Pensieve header: Mathematica notebook for Talks: Groningen-240530.

Ancestors in Projects/HigherRank.

exec

```
nb2tex$TeXFileName = "IType1.tex";
```

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Groningen-240530"];
```

Pensieve header: Implementing ρ_1, and also ρ_d.

pdf

Preliminaries

tex

This is IType.nb of \surl{drorbn.net/g24/ap}.

pdf

In[∘]:=
```
Once[<< KnotTheory`; << Rot.m];
```

pdf

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/AP/Talks/Groningen-240530 to compute rotation numbers.
```

pdf

In[∘]:=
```
CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
   vs = Cases[ℰ, (x | p | ξ | π)__, ∞] ⋃ {x, p, ϵ};
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ Factor[c] (Times @@ vs^ps)] ];
```

tex

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pdf

Integration

pdf

Using Picard Iteration!

pdf

In[∘]:=
```
𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B];
```

pdf

In[∘]:=
```
$π = Identity; (* hacks in pink *)
```

pdf

In[∘]:=
```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, FZ, a, b},
   n = Length@vs; L0 = L /. ϵ → 0;
   Q = Table[(-∂_{vs[[a]],vs[[b]]} L0) /. Thread[vs → 0] /. (p | x)__ → 0, {a, n}, {b, n}];
   If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
   Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
   DZ_a_ := ∂_{vs[[a]]} Z; DZ_{a_,b_} := ∂_{vs[[b]]} DZ_a;
   FZ := CF@$π[1/2 Sum_{a=1}^{n} Sum_{b=1}^{n} G[[a, b]] (DZ_{a,b} + DZ_a DZ_b)];
   FixedPoint[(Z = Z0 + ∫_0^λ FZ ⅆλ) &, Z];
   PowerExpand@Factor[ω Δ^{-1/2}] 𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

pdf

In[∘]:= $\int \mathbb{E}\left[-\mu x^2/2 + i\,\xi\, x\right] ⅆ\{x\}$

Out[∘]=
pdf

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

pdf

In[∘]:= $L = -\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$

$Z12 = \int \mathbb{E}[L]\, ⅆ\{x_1, x_2\}$

Out[∘]=
pdf

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\left(-b^2+a\,c\right)} + \frac{b\,\xi_1\,\xi_2}{b^2-a\,c} + \frac{a\,\xi_2^2}{2\left(-b^2+a\,c\right)}\right]}{\sqrt{-b^2+a\,c}}$$

pdf

In[∘]:= $\left\{Z1 = \int \mathbb{E}[L]\, ⅆ\{x_1\},\ Z12 == \int Z1\, ⅆ\{x_2\}\right\}$

Out[∘]=
pdf

$$\left\{\frac{\mathbb{E}\left[-\frac{\left(-b^2+a\,c\right)x_2^2}{2\,a} - \frac{b\,x_2\,\xi_1}{a} + \frac{\xi_1^2}{2\,a} + x_2\,\xi_2\right]}{\sqrt{a}},\ \text{True}\right\}$$

pdf

In[∘]:=
```
$π = Normal[# + O[ϵ]^13] &; ∫𝔼[-ϕ^2 / 2 + ϵ ϕ^3 / 6] ⅆ{ϕ}
```

Out[∘]=
pdf

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24}+\frac{5\,\epsilon^4}{16}+\frac{1105\,\epsilon^6}{1152}+\frac{565\,\epsilon^8}{128}+\frac{82\,825\,\epsilon^{10}}{3072}+\frac{19\,675\,\epsilon^{12}}{96}\right]$$

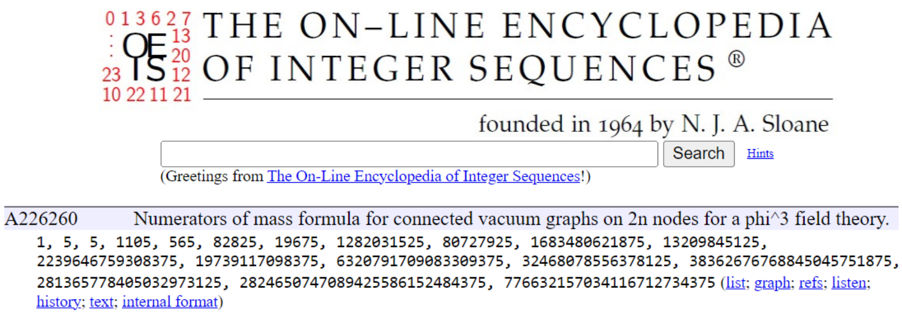
tex

\vskip 1mm

From \surl{oeis.org/A226260}:

\vskip 1mm

\includegraphics[width=\linewidth]{OEIS.png}



tex

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pdf

The Right-Handed Trefoil

pdf

In[∘]:=
```
K = Mirror@Knot[3, 1]; Features[K]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$\text{Features}[7, C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]]$

pdf

In[∘]:=
```
ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[
   x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
    (ϵ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
ℒ[C_{i_}[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
ℒ[K_] := CF[ℒ /@ Features[K][[2]]]
vs[K_] := Join @@ Table[{p_i, x_i}, {i, Features[K][[1]]}]
```

tex

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pdf

In[∘]:= `{vs[K], ℒ[K]}`

Out[∘]=
pdf

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-2\,\epsilon - p_1 x_1 + \epsilon\, p_1 x_1 + T\, p_2 x_1 - \epsilon\, p_5 x_1 + (1-T)\, p_6 x_1 + \frac{1}{2}(-1+T)\,\epsilon\, p_1 p_5 x_1^2 + \frac{1}{2}(1-T)\,\epsilon\, p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 - p_3 x_3 + \epsilon\, p_3 x_3 + T\, p_4 x_3 - \epsilon\, p_7 x_3 + (1-T)\, p_8 x_3 + \frac{1}{2}(-1+T)\,\epsilon\, p_3 p_7 x_3^2 + \frac{1}{2}(1-T)\,\epsilon\, p_7^2 x_3^2 - p_4 x_4 + \epsilon\, p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \epsilon\, p_1 p_5 x_1 x_5 + \epsilon\, p_5^2 x_1 x_5 - \epsilon\, p_2 x_6 + (1-T)\, p_3 x_6 - p_6 x_6 + \epsilon\, p_6 x_6 + T\, p_7 x_6 + \epsilon\, p_2^2 x_2 x_6 - \epsilon\, p_2 p_6 x_2 x_6 + \frac{1}{2}(1-T)\,\epsilon\, p_2^2 x_6^2 + \frac{1}{2}(-1+T)\,\epsilon\, p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \epsilon\, p_3 p_7 x_3 x_7 + \epsilon\, p_7^2 x_3 x_7\Big]\Big\}$$

tex

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pdf

In[∘]:= `$π = Normal[# + O[ϵ]^2] &; ∫ℒ[K] ⅆ(vs@K)`

Out[∘]=
pdf

$$-\frac{\mathbb{i}\, T\, \mathbb{E}\left[-\frac{(-1+T)^2 (1+T^2)\,\epsilon}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

In[∘]:= `∫(ℒ[K] /. x_i_ :> ⅈ x_i) ⅆ(vs@K)`

Out[∘]=

$$\frac{T\, \mathbb{E}\left[-\frac{(-1+T)^2 (1+T^2)\,\epsilon}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

tex

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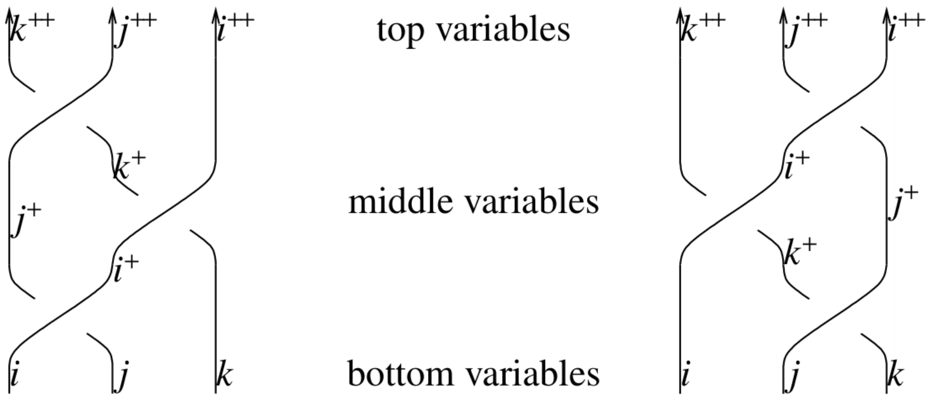
pdf

Invariance Under Reidemeister 3

tex

\def\ip{{i^+}} \def\jp{{j^+}} \def\kp{{k^+}}
\def\ipp{{i^{+\!+}}} \def\jpp{{j^{+\!+}}} \def\kpp{{k^{+\!+}}}
\input{figs/R3.pdf_t}



pdf

In[∘]:=
```
lhs = ∫(ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])) ⅆ{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
rhs = ∫(ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1])) ⅆ{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}};
lhs === rhs
```

Out[∘]=
pdf

```
False
```

tex

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pdf

Invariance Under Reidemeister 3, Take 2

pdf

In[∘]:=
```
lhs = ∫(ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])) ⅆ{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
rhs = ∫(ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1])) ⅆ{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}};
lhs === rhs
```

Out[∘]=
pdf

```
True
```

pdf

In[∘]:=
```
lhs
```

Out[∘]=
pdf

```
Degenerate Q!
```

pdf

Invariance Under Reidemeister 3, Take 3

pdf

In[∘]:=
```
lhs = ∫(𝔼[𝕚 π_i p_i + 𝕚 π_j p_j + 𝕚 π_k p_k] ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
    d{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
rhs = ∫(𝔼[𝕚 π_i p_i + 𝕚 π_j p_j + 𝕚 π_k p_k] ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
    d{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
lhs == rhs
```

Out[∘]=
pdf

True

tex

\needspace{20mm}

pdf

In[∘]:= `lhs`

Out[∘]=
pdf

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\,\epsilon}{2}+\mathbb{i}\,T^2\,p_{2+i}\,\pi_i-\mathbb{i}\,(-1+T)\,T\,p_{2+j}\,\pi_i+\mathbb{i}\,T^2\,\epsilon\,p_{2+j}\,\pi_i-\mathbb{i}\,(-1+T)\,p_{2+k}\,\pi_i+\mathbb{i}\,T\,\epsilon\,p_{2+k}\,\pi_i-$$
$$\frac{1}{2}\,(-1+T)\,T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T^3\,\epsilon\,p_{2+j}^2\,\pi_i^2-\frac{1}{2}\,(-1+T)\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i^2+$$
$$\frac{1}{2}\,(-1+T)^2\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_i^2+\mathbb{i}\,T\,p_{2+j}\,\pi_j-\mathbb{i}\,T\,\epsilon\,p_{2+j}\,\pi_j-$$
$$\mathbb{i}\,(-1+T)\,p_{2+k}\,\pi_j+\mathbb{i}\,(-1+2\,T)\,\epsilon\,p_{2+k}\,\pi_j+T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,\pi_j-T^3\,\epsilon\,p_{2+j}^2\,\pi_i\,\pi_j-$$
$$(-1+T)\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,\pi_j+(-1+T)^2\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_j+(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_i\,\pi_j-$$
$$\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_j^2+\mathbb{i}\,p_{2+k}\,\pi_k-2\,\mathbb{i}\,\epsilon\,p_{2+k}\,\pi_k+T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,\pi_k-$$
$$(-1+T)\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_k-T\,\epsilon\,p_{2+k}^2\,\pi_i\,\pi_k+T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j\,\pi_k-T\,\epsilon\,p_{2+k}^2\,\pi_j\,\pi_k\Big]$$

tex

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at \surl{drorbn.net/g24/ap}.

Invariance Under Reidemeister 3, Take 4 (just for fun)

In[∘]:=
```
lhs = ∫(𝔼[ⅈ πi pi + ⅈ πj pj + ⅈ πk pk + ⅈ πi+2 pi+2 + ⅈ πj+2 pj+2 + ⅈ πk+2 pk+2 +
      ⅈ ξi+2 xi+2 + ⅈ ξj+2 xj+2 + ⅈ ξk+2 xk+2] ℒ /@ (Xi,j[1] Xi+1,k[1] Xj+1,k+1[1]))
    ⅆ{pi, pj, pk, xi, xj, xk, pi+1, pj+1, pk+1, xi+1, xj+1, xk+1, pi+2, pj+2, pk+2, xi+2, xj+2, xk+2};
rhs = ∫(𝔼[ⅈ πi pi + ⅈ πj pj + ⅈ πk pk + ⅈ πi+2 pi+2 + ⅈ πj+2 pj+2 + ⅈ πk+2 pk+2 +
      ⅈ ξi+2 xi+2 + ⅈ ξj+2 xj+2 + ⅈ ξk+2 xk+2] ℒ /@ (Xj,k[1] Xi,k+1[1] Xi+1,j+1[1]))
    ⅆ{pi, pj, pk, xi, xj, xk, pi+1, pj+1, pk+1, xi+1, xj+1, xk+1, pi+2, pj+2, pk+2, xi+2, xj+2, xk+2};
lhs == rhs
```

Out[∘]=

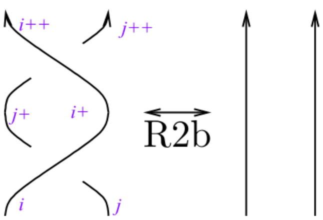
True

In[∘]:= `lhs`

Out[∘]=

Degenerate Q!

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=
```
lhs = ∫𝔼[ⅈ πi pi + ⅈ πj pj] ℒ /@ (Xi,j[1] Xi+1,j+1[-1]) ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1}
rhs =
  ∫𝔼[ⅈ πi pi + ⅈ πj pj] ℒ /@ (Ci[0] Ci+1[0] Cj[0] Cj+1[0]) ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1};
lhs == rhs
```

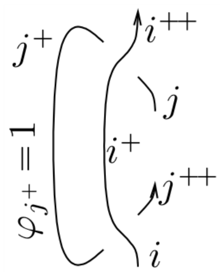
Out[∘]=

$\mathbb{E}[\mathbb{i}\, p_{2+i}\, \pi_i + \mathbb{i}\, p_{2+j}\, \pi_j]$

Out[∘]=

True

Invariance Under R2c



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}\ /@\ (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])$$
$$\mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\}$$
$$\text{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])$$
$$\mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\};$$
$$\text{lhs} == \text{rhs}$$

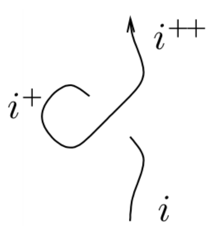
Out[∘]=

$$-\mathbb{i}\,\sqrt{T}\ \mathbb{E}\left[-\frac{\epsilon}{2} + \mathbb{i}\,p_{2+i}\,\pi_i + \mathbb{i}\,p_{3+j}\,\pi_j - \mathbb{i}\,\epsilon\,p_{3+j}\,\pi_j\right]$$

Out[∘]=

True

Invariance Under R1l



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}\ /@\ (X_{i+2,i}[1]\ C_{i+1}[1])\ \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$$
$$\text{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\};$$
$$\text{lhs} == \text{rhs}$$

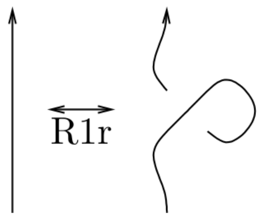
Out[∘]=

$$-\mathbb{i}\ \mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Out[∘]=

True

Invariance Under R1r



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}\ \text{/@}\ (X_{i,i+2}[1]\ C_{i+1}[-1])\ \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$$

$$\text{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}\ \text{/@}\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\};$$

$$\text{lhs} == \text{rhs}$$

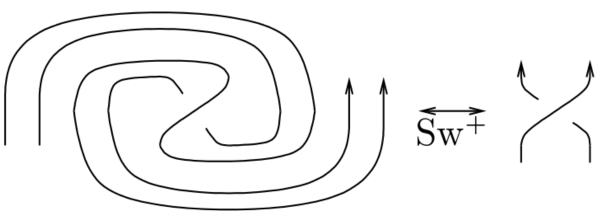
Out[∘]=

$$-\mathbb{i}\ \mathbb{E}[\mathbb{i}\ p_{3+i}\ \pi_i]$$

Out[∘]=

True

Invariance Under Sw



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}\ \text{/@}\ (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])$$
$$\mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\}$$

$$\text{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}\ \text{/@}\ (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])$$
$$\mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\};$$

$$\text{lhs} == \text{rhs}$$

Out[∘]=

$$\sqrt{T}\ \mathbb{E}\Big[-\frac{\epsilon}{2} + \mathbb{i}\,T\,p_{3+i}\,\pi_i - \mathbb{i}\,(-1+T)\,p_{3+j}\,\pi_i + \mathbb{i}\,T\,\epsilon\,p_{3+j}\,\pi_i - \frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i^2 +$$
$$\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+j}^2\,\pi_i^2 + \mathbb{i}\,p_{3+j}\,\pi_j - \mathbb{i}\,\epsilon\,p_{3+j}\,\pi_j + T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j - T\,\epsilon\,p_{3+j}^2\,\pi_i\,\pi_j\Big]$$

Out[∘]=

True