knot[#]]] & /@
  {"3_1", "K11n34", "K11n42"}]
```

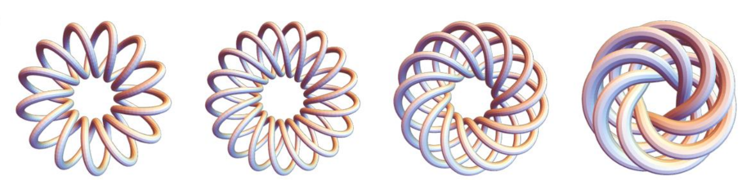


(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

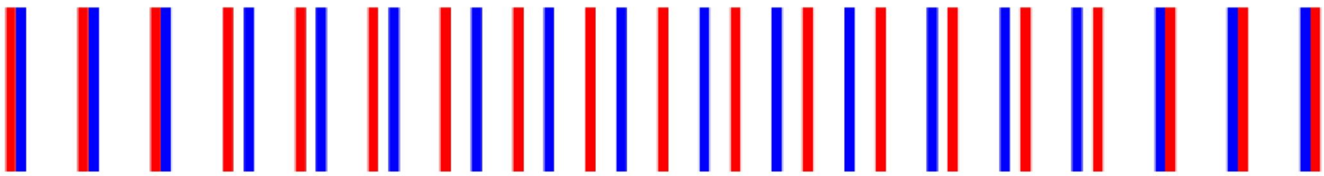
Some Torus Knots

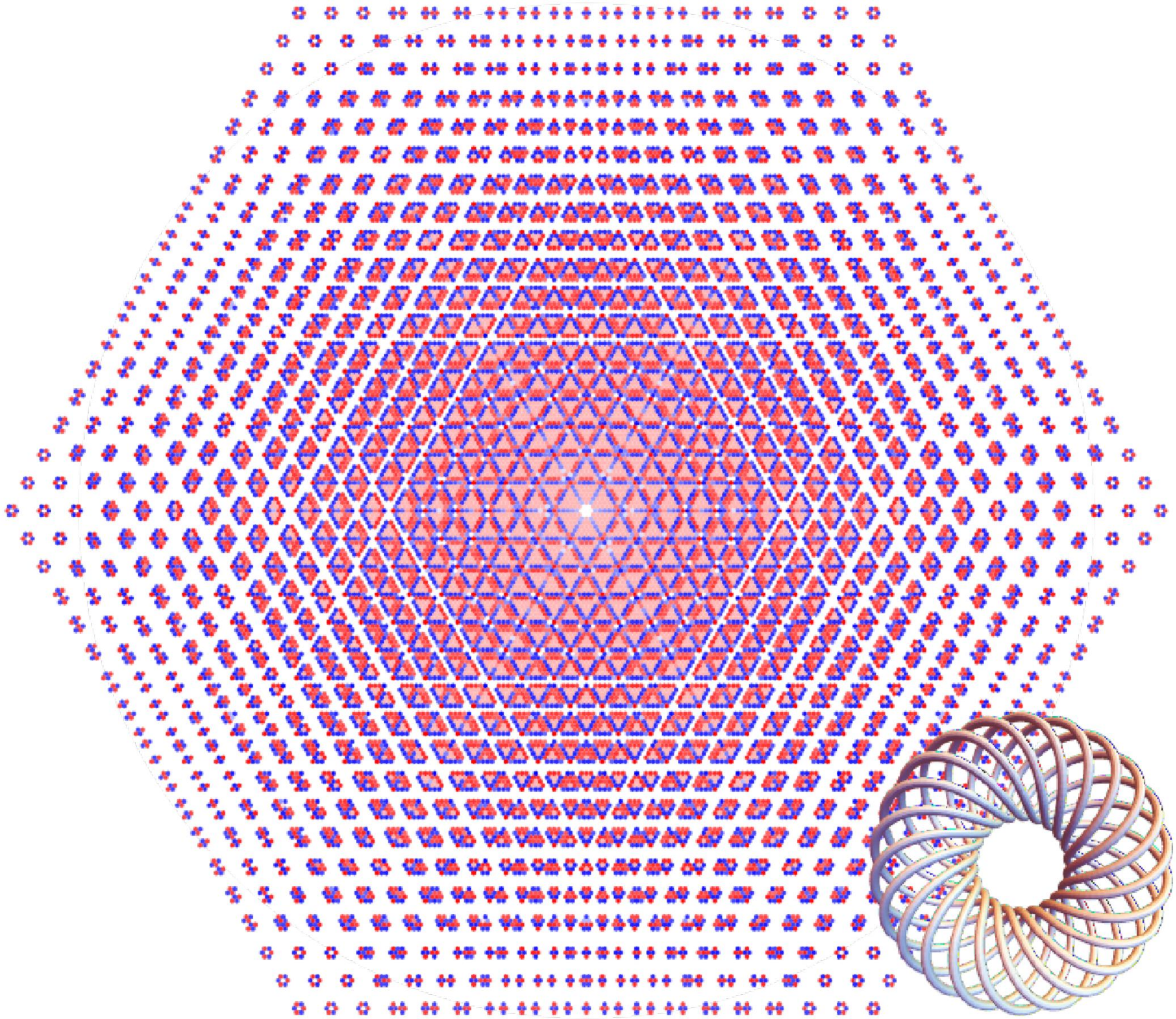
```
TKs = {{13, 2}, {17, 3}, {13, 5}, {7, 6}};
GraphicsRow[PolyPlot[Θ[TorusKnot @@ #]] & /@ TKs]
GraphicsRow[TubePlot[TorusKnot @@ #] & /@ TKs]
```





The 132-crossing torus knot $T_{22/7}$: (many more at ωεβ/TK)





Random knots from [DHOEBL], with 50-73 crossings: (many more at ωεβ/DK)