Pensieve header: Will it work with an imaginary integrand? (Maybe, but not here).

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Groningen-240530"];
```

pdf

Preliminaries

pdf

In[∘]:=
```
Once[<< KnotTheory`; << Rot.m];
```

pdf

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/AP/Talks/Groningen-240530 to compute rotation numbers.
```

pdf

In[∘]:=
```
CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
   vs = Cases[ℰ, (x | p | ξ | π)__, ∞] ⋃ {x, p, ϵ};
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ↦ Factor[c] (Times @@ vs^ps)] ];
```

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Integration

pdf

In[∘]:=
```
𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B];
```

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In[∘]:=
```
$π = Identity; (* hacks in pink *)
```

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In[∘]:=
```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, FZ, a, b},
    n = Length@vs; L0 = L /. ϵ → 0;
    Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /. (p | x)__ → 0, {a, n}, {b, n}];
    If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
    Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
    DZ_a_ := ∂_{vs⟦a⟧} Z; DZ_{a_,b_} := ∂_{vs⟦b⟧} DZ_a;
    FZ := CF@$π[1/2 Σ_{a=1}^{n} Σ_{b=1}^{n} G⟦a, b⟧ (DZ_{a,b} + DZ_a DZ_b)];
    FixedPoint[(Z = Z0 + ∫_0^λ FZ ⅆλ) &, Z];
    PowerExpand@Factor[ω Δ^{-1/2}] 𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

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The Right-Handed Trefoil

pdf

In[∘]:=
```
K = Mirror@Knot[3, 1]; Features[K]
```

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KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

Features[7, $C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]$]

pdf

In[∘]:=
```
ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[
    x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
     (ϵ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (-x_j p_j)) + 1)]
ℒ[C_i_[0]] := 𝔼[x_i (p_{i+1} - p_i)];
ℒ[C_i_[1]] := T^{1/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ (c_1 + c_2 x_i^2 p_i^2)];
ℒ[C_i_[-1]] := T^{-1/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ (c_3 + c_4 x_i^2 p_i^2)];
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧]
vs[K_] := Join @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}]
```

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In[∘]:= {vs[K], ℒ[K]}

Out[∘]=
pdf

$$\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[\frac{1}{2}\in(3+2c_3) - p_1 x_1 + T p_2 x_1 + (1-T)\,p_6 x_1 + \frac{1}{2}(-1+T)\in p_1 p_5 x_1^2 + \frac{1}{2}(1-T)\in p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 - p_3 x_3 + T p_4 x_3 + (1-T)\,p_8 x_3 + \frac{1}{2}(-1+T)\in p_3 p_7 x_3^2 + \frac{1}{2}(1-T)\in p_7^2 x_3^2 - p_4 x_4 + p_5 x_4 + \in c_4 p_4^2 x_4^2 - p_5 x_5 + p_6 x_5 - \in p_1 p_5 x_1 x_5 + \in p_5^2 x_1 x_5 + (1-T)\,p_3 x_6 - p_6 x_6 + T p_7 x_6 + \in p_2^2 x_2 x_6 - \in p_2 p_6 x_2 x_6 + \frac{1}{2}(1-T)\in p_2^2 x_6^2 + \frac{1}{2}(-1+T)\in p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \in p_3 p_7 x_3 x_7 + \in p_7^2 x_3 x_7\Big]\}$$

pdf

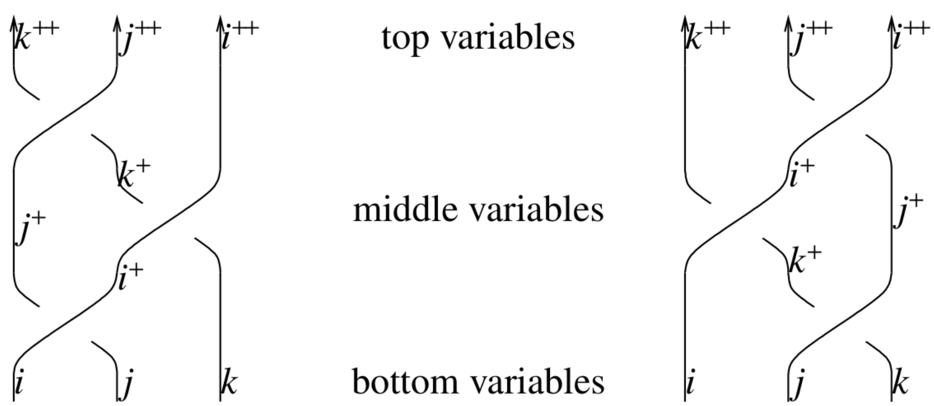
In[∘]:= $π = Normal[# + O[ϵ]^2] &; ∫ℒ[K] ⅆ(vs@K)

Out[∘]=
pdf

$$-\frac{\mathbb{i}\,T\,\mathbb{E}\left[\frac{\in(-3+2T+3T^2-4T^3+3T^4+2c_3-4Tc_3+6T^2c_3-4T^3c_3+2T^4c_3+4c_4)}{2(1-T+T^2)^2}\right]}{1-T+T^2}$$

pdf

Invariance Under Reidemeister 3



pdf

In[∘]:= lhs = $\int(\mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j + \mathbb{i}\,\pi_k\,p_k]$ ℒ /@ $(X_{i,j}[1]\;X_{i+1,k}[1]\;X_{j+1,k+1}[1]))$
ⅆ{$p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}$};
rhs = $\int(\mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j + \mathbb{i}\,\pi_k\,p_k]$ ℒ /@ $(X_{j,k}[1]\;X_{i,k+1}[1]\;X_{i+1,j+1}[1]))$
ⅆ{$p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}$};
lhs == rhs

Out[∘]=
pdf

True

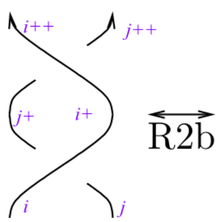
pdf

In[∘]:=
```
lhs
```

Out[∘]=
pdf

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2}+i\,T^2 p_{2+i}\pi_i-2\,i\,T^2\epsilon\,p_{2+i}\pi_i-i\,(-1+T)\,T\,p_{2+j}\pi_i+i\,T\,(-1+3T)\,\epsilon\,p_{2+j}\pi_i-i\,(-1+T)\,p_{2+k}\pi_i+2\,i\,T\,\epsilon\,p_{2+k}\pi_i-\frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+i}p_{2+j}\pi_i^2+\frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+j}^2\pi_i^2-\frac{1}{2}(-1+T)\,T^2\epsilon\,p_{2+i}p_{2+k}\pi_i^2+\frac{1}{2}(-1+T)^2\,T\,\epsilon\,p_{2+j}p_{2+k}\pi_i^2+\frac{1}{2}(-1+T)\,T\,\epsilon\,p_{2+k}^2\pi_i^2+i\,T\,p_{2+j}\pi_j-2\,i\,T\,\epsilon\,p_{2+j}\pi_j-i\,(-1+T)\,p_{2+k}\pi_j+i\,(-1+3T)\,\epsilon\,p_{2+k}\pi_j+T^3\epsilon\,p_{2+i}p_{2+j}\pi_i\pi_j-T^3\epsilon\,p_{2+j}^2\pi_i\pi_j-(-1+T)\,T^2\epsilon\,p_{2+i}p_{2+k}\pi_i\pi_j+(-1+T)^2\,T\,\epsilon\,p_{2+j}p_{2+k}\pi_i\pi_j+(-1+T)\,T\,\epsilon\,p_{2+k}^2\pi_i\pi_j-\frac{1}{2}(-1+T)\,T\,\epsilon\,p_{2+j}p_{2+k}\pi_j^2+\frac{1}{2}(-1+T)\,T\,\epsilon\,p_{2+k}^2\pi_j^2+i\,p_{2+k}\pi_k-2\,i\,\epsilon\,p_{2+k}\pi_k+T^2\epsilon\,p_{2+i}p_{2+k}\pi_i\pi_k-(-1+T)\,T\,\epsilon\,p_{2+j}p_{2+k}\pi_i\pi_k-T\,\epsilon\,p_{2+k}^2\pi_i\pi_k+T\,\epsilon\,p_{2+j}p_{2+k}\pi_j\pi_k-T\,\epsilon\,p_{2+k}^2\pi_j\pi_k\Big]$$

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=
```
lhs = ∫𝔼[i πi pi + i πj pj] ℒ /@ (Xi,j[1] Xi+1,j+1[-1]) d{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1}
rhs =
  ∫𝔼[i πi pi + i πj pj] ℒ /@ (Ci[0] Ci+1[0] Cj[0] Cj+1[0]) d{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1};
lhs == rhs
```

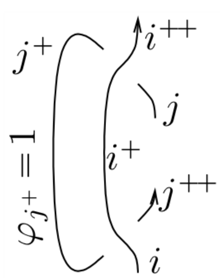
Out[∘]=

$$\mathbb{E}[i\,p_{2+i}\pi_i+i\,p_{2+j}\pi_j]$$

Out[∘]=

True

Invariance Under R2c



In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
lhs == rhs
```

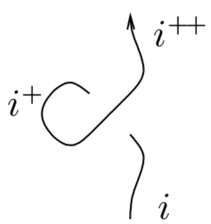
Out[∘]=

$$-\mathbb{i}\sqrt{T}\,\mathbb{E}\Big[\in(c_1+2c_2)+\mathbb{i}\,p_{2+i}\,\pi_i-\frac{\mathbb{i}\,(-1+T)\in(1+4c_2)\,p_{3+j}\,\pi_i}{T}-\frac{(-1+T)^2\in c_2\,p_{3+j}^2\,\pi_i^2}{T^2}+\mathbb{i}\,p_{3+j}\,\pi_j+4\,\mathbb{i}\in c_2\,p_{3+j}\,\pi_j+\frac{2\,(-1+T)\in c_2\,p_{3+j}^2\,\pi_i\,\pi_j}{T}-\in c_2\,p_{3+j}^2\,\pi_j^2\Big]$$

Out[∘]=

$$-\mathbb{i}\sqrt{T}\,\mathbb{E}\Big[\in(c_1+2c_2)+\mathbb{i}\,p_{2+i}\,\pi_i-\frac{\mathbb{i}\,(-1+T)\in(1+4c_2)\,p_{3+j}\,\pi_i}{T}-\frac{(-1+T)^2\in c_2\,p_{3+j}^2\,\pi_i^2}{T^2}+\mathbb{i}\,p_{3+j}\,\pi_j+4\,\mathbb{i}\in c_2\,p_{3+j}\,\pi_j+\frac{2\,(-1+T)\in c_2\,p_{3+j}^2\,\pi_i\,\pi_j}{T}-\in c_2\,p_{3+j}^2\,\pi_j^2\Big] ==$$
$$-\mathbb{i}\sqrt{T}\,\mathbb{E}\big[\in(c_1+2c_2)+\mathbb{i}\,p_{2+i}\,\pi_i+\mathbb{i}\,p_{3+j}\,\pi_j+4\,\mathbb{i}\in c_2\,p_{3+j}\,\pi_j-\in c_2\,p_{3+j}^2\,\pi_j^2\big]$$

Invariance Under R1l



In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i] ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[ⅈ π_i p_i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs
```

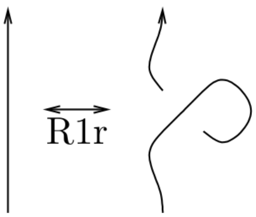
Out[∘]=

$$-\mathbb{i}\,\mathbb{E}\Big[\frac{\in\left(2T+T^2+2T^2c_1+4c_2\right)}{2T^2}+\mathbb{i}\,p_{3+i}\,\pi_i+\frac{\mathbb{i}\in(T+4c_2)\,p_{3+i}\,\pi_i}{T^2}-\frac{\in c_2\,p_{3+i}^2\,\pi_i^2}{T^2}\Big]$$

Out[∘]=

$$-\mathbb{i}\,\mathbb{E}\Big[\frac{\in\left(2T+T^2+2T^2c_1+4c_2\right)}{2T^2}+\mathbb{i}\,p_{3+i}\,\pi_i+\frac{\mathbb{i}\in(T+4c_2)\,p_{3+i}\,\pi_i}{T^2}-\frac{\in c_2\,p_{3+i}^2\,\pi_i^2}{T^2}\Big] == -\mathbb{i}\,\mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Invariance Under R1r



In[ ]:=
```
lhs = ∫𝔼[𝕚 π_i p_i] ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) 𝕕{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[𝕚 π_i p_i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) 𝕕{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs
```

Out[ ]=

$$-i\,\mathbb{E}\left[\frac{1}{2}\in(-1+2c_3+4c_4)+i\,p_{3+i}\,\pi_i+i\,T\in(-1+4c_4)\,p_{3+i}\,\pi_i-T^2\in c_4\,p_{3+i}^2\,\pi_i^2\right]$$

Out[ ]=

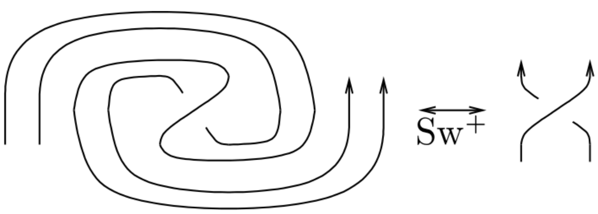
$$-i\,\mathbb{E}\left[\frac{1}{2}\in(-1+2c_3+4c_4)+i\,p_{3+i}\,\pi_i+i\,T\in(-1+4c_4)\,p_{3+i}\,\pi_i-T^2\in c_4\,p_{3+i}^2\,\pi_i^2\right]==-i\,\mathbb{E}[i\,p_{3+i}\,\pi_i]$$

In[ ]:=
```
(lhs == rhs) /. {c_5 → 0}
```

Out[ ]=

$$-i\,\mathbb{E}\left[\frac{1}{2}\in(-1+2c_3+4c_4)+i\,p_{3+i}\,\pi_i+i\,T\in(-1+4c_4)\,p_{3+i}\,\pi_i-T^2\in c_4\,p_{3+i}^2\,\pi_i^2\right]==-i\,\mathbb{E}[i\,p_{3+i}\,\pi_i]$$

Invariance Under Sw



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[i\,\pi_i\,p_i + i\,\pi_j\,p_j]\ \mathcal{L}\ \text{/@}\ (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])\ d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\}$$

$$\text{rhs} = \int \mathbb{E}[i\,\pi_i\,p_i + i\,\pi_j\,p_j]\ \mathcal{L}\ \text{/@}\ (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])\ d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\};$$

$$\text{lhs} == \text{rhs}$$

Out[∘]=

$$\sqrt{T}\ \mathbb{E}\Big[\frac{1}{2}\in(-1+4c_1+8c_2+4c_3+8c_4) + i\,T\,p_{3+i}\,\pi_i + i\,T\in(-1+4c_2+4c_4)\,p_{3+i}\,\pi_i - i\,(-1+T)\,p_{3+j}\,\pi_i - 2\,i\in(-T-2c_2+2Tc_2-2c_4+2Tc_4)\,p_{3+j}\,\pi_i - T^2\in(c_2+c_4)\,p_{3+i}^2\,\pi_i^2 + \frac{1}{2}(-1+T)\,T\in(-1+4c_4)\,p_{3+i}\,p_{3+j}\,\pi_i^2 - \frac{1}{2}(-1+T)\in(-T-2c_2+2Tc_2-2c_4+2Tc_4)\,p_{3+j}^2\,\pi_i^2 + i\,p_{3+j}\,\pi_j + i\in(-1+4c_2+4c_4)\,p_{3+j}\,\pi_j + T\in p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j + \in(-T-2c_2+2Tc_2)\,p_{3+j}^2\,\pi_i\,\pi_j + \in(-c_2-c_4)\,p_{3+j}^2\,\pi_j^2\Big]$$

Out[∘]=

$$\sqrt{T}\ \mathbb{E}\Big[\frac{1}{2}\in(-1+4c_1+8c_2+4c_3+8c_4) + i\,T\,p_{3+i}\,\pi_i + i\,T\in(-1+4c_2+4c_4)\,p_{3+i}\,\pi_i - i\,(-1+T)\,p_{3+j}\,\pi_i - 2\,i\in(-T-2c_2+2Tc_2-2c_4+2Tc_4)\,p_{3+j}\,\pi_i - T^2\in(c_2+c_4)\,p_{3+i}^2\,\pi_i^2 + \frac{1}{2}(-1+T)\,T\in(-1+4c_4)\,p_{3+i}\,p_{3+j}\,\pi_i^2 - \frac{1}{2}(-1+T)\in(-T-2c_2+2Tc_2-2c_4+2Tc_4)\,p_{3+j}^2\,\pi_i^2 + i\,p_{3+j}\,\pi_j + i\in(-1+4c_2+4c_4)\,p_{3+j}\,\pi_j + T\in p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j + \in(-T-2c_2+2Tc_2)\,p_{3+j}^2\,\pi_i\,\pi_j + \in(-c_2-c_4)\,p_{3+j}^2\,\pi_j^2\Big] ==$$

$$\sqrt{T}\ \mathbb{E}\Big[-\frac{\in}{2} + i\,T\,p_{3+i}\,\pi_i - i\,T\in p_{3+i}\,\pi_i - i\,(-1+T)\,p_{3+j}\,\pi_i + 2\,i\,T\in p_{3+j}\,\pi_i - \frac{1}{2}(-1+T)\,T\in p_{3+i}\,p_{3+j}\,\pi_i^2 + \frac{1}{2}(-1+T)\,T\in p_{3+j}^2\,\pi_i^2 + i\,p_{3+j}\,\pi_j - i\in p_{3+j}\,\pi_j + T\in p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j - T\in p_{3+j}^2\,\pi_i\,\pi_j\Big]$$