# Locally Euclidean Knotted Objects, Meta-Hopf Algebras, and Circuit Algebras

Handout, video, links at ωεβ:=http://drorbn.net/hef18/

Thanks for inviting me to Hefei!

**Abstract.** Seeing that I have nothing to say about operads, I'll talk about other "generalized algebraic structures" that I like. Specifically, I will explain what are locally Euclidean knotted objects and how they form a "meta-Hopf algebra" (along the way explaining what is that latter notion). I will then describe the "circuit algebra" of linearized circuits and explain how to use it to construct a "Yang-Baxter meta-Hopf algebra" which generalizes the Alexander polynomial. I will have no time to explain, yet I'll sketch, how "solvable approximation of semi-simple Lie algebras" lead to more sophisticated Yang-Baxter meta-Hopf algebras which lead to very powerful poly-time computable knot polynomials.

## What we care for in knot theory.

Knotted Candies by Dror Bar-Natan (2008)
based on data by Brian Gilbert
http://drorbn.net/ap/2008-12/

Pensieve header: Derived from pensieve://Projects/SL2Invariant/Archive/nb/SL2Invariant-180811.pdf.

# Program

## Internal Utilities

Canonical Form:

```
CF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CF[x]];
```

The Kronecker $\delta$:

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] :=
  𝔼[L1 + L2, Q1 + Q2, P1 * P2];
```

## Zip and Bind

Variables and their duals:

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};
(u_i_)* := (u*)_i;
```

Finite Zips:

```
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P; Zip_{ζ_,ζs___}[P_] :=
  (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :> ∂_{ζ*,d} f) /. ζ* → 0
```

QZip implements the "$Q$-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the $L$ variables as scalars.

```
QZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   CF /@ 𝔼[L, Q2, Det[qt] e^(-Q2) Zip_{ζs}[e^Q1 (P /. zrule)]] ];
```

Upper to lower and lower to Upper:

```
U2l = {B_i_^p_. → e^(-p ħ γ b_i), B^p_. → e^(-p ħ γ b), T_i_^p_. → e^(p ħ t_i),
   T^p_. → e^(p ħ t), 𝒜_i_^p_. → e^(p γ α_i), 𝒜^p_. → e^(p γ α)};
l2U = {e^(c_. b_i_ + d_.) :> B_i^(-c/(ħ γ)) e^d, e^(c_. b + d_.) :> B^(-c/(ħ γ)) e^d,
   e^(c_. t_i_ + d_.) :> T_i^(c/ħ) e^d, e^(c_. t + d_.) :> T^(c/ħ) e^d,
   e^(c_. α_i_ + d_.) :> 𝒜_i^(c/γ) e^d, e^(c_. α + d_.) :> 𝒜^(c/γ) e^d,
   e^ℰ_ :> e^Expand@ℰ};
```

LZip implements the "$L$-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of $Pe^{Q}$ as a single"$P$". Here the $z$'s are $b$ and $\alpha$ and the $\zeta$'s are $\beta$ and $a$.

```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
   CF /@ 𝔼[L2, Q2, Det[lt] e^(-L2-Q2)
      Zip_{ζs}[e^(L1+Q1) (P /. U2l /. zrule)]] //. l2U ];
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n},
  Times[
    L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i},
     {i, {is}}],
    R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
   ] // LZip_{Join@@Table[{β_{n@i}, τ_{n@i}, a_{n@i}}, {i, {is}}]} //
   QZip_{Join@@Table[{ξ_{n@i}, y_{n@i}}, {i, {is}}]} ];
B_is___[L_, R_] := B_{is}[L, R];
m_{i_,j_→k_} := 𝔼[a_k α_i + a_k α_j + b_k β_i + b_k β_j,
   1/(ħ 𝒜_i 𝒜_j) (ħ y_k 𝒜_i 𝒜_j η_i + ħ y_k 𝒜_j η_j + ħ x_k 𝒜_i ξ_i + 𝒜_i 𝒜_j η_j ξ_i -
     B_k 𝒜_i 𝒜_j η_j ξ_i + ħ x_k 𝒜_i 𝒜_j ξ_j), 1];
R_{i_,j_} := 𝔼[ħ a_j b_i, ħ x_j y_i, 1];
R̄_{i_,j_} := 𝔼[-ħ a_j b_i, -(ħ x_j y_i)/B_i, 1];
S_{i_} := 𝔼[-a_i α_i - b_i β_i,
   1/(ħ B_i) (-ħ y_i 𝒜_i η_i - ħ B_i x_i 𝒜_i ξ_i + 𝒜_i η_i ξ_i - B_i 𝒜_i η_i ξ_i), 1];
Δ_{i_→j_,k_} := 𝔼[a_j α_i + a_k α_i + b_j β_i + b_k β_i,
   y_j η_i + B_j y_k η_i + x_j ξ_i + x_k ξ_i, 1];
C_{i_} := 𝔼[0, 0, B_i^(1/2)];
C̄_{i_} := 𝔼[0, 0, B_i^(-1/2)];
Kink_{i_} := (R_{1,3} C̄_2) ~B_{1,2}~ m_{1,2→1} ~B_{1,3}~ m_{1,3→i};
K̄ink_{i_} := (R̄_{1,3} C_2) ~B_{1,2}~ m_{1,2→1} ~B_{1,3}~ m_{1,3→i};
```

# Testing

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

(co)-associativity

```
HL /@ {(Δ_{1→1,2} ~B_2~ Δ_{2→2,3}) ≡ (Δ_{1→1,3} ~B_1~ Δ_{1→1,2}),
   (m_{1,2→1} ~B_1~ m_{1,3→1}) ≡ (m_{2,3→2} ~B_2~ m_{1,2->1})}
```

{True, True}

$\Delta$ is an algebra morphism

```
HL[m_{1,2→1} ~B_1~ Δ_{1→1,2} ≡ (Δ_{1→1,3} Δ_{2→2,4}) ~B_{1,2,3,4}~ (m_{3,4→2} m_{1,2→1})]
```

True

$S_2$ inverts $R$, but not $S_1$:

```
{R_{1,2} ~B_1~ S_1 ≡ R̄_{1,2}, HL[R_{1,2} ~B_2~ S_2 ≡ R̄_{1,2}]}
```

{True, True}

$S$ is convolution inverse of id

```
HL[# ≡ 𝔼[0, 0, 1]] & /@
  {(Δ_{1→1,2} ~B_1~ S_1) ~B_{1,2}~ m_{1,2→1}, (Δ_{1→1,2} ~B_2~ S_2) ~B_{1,2}~ m_{1,2→1}}
```

{True, True}

$S$ is a (co)-algebra anti-morphism

$$\mathrm{HL}\ /@\ \mathrm{Expand}\ /@\ \{m_{1,2\to1}\sim B_1\sim S_1 \equiv (S_1\, S_2)\sim B_{1,2}\sim m_{2,1\to1},\ S_1\sim B_1\sim \Delta_{1\to1,2} \equiv \Delta_{1\to2,1}\sim B_{1,2}\sim (S_1\, S_2)\}$$

{True, True}

Quasi-triangular axiom 1:

$$\mathrm{HL}[R_{1,2}\sim B_1\sim \Delta_{1\text{->}1,3} \equiv (R_{1,4}\, R_{3,2})\sim B_{2,4}\sim m_{2,4\to2}]$$

True

Quasi-triangular axiom 2:

$$\mathrm{HL}[((\Delta_{1\text{->}1,2}\, R_{3,4})\sim B_{1,2,3,4}\sim (m_{1,3\to1}\, m_{2,4\to2})) \equiv ((\Delta_{1\to2,1}\, R_{3,4})\sim B_{1,2,3,4}\sim (m_{3,1\to1}\, m_{4,2\to2}))]$$

True

$$\mathrm{HL}\ /@\ \{(C_i\, \overline{C}_j)\sim B_{i,j}\sim m_{i,j\to i} \equiv \mathbb{E}[0,0,1],\ (\overline{C}_i\, \overline{C}_j)\sim B_{i,j}\sim m_{i,j\to i} \equiv (((R_{1,2}\sim B_1\sim S_1\sim B_{1,2}\sim m_{2,1\to i})\sim B_i\sim S_i)\ (R_{1,2}\sim B_2\sim S_2\sim B_2\sim S_2\sim B_{1,2}\sim m_{2,1\to j}))\sim B_{i,j}\sim m_{i,j\to i}\}$$

{True, True}

Reidemeister 2:

$$\mathrm{HL}[\# \equiv \mathbb{E}[0,0,1]\ \&\ /@\ \{(\overline{R}_{1,2}\, R_{3,4})\sim B_{1,2,3,4}\sim (m_{1,3\to1}\, m_{2,4\to2}),\ (R_{1,2}\, \overline{R}_{3,4})\sim B_{1,2,3,4}\sim (m_{1,3\to1}\, m_{2,4\to2})\}$$

{True, True}

Cyclic Reidemeister 2:

$$\mathrm{HL}[(R_{1,4}\, \overline{R}_{5,2}\, \overline{C}_3)\sim B_{2,4}\sim m_{2,4\to2}\sim B_{1,3}\sim m_{1,3\to1}\sim B_{1,5}\sim m_{1,5\to1} \equiv \overline{C}_1]$$

True

Reidemeister 3:

$$\mathrm{HL}[((R_{1,2}\, R_{4,3}\, R_{5,6})\sim B_{1,4}\sim m_{1,4\to1}\sim B_{2,5}\sim m_{2,5\to2}\sim B_{3,6}\sim m_{3,6\to3}) \equiv ((R_{1,6}\, R_{2,3}\, R_{4,5})\sim B_{1,4}\sim m_{1,4\to1}\sim B_{2,5}\sim m_{2,5\to2}\sim B_{3,6}\sim m_{3,6\to3})]$$

True

The Trefoil

$$Z = R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{C}_4\, \overline{\mathrm{Kink}}_8\, \overline{\mathrm{Kink}}_9\, \overline{\mathrm{Kink}}_{10};$$

$$\mathrm{Do}[Z = Z\sim B_{1,r}\sim m_{1,r\to1},\ \{r, 2, 10\}];$$

Simplify /@ Z

$$\mathbb{E}\left[0,\ 0,\ \frac{B_1}{1 - B_1 + B_1^2}\right]$$