Knot Invariants from Finite Dimensional Integration

Abstract. For the purpose of today, an "I-Type Knot Invariant" is a knot invariant computed from a knot diagram by integrating the exponential of a Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.

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A. Yes.
Q. Are they any good?
A. They are the strongest we know per CPU cycle, and are excellent in other ways too.
Q. Didn't Witten do that back in 1988 with path integrals?
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Q. But integrals belong in analysis!
A. Ours only use squeaky-clean algebra.

∃ $V: \mathbb{R}^4 \to \mathbb{R}$, $r^{\pm}: \mathbb{R}^8 \to \mathbb{R}$
s.t.

$\longrightarrow \int_{\mathbb{R}^{28}_{x_i p_i \xi_i \pi_i}} \exp\left(\sum_{c\,(s,i,j)} + \sum_{k}\right)$ is an invariant.

need an explain

Thm as in Oaxaca

Formulas of $V, r^{\pm}$.

So what? * yet another philosophy for invariants.
* strongest per CPU cycle?
* Easy, despite appearances.
* Has applications to topology, may have crazy good ones (not today, but see ...)

Knots:  /R123 $\xrightarrow{\text{invariants}}$ something simple

Knot table in background.

The good: 1. At the centre of low dim top
2. "Invariants" connect to pretty much all of algebra

The agony: 1 & 2 don't talk well to each other
* Not enough topological applications of all these invariants
* The fancy algebra doesn't come naturally to a topologist.

$\Longrightarrow$ we're still missing something about the relationship between knots & algebra

Thanks for having me in Groningen!

Knot Invariants from Finite Dimensional Integration

http://drorbn.net/g24

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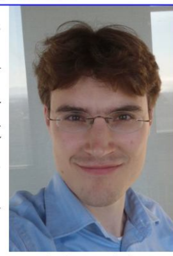

joint with R. van der Veen

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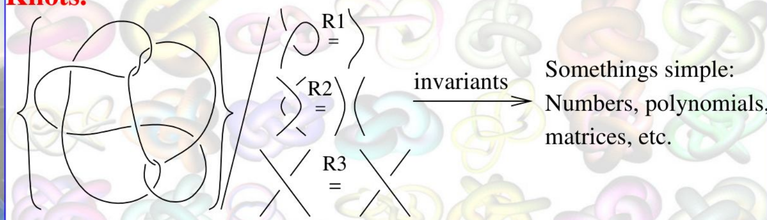

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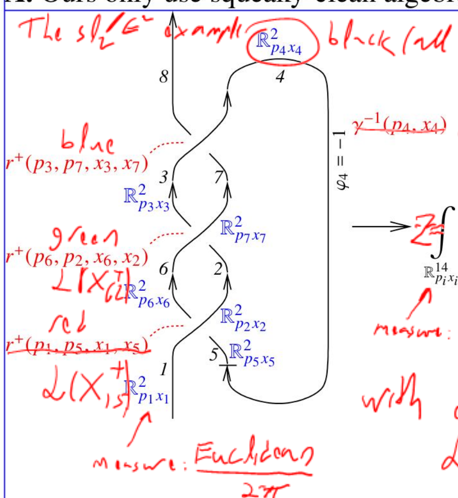
The sl_2^{ϵ} example



$$\longrightarrow Z = \int_{\mathbb{R}^{14}_{p_ix_i}} \exp i\left(\mathcal{L}(X)+\mathcal{L}(X)+\mathcal{L}(C)+\mathcal{L}(C)\right)$$

optional measure: Euclidean / $(2\pi)^7$

with $\mathcal{L}(X) = \ldots$

$\mathcal{L}(C) = \ldots$

So $Z = \int_{\mathbb{R}^{14}} \exp(\quad) =$

(Alternative) Gausssian Integration.

Goal. Compute

$$I_1(0) := \int_{\mathbb{R}^n} d^{\cancel{n}}x \exp\left(-\frac{1}{2}a^{ij}x_ix_j + V(x)\right).$$

Solution. Set

$$I_\lambda(x) := \int_{\mathbb{R}^n} d^{\cancel{n}}y \exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j + V(x+y)\right).$$

Then $I_1(0)$ is what we want, $I_0(x) = (\det A)^{-1/2}\exp V(x)$, and

$$\partial_\lambda I_\lambda(x) = \frac{1}{2\lambda^2}\int_{\mathbb{R}^n} d^{\cancel{n}}y a^{ij}y_iy_j \exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j + V(x+y)\right)$$

While with g_{ij} the inverse matrix of a^{ij}, and noting that under the dy integral $\partial_y = 0$,

$$\frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}I_\lambda(x) =$$

$$\frac{1}{2}\int_{\mathbb{R}^n} d^{\cancel{n}}y g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j}) \exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j + V(x+y)\right)$$

$$= \frac{1}{2\lambda^2}\int_{\mathbb{R}^n} d^{\cancel{n}}y a^{ij}y_iy_j \exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j + V(x+y)\right).$$

Hence

$$\partial_\lambda I_\lambda(x) = \frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}I_\lambda(x),$$

and therefore

$$I_\lambda(x) = (\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$

Better the Goal.

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[CC] D. Cimasoni, A. Conway, *Colored Tangles and Signatures,* Math. Proc. Camb. Phil. Soc. **164** (2018) 493–530, arXiv:1507.07818.

[Co] A. Conway, *The Levine-Tristram Signature: A Survey,* arXiv:1903.04477.

[GG] J-M. Gambaudo, É. Ghys, *Braids and Signatures,* Bull. Soc. Math. France **133-4** (2005) 541–579.

[Ka] R. Kashaev, *On Symmetric Matrices Associated with Oriented Link Diagrams,* in *Topology and Geometry, A Collection of Essays Dedicated to Vladimir G. Turaev,* EMS Press 2021, arXiv:1801.04632.

[Li] J. Liu, *A Proof of the Kashaev Signature Conjecture,* arXiv:2311.01923.

[Me] A. Merz, *An Extension of a Theorem by Cimasoni and Conway,* arXiv:2104.02993.

Acknowledgement. This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

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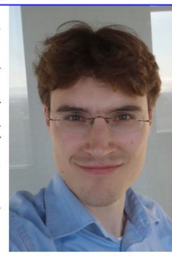

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Continues  Rozansky  Overbay

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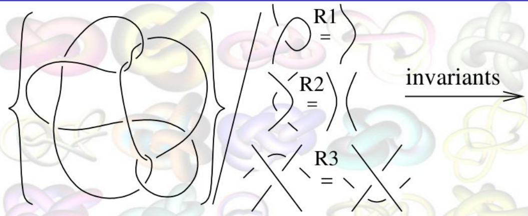

Somethings simple: Numbers, polynomials, matrices, etc.

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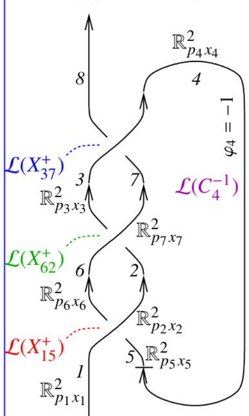
The $sl_2^{/\epsilon^2}$ Example. With T an indeterminate and with $\epsilon^2=0$:



$$\longrightarrow Z=\int_{\mathbb{R}^{14}_{p_ix_i}}\mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

measure on $\mathbb{R}$ is $(2\pi)^{-1/2}\cdot$*standard*

improve the green

where $\mathcal{L}(X^s_{ij})=T^{s/2}e^{iL(X^s_{ij})}$ and $\mathcal{L}(C_i^\varphi)=T^{\varphi/2}e^{iL(C_i^\varphi)}$ ($i=\sqrt{-1}$ is optional), and

$$L(X^s_{ij})=x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j)+(T^s-1)x_i(p_{i+1}-p_{j+1})+\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\left(\begin{array}{c}(T^s-1)x_ip_j\\+2(1-x_jp_j)\end{array}\right)-1\right)$$

flash right?

$$L(C_i^\varphi)=x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

So $Z=T\int e^{iL(\circledast)}dp_1\dots dp_7dx_1\dots dx_7$, where $L(\circledast)=$

$$\sum_{i=1}^7 x_i(p_{i+1}-p_i)+(T-1)(x_1(p_2-p_6)+x_6(p_7-p_3)+x_3(p_4-p_8))$$

$$+\frac{\epsilon}{2}\left(\begin{array}{c}x_1(p_1-p_5)\left((T^s-1)x_1p_5+2(1-x_5p_5)\right)-1\\+x_6(p_6-p_2)\left((T^s-1)x_6p_2+2(1-x_2p_2)\right)-1\\+x_3(p_3-p_7)\left((T^s-1)x_3p_7+2(1-x_7p_7)\right)-1\\+2x_4p_4-1\end{array}\right)$$

flash right?

and so $Z=(T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right)=\Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^{-2}}\right)$. Here Δ is the Alexander polynomial and ρ_1 is the Rozansky-Overbay polynomial [Ro1, Ro2, Ro3, Ov].

Theorem. Z is a knot invariant.

Proof. Use Fubini (details later).


Guido Fubini

To do. * Human hard yet computer very easy (polynomial time!).
* strong?
* Proof of Theorem.
* Where is it coming from?
* Is there more like it?
* philosophical point: universal invariants are qualitatively better than rep-theory ones.

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Goal. Compute $$I_1(0) := \int_{\mathbb{R}^n} dx \exp\left(-\frac{1}{2}a^{ij}x_ix_j + V(x)\right).$$

change I to $\mathcal{Z}$

Move to first page.

Solution. Set $$I_\lambda(x) := \int_{\mathbb{R}^n} dy \exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j + V(x+y)\right).$$

Then $I_1(0)$ is what we want, $I_0(x) = (\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y = 0$,

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$$= \frac{1}{2\lambda^2}\int_{\mathbb{R}^n} dy\, a^{ij}y_iy_j\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j + V(x+y)\right) = \partial_\lambda I_\lambda(x).$$

Include pictures of Gauss & Feynman.

Hence $$\partial_\lambda I_\lambda(x) = \frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}I_\lambda(x),$$

and therefore $$I_\lambda(x) = (\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$

and we've just witnessed the birth of Feynman diagrams.

Even better with $Z = \log\mathcal{Z}$, by a simple substitution, [The Synthesis eqn]

References.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.

[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

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Include a clean implementation of ρ_1, up to R3.

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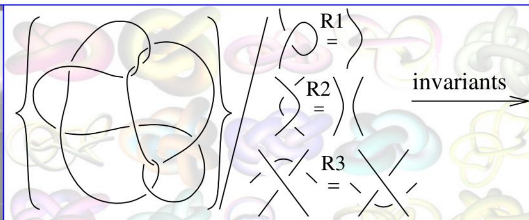

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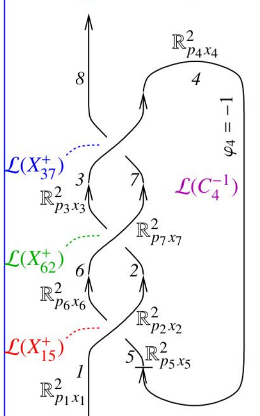

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measure on $\mathbb{R}$ is $(2\pi)^{-1/2}\cdot$*standard*

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and so $Z = (T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right) = \Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^{-2}}\right)$. Here Δ is the Alexander polynomial and ρ_1 is the Rozansky-Overbay polynomial [Ro, Ov, BV].

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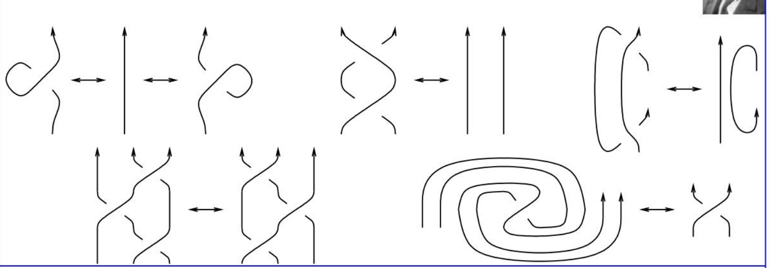
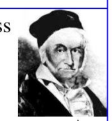
Proof. Use Fubini (details later).

Guido Fubini 



To Do. • Human-hard but computer-very-easy (poly time!). • Strong! • Details of the proof. • Where is it coming from? • A philosophical point: "Universal invariants" are qualitatively better than representation theory ones.

(Alternative) Gaussian Integration.

Gauss 

Goal. Compute $\displaystyle\int_{\mathbb{R}^n} dx\exp\left(-\frac{1}{2}a^{ij}x_ix_j + V(x)\right)$.

Solution. Set $\displaystyle\mathcal{Z}_\lambda(x) := \int_{\mathbb{R}^n} dy\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j + V(x+y)\right)$.

Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x) = (\det A)^{-1/2}\exp V(x)$, ~~and~~ ex ~~with~~ g_{ij} the inverse matrix of a^{ij} and ~~noting~~ note that under the dy integral ∂_y be $= 0$, so,

$$\frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)$$
$$= \frac{1}{2}\int_{\mathbb{R}^n} dy\, g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j + V(x+y)\right)$$
$$= \frac{1}{2\lambda^2}\int_{\mathbb{R}^n} dy\, a^{ij}y_iy_j\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j + V(x+y)\right) = \partial_\lambda\mathcal{Z}_\lambda(x).$$

Hence $\quad (*)\quad \partial_\lambda\mathcal{Z}_\lambda(x) = \frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x),$

and therefore $\quad \mathcal{Z}_\lambda(x) = (\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$

We've just witnessed the birth of "Feynman Diagrams".

Even better. With $Z := \log(\sqrt{\det A}\,\mathcal{Z})$, by a simple substitution into (*), we get the "Synthesis Equation":

Feynman

$$Z_0 = V, \qquad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j=1}^n g_{ij}\left(\partial_{x_i,x_j}Z_\lambda + (\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right),$$

an ODE (in λ) whose solution is pure algebra.

Picard's Method

Picard

Acknowledgement. This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Preliminaries

This is IType.nb of http://drorbn.net/g24/ap.

```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from
  http://drorbn.net/AP/Talks/Groningen-240530
  to compute rotation numbers.
```

```
CF[ω_. ℰ_𝔼] := CF[ω] × CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] :=
  Module[
    {vs = Cases[ℰ, (x | p | ξ | π)__, ∞] ⋃ {x, p, ϵ},
     ps, c},
    Total[CoefficientRules[Expand[ℰ], vs] /.
      (ps_ → c_) ⧴ Factor[c] (Times @@ vs^ps)] ];
```

Integration

Using Picard Iteration!

```
𝔼 /: 𝔼[A_] × 𝔼[B_] := 𝔼[A + B];
```

```
$π = Identity; (* hacks in pink *)
```

```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) :=
  Module[{n, L0, Q, Δ, G, Z, e, λ, DZ, a, b},
    n = Length@vs; L0 = L /. ϵ → 0;
    Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /.
        (p | x)__ → 0, {a, n}, {b, n}];
    If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
    Z = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
    DZ_a_ := ∂_{vs⟦a⟧} Z; DZ_{a_,b_} := ∂_{vs⟦b⟧} DZ_a;
    While[e = CF@$π[
        (∂_λ Z) - 1/2 ∑_{a=1}^n ∑_{b=1}^n G⟦a, b⟧ (DZ_{a,b} + DZ_a DZ_b)];
      0 =!= e, Z -= ∫_0^λ e ⅆλ
    ];
    PowerExpand@Factor[ω Δ^(-1/2)] ×
      𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]
  ];
Protect[Integrate];
```

$$\int \mathbb{E}\left[\mathbb{i}\,\mu\, x^2/2 + \mathbb{i}\,\xi\, x\right] \mathbb{d}\{x\}$$

$$\frac{(-1)^{1/4}\,\mathbb{E}\left[-\frac{\mathbb{i}\,\xi^2}{2\,\mu}\right]}{\sqrt{\mu}}$$

$$L = -\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$$

$$Z12 = \int \mathbb{E}[L]\, \mathbb{d}\{x_1, x_2\}$$

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\,(-b^2+a\,c)} + \frac{b\,\xi_1\,\xi_2}{b^2-a\,c} + \frac{a\,\xi_2^2}{2\,(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$$

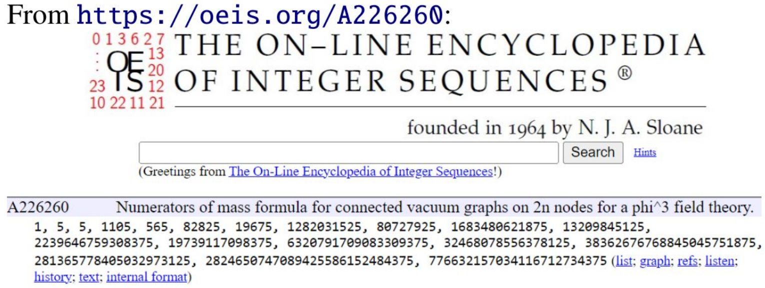
$$\left\{Z1 = \int \mathbb{E}[L]\, \mathbb{d}\{x_1\},\ Z12 == \int Z1\, \mathbb{d}\{x_2\}\right\}$$

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\,x_2^2}{2\,a} - \frac{b\,x_2\,\xi_1}{a} + \frac{\xi_1^2}{2\,a} + x_2\,\xi_2\right]}{\sqrt{a}},\ \mathrm{True}\right\}$$

$$\$\pi = \mathrm{Normal}\left[\# + O[\epsilon]^{13}\right]\ \&;\ \int \mathbb{E}\left[-x^2/2 + \epsilon\, x^3/6\right] \mathbb{d}\{x\}$$

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24} + \frac{5\,\epsilon^4}{16} + \frac{1105\,\epsilon^6}{1152} + \frac{565\,\epsilon^8}{128} + \frac{82\,825\,\epsilon^{10}}{3072} + \frac{19\,675\,\epsilon^{12}}{96}\right]$$

From https://oeis.org/A226260:


THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.

1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 3246807855637812 5, 38362676768845045751875, 2813657784050329731 25, 2824650747089425586152484375, 7766321570341167127343 75 (list; graph; refs; listen; history; text; internal format)

```
K = Knot[3, 1]; Features[K]
```

$$\mathrm{Features}[7, C_4[-1]\, X_{2,6}[-1]\, X_{5,1}[-1]\, X_{7,3}[-1]]$$

```
ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[
    x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) +
    (T^s - 1) x_i (p_{i+1} - p_{j+1})
    +
    (ϵ s)/2 (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) -
      1)];
ℒ[C_i_[φ_]] :=
  T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)];
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧];
vs[K_] :=
  Union @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}];
```

```
{vs[K], ℒ[K]}
```

$$\Big\{\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\},\ \frac{1}{T^2}\,\mathbb{E}\Big[\epsilon - p_1 x_1 + p_2 x_1 - p_2 x_2 - \epsilon p_2 x_2 + \frac{p_3 x_2}{T} + \epsilon p_6 x_2 + \frac{(-1+T)\,p_7 x_2}{T} + \frac{(-1+T)\,\epsilon p_2 p_6 x_2^2}{2T} - \frac{(-1+T)\,\epsilon p_6^2 x_2^2}{2T} - p_3 x_3 + p_4 x_3 - p_4 x_4 + \epsilon p_4 x_4 + p_5 x_4 + \epsilon p_1 x_5 + \frac{(-1+T)\,p_2 x_5}{T} - p_5 x_5 - \epsilon p_5 x_5 + \frac{p_6 x_5}{T} - \epsilon p_1^2 x_1 x_5 + \epsilon p_1 p_5 x_1 x_5 - \frac{(-1+T)\,\epsilon p_1^2 x_5^2}{2T} + \frac{(-1+T)\,\epsilon p_1 p_5 x_5^2}{2T} - p_6 x_6 + p_7 x_6 + \epsilon p_2 p_6 x_2 x_6 - \epsilon p_6^2 x_2 x_6 + \epsilon p_3 x_7 + \frac{(-1+T)\,p_4 x_7}{T} - p_7 x_7 - \epsilon p_7 x_7 + \frac{p_8 x_7}{T} - \epsilon p_3^2 x_3 x_7 + \epsilon p_3 p_7 x_3 x_7 - \frac{(-1+T)\,\epsilon p_3^2 x_7^2}{2T} + \frac{(-1+T)\,\epsilon p_3 p_7 x_7^2}{2T}\Big]\Big\}$$

```
$π = Normal[# + O[∈]^2] &; ∫ℒ[K] ⅆ(vs@K)
```

$$-\frac{i\,T\,\mathbb{E}\left[\frac{(-1+T)^2\,(1+T^2)\,\epsilon}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

```
lhs = ∫(ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
    ⅆ{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}}
rhs = ∫(ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
    ⅆ{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}};
lhs == rhs
```

Degenerate Q!

True

```
lhs =
  ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] ×
      ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
    ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1},
      x_{j+1}, x_{k+1}}
rhs =
  ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] ×
      ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
    ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1},
      x_{j+1}, x_{k+1}};
lhs == rhs
```

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2} + T^2 p_{2+i}\pi_i - (-1+T)\,T p_{2+j}\pi_i + T^2\epsilon p_{2+j}\pi_i + (1-T)\,p_{2+k}\pi_i + T\epsilon p_{2+k}\pi_i + \frac{1}{2}(-1+T)\,T^3\epsilon p_{2+i}p_{2+j}\pi_i^2 - \frac{1}{2}(-1+T)\,T^3\epsilon p_{2+j}^2\pi_i^2 + \frac{1}{2}(-1+T)\,T^2\epsilon p_{2+i}p_{2+k}\pi_i^2 - \frac{1}{2}(-1+T)^2\,T\epsilon p_{2+j}p_{2+k}\pi_i^2 - \frac{1}{2}(-1+T)\,T\epsilon p_{2+k}^2\pi_i^2 + T p_{2+j}\pi_j - T\epsilon p_{2+j}\pi_j + (1-T)\,p_{2+k}\pi_j + (-1+2T)\,\epsilon p_{2+k}\pi_j - T^3\epsilon p_{2+i}p_{2+j}\pi_i\pi_j + T^3\epsilon p_{2+j}^2\pi_i\pi_j + (-1+T)\,T^2\epsilon p_{2+i}p_{2+k}\pi_i\pi_j - (-1+T)^2\,T\epsilon p_{2+j}p_{2+k}\pi_i\pi_j - (-1+T)\,T\epsilon p_{2+k}^2\pi_i\pi_j + \frac{1}{2}(-1+T)\,T\epsilon p_{2+j}p_{2+k}\pi_j^2 - \frac{1}{2}(-1+T)\,T\epsilon p_{2+k}^2\pi_j^2 + p_{2+k}\pi_k - 2\epsilon p_{2+k}\pi_k - T^2\epsilon p_{2+i}p_{2+k}\pi_i\pi_k + (-1+T)\,T\epsilon p_{2+j}p_{2+k}\pi_i\pi_k + T\epsilon p_{2+k}^2\pi_i\pi_k - T\epsilon p_{2+j}p_{2+k}\pi_j\pi_k + T\epsilon p_{2+k}^2\pi_j\pi_k\Big]$$

True

References.

[BV] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* Proc. Amer. Math. Soc. **147** (2019) 377–397, arXiv:1708.04853; *A Perturbed-Alexander Invariant,* to appear in Quantum Topology, ωεβ/APAI.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.

[Ro] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061; *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005; *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

Knot Invariants from Finite Dimensional Integration

Abstract. For the purpose of today, an "I-Type Knot Invariant" is a knot invariant computed from a knot diagram by integrating the exponential of a Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.

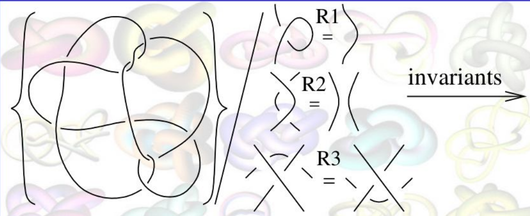

joint with R. van der Veen

Q. Are there any such things? **A.** Yes.
Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.
Q. Didn't Witten do that back in 1988 with path integrals?
A. No. His constructions are infinite dimensional and far from rigorous.
Q. But integrals belong in analysis!
A. Ours only use squeaky-clean algebra.


Rozansky

Overbay
Continues

Knots.



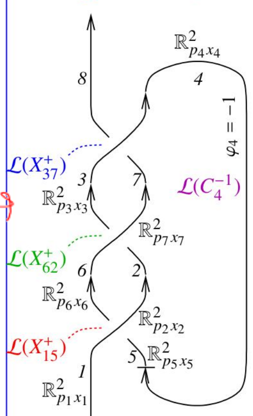
Something simple: numbers, polynomials, matrices, etc.

The Good. 1. At the centre of low dimensional topology.
2. "Invariants" connect to pretty much all of algebra.
The Agony. 1&2 don't talk to each other.
- Not enough topological applications for all these invariants.
- The fancy algebra doesn't arise naturally within topology.

$\Longrightarrow$ We're still missing something about the relationship between knots and algebra.

The $sl_2^{\epsilon^2}$ Example.

With T an indeterminate and with $\epsilon^2=0$:



$$\longrightarrow Z=\int_{\mathbb{R}^{14}_{p_ix_i}} \mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C^{-1}_4)$$

measure on $\mathbb{R}$ is $(2\pi)^{-1/2}\cdot$*standard*

where $\mathcal{L}(X^s_{ij}) = T^{s/2}e^{\mathbb{i}L(X^s_{ij})}$ and $\mathcal{L}(C^\varphi_i) = T^{\varphi/2}e^{\mathbb{i}L(C^\varphi_i)}$ ($\mathbb{i}=\sqrt{-1}$ is optional), and

$$L(X^s_{ij}) = x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j) +(T^s-1)x_i(p_{i+1}-p_{j+1}) +\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\ +2(1-x_jp_j)\end{pmatrix}-1\right)$$

$$L(C^\varphi_i) = x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

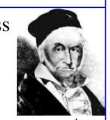
better spacing

So $Z=T\int e^{\mathbb{i}L(\circledast)}dp_1\dots dp_7dx_1\dots dx_7$, where $L(\circledast)=$

$$\sum_{i=1}^7 x_i(p_{i+1}-p_i)+(T-1)(x_1(p_2-p_6)+x_6(p_7-p_3)+x_3(p_4-p_8))$$
$$+\frac{\epsilon}{2}\begin{pmatrix} x_1(p_1-p_5)\left((T^s-1)x_1p_5+2(1-x_5p_5)\right)-1\\ +x_6(p_6-p_2)\left((T^s-1)x_6p_2+2(1-x_2p_2)\right)-1\\ +x_3(p_3-p_7)\left((T^s-1)x_3p_7+2(1-x_7p_7)\right)-1\\ +2x_4p_4-1\end{pmatrix},$$

and so $Z=(T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right)=\Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^{-2}}\right)$. Here Δ is the Alexander polynomial and ρ_1 is the Rozansky-Overbay polynomial [Ro, Ov, BV].

(Alternative) Gaussian Integration.


Gauss

Goal. Compute $\displaystyle\int_{\mathbb{R}^n} dx\exp\left(-\frac12 a^{ij}x_ix_j+V(x)\right)$.

Solution. Set $\displaystyle\mathcal{Z}_\lambda(x):=\int_{\mathbb{R}^n}dy\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$.

Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x)=(\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y=0$,

$$\frac12 g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x) = \frac12\int_{\mathbb{R}^n}dy\, g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\frac{1}{2\lambda^2}\int_{\mathbb{R}^n}dy\, a^{ij}y_iy_j\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)=\partial_\lambda\mathcal{Z}_\lambda(x).$$

Hence
$$(*)\qquad \partial_\lambda\mathcal{Z}_\lambda(x)=\frac12 g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x),$$

and therefore
$$\mathcal{Z}_\lambda(x)=(\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$

We've just witnessed the birth of "Feynman Diagrams".


Feynman

Even better. With $Z:=\log(\sqrt{\det A}\mathcal{Z})$, by a simple substitution into (*), we get the "Synthesis Equation":

$$Z_0=V,\qquad \partial_\lambda Z_\lambda=\frac12\sum_{i,j=1}^n g_{ij}\left(\partial_{x_i,x_j}Z_\lambda+(\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right),$$

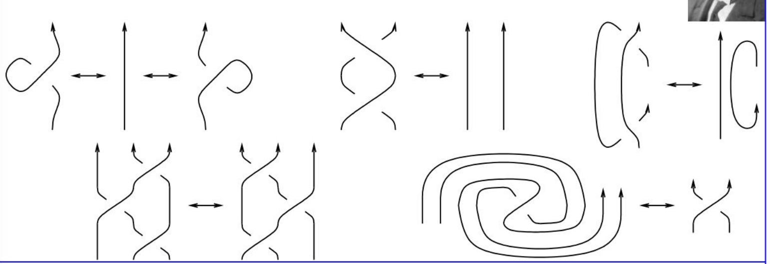
an ODE (in λ) whose solution is pure algebra.

Picard Iteration (used to prove the existance and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda=F(f_\lambda)$ with a given f_0, start with f_0, iterate $f\mapsto f_0+\int_0^\lambda F(f_\lambda)d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!


Picard

Theorem. Z is a knot invariant.
Proof. Use Fubini (details later).


Guido Fubini



Disclaimer! It's fun, but not fully ready

To Do. • Human-hard but computer-very-easy (poly time!).
- Strong!
- Details of the proof.
- Where is it coming from?
- A philosophical point: "Universal invariants" are qualitatively better than representation theory ones.

Acknowledgement. This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Preliminaries

This is IType.nb of http://drorbn.net/g24/ap.

```
Once[<< KnotTheory`; << Rot.m];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/AP/Talks/Groningen-240530 to compute rotation numbers.

```
CF[ω_. ℰ_𝔼] := CF[ω] × CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] :=
 Module[
  {vs = Cases[ℰ, (x | p | ξ | π)__, ∞] ∪ {x, p, ϵ},
   ps, c},
  Total[CoefficientRules[Expand[ℰ], vs] /.
    (ps_ → c_) :> Factor[c] (Times @@ vs^ps)] ];
```

clean

Integration

Using Picard Iteration!

mirror

```
𝔼 /: 𝔼[A_] × 𝔼[B_] := 𝔼[A + B];
$π = Identity; (* hacks in pink *)
Unprotect[Integrate];
∫ω_. 𝔼[L_] 𝕕(vs_List) :=
 Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, FZ, a, b},
  n = Length@vs; L0 = L /. ϵ → 0;
  Q = Table[(-∂_{vs[[a]],vs[[b]]} L0) /. Thread[vs → 0] /.
     (p | x)__ → 0, {a, n}, {b, n}];
  If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
  Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
  DZ_a_ := ∂_{vs[[a]]} Z; DZ_{a_,b_} := ∂_{vs[[b]]} DZ_a;
  FZ := CF@$π[1/2 Sum[Sum[G[[a, b]] (DZ_{a,b} + DZ_a DZ_b), {b, 1, n}], {a, 1, n}]];
  FixedPoint[(Z = Z0 + ∫_0^λ FZ 𝕕λ) &, Z];
  PowerExpand@Factor[ω Δ^(-1/2)] ×
   𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]
 ];
Protect[Integrate];
```

```
∫𝔼[i μ x^2 / 2 + i ξ x] 𝕕{x}
```

not here

$$\frac{(-1)^{1/4}\,\mathbb{E}\left[-\frac{i\,\xi^2}{2\,\mu}\right]}{\sqrt{\mu}}$$

```
L = -1/2 {x1, x2}.({{a, b}, {b, c}}).{x1, x2} + {ξ1, ξ2}.{x1, x2};
Z12 = ∫𝔼[L] 𝕕{x1, x2}
```

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\,(-b^2+a\,c)}+\frac{b\,\xi_1\,\xi_2}{b^2-a\,c}+\frac{a\,\xi_2^2}{2\,(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$$

```
{Z1 = ∫𝔼[L] 𝕕{x1}, Z12 == ∫Z1 𝕕{x2}}
```

$x \to \emptyset$

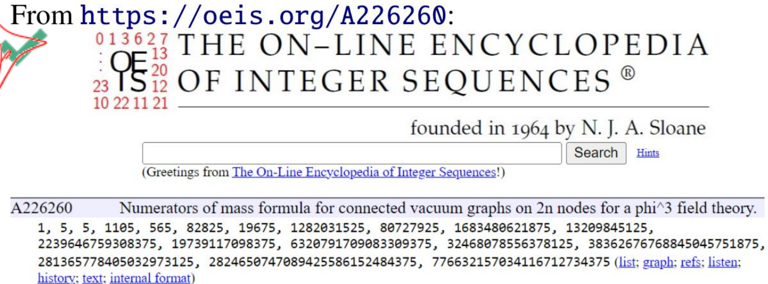
$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\,x_2^2}{2\,a}-\frac{b\,x_2\,\xi_1}{a}+\frac{\xi_1^2}{2\,a}+x_2\,\xi_2\right]}{\sqrt{a}},\ \text{True}\right\}$$

```
$π = Normal[# + O[ϵ]^13] &; ∫𝔼[-x^2 / 2 + ϵ x^3 / 6] 𝕕{x}
```

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24}+\frac{5\,\epsilon^4}{16}+\frac{1105\,\epsilon^6}{1152}+\frac{565\,\epsilon^8}{128}+\frac{82825\,\epsilon^{10}}{3072}+\frac{19675\,\epsilon^{12}}{96}\right]$$

From https://oeis.org/A226260:

vskip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founded in 1964 by N. J. A. Sloane

Search Hints

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```
K = Knot[3, 1]; Features[K]
```

The right-handed trefoil

```
Features[7, C4[-1] X2,6[-1] X5,1[-1] X7,3[-1]]
```

```
ℒ[X_{i_,j_}[s_]] := T^(s/2) 𝔼[
   x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) +
    (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
    (ϵ s / 2) ×
     (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
ℒ[C_i_[φ_]] := T^(φ/2) 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
ℒ[K_] := CF[ℒ /@ Features[K][[2]]]
vs[K_] :=
 Join @@ Table[{p_i, x_i}, {i, Features[K][[1]]}]
```

```
{vs[K], ℒ[K]}
```

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$\frac{1}{T^2}\,\mathbb{E}\Big[\epsilon - p_1 x_1 + p_2 x_1 - p_2 x_2 - \epsilon\, p_2 x_2 + \frac{p_3 x_2}{T} + \epsilon\, p_6 x_2 + \frac{(-1+T)\,p_7 x_2}{T} + \frac{(-1+T)\,\epsilon\, p_2 p_6 x_2^2}{2\,T} - \frac{(-1+T)\,\epsilon\, p_6^2 x_2^2}{2\,T} - p_3 x_3 + p_4 x_3 - p_4 x_4 + \epsilon\, p_4 x_4 +$$
$$p_5 x_4 + \epsilon\, p_1 x_5 + \frac{(-1+T)\,p_2 x_5}{T} - p_5 x_5 - \epsilon\, p_5 x_5 + \frac{p_6 x_5}{T} - \epsilon\, p_1^2 x_1 x_5 + \epsilon\, p_1 p_5 x_1 x_5 - \frac{(-1+T)\,\epsilon\, p_1^2 x_5^2}{2\,T} + \frac{(-1+T)\,\epsilon\, p_1 p_5 x_5^2}{2\,T} - p_6 x_6 + p_7 x_6 + \epsilon\, p_2 p_6 x_2 x_6 -$$
$$\epsilon\, p_6^2 x_2 x_6 + \epsilon\, p_3 x_7 + \frac{(-1+T)\,p_4 x_7}{T} - p_7 x_7 - \epsilon\, p_7 x_7 + \frac{p_8 x_7}{T} - \epsilon\, p_3^2 x_3 x_7 + \epsilon\, p_3 p_7 x_3 x_7 - \frac{(-1+T)\,\epsilon\, p_3^2 x_7^2}{2\,T} + \frac{(-1+T)\,\epsilon\, p_3 p_7 x_7^2}{2\,T}\Big]\Big\}$$

```
$π = Normal[# + O[ϵ]^2] &; ∫ℒ[K] 𝕕(vs@K)
```

$Aborted

Invariance Under Reidemeister 3b

Picture

bottom vars
middle vars
top variables.

```
lhs = ∫(ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
   𝕕{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}}
rhs = ∫(ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
   𝕕{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}};
lhs == rhs
```

Degenerate Q!

True

Add a version where only mid-vars integrated?

Invariance under R3, take 2.

```
lhs =
 ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] ×
    ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
  𝕕{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1},
    x_{j+1}, x_{k+1}}
rhs =
 ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] ×
    ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
  𝕕{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1},
    x_{j+1}, x_{k+1}};
lhs == rhs
```

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2}+T^2 p_{2+i}\pi_i-(-1+T)\,T\,p_{2+j}\pi_i+T^2\epsilon\,p_{2+j}\pi_i+(1-T)\,p_{2+k}\pi_i+T\epsilon\,p_{2+k}\pi_i+\frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+i}p_{2+j}\pi_i^2-\frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+j}^2\pi_i^2+\frac{1}{2}(-1+T)\,T^2\epsilon\,p_{2+i}p_{2+k}\pi_i^2-\frac{1}{2}(-1+T)^2\,T\epsilon\,p_{2+j}p_{2+k}\pi_i^2-\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+k}^2\pi_i^2+T\,p_{2+j}\pi_j-T\epsilon\,p_{2+j}\pi_j+(1-T)\,p_{2+k}\pi_j+(-1+2T)\,\epsilon\,p_{2+k}\pi_j-T^3\epsilon\,p_{2+i}p_{2+j}\pi_i\pi_j+T^3\epsilon\,p_{2+j}^2\pi_i\pi_j+(-1+T)\,T^2\epsilon\,p_{2+i}p_{2+k}\pi_i\pi_j-(-1+T)^2\,T\epsilon\,p_{2+j}p_{2+k}\pi_i\pi_j-(-1+T)\,T\epsilon\,p_{2+k}^2\pi_i\pi_j+\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+j}p_{2+k}\pi_j^2-\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+k}^2\pi_j^2+p_{2+k}\pi_k-2\epsilon\,p_{2+k}\pi_k-T^2\epsilon\,p_{2+i}p_{2+k}\pi_i\pi_k+(-1+T)\,T\epsilon\,p_{2+j}p_{2+k}\pi_i\pi_k+T\epsilon\,p_{2+k}^2\pi_i\pi_k-T\epsilon\,p_{2+j}p_{2+k}\pi_j\pi_k+T\epsilon\,p_{2+k}^2\pi_j\pi_k\Big]$$

True

The rest of the invariance proof is at ...

Sort out the i factors.

Add in/out arrows.

Spellcheck!

strong

A: A much faster program is at [APAI]. Over there we compute [deg 12].

Where is it coming from?

Principles
* Morphisms are generating functions.
* Composition is integration
* Use universal invariants.
* "Solvable Approximation"

⇝ Gaussian generating functions

There's more where this came from

universal is better than rep theoretical

References.

[BV] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* Proc. Amer. Math. Soc. **147** (2019) 377–397, arXiv:1708.04853; *A Perturbed-Alexander Invariant,* to appear in Quantum Topology, ωεβ/APAI.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.

[Ro] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061; *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005; *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

Knot Invariants from Finite Dimensional Integration

Abstract. For the purpose of today, an "I-Type Knot Invariant" is a knot invariant computed from a knot diagram by integrating the exponential of a Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.

joint with R. van der Veen

Q. Are there any such things? **A.** Yes.

Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.

Q. Didn't Witten do that back in 1988 with path integrals?

A. No. His constructions are infinite dimensional and far from rigorous.

Q. But integrals belong in analysis!

A. Ours only use squeaky-clean algebra.

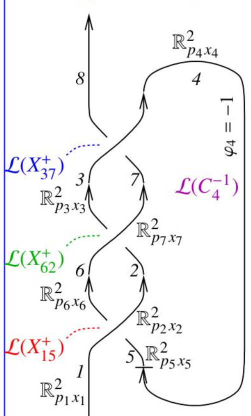
Continues

Rozansky

Overbay

The $sl_2^{/\epsilon^2}$ Example.

With T an indeterminate and with $\epsilon^2=0$:



$$\longrightarrow Z=\int_{\mathbb{R}^{14}_{p_ix_i}}\mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

measure on $\mathbb{R}$ is $(2\pi)^{-1/2}\cdot$*standard*

where $\mathcal{L}(X^s_{ij})=T^{s/2}\mathbb{e}^{\mathbb{i}L(X^s_{ij})}$ and $\mathcal{L}(C_i^\varphi)=T^{\varphi/2}\mathbb{e}^{\mathbb{i}L(C_i^\varphi)}$ ($\mathbb{i}=\sqrt{-1}$ is optional), and

$$L(X^s_{ij})=x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j)+(T^s-1)x_i(p_{i+1}-p_{j+1})+\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\left(\begin{array}{c}(T^s-1)x_ip_j\\+2(1-x_jp_j)\end{array}\right)-1\right)$$

$$L(C_i^\varphi)=x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

better spacing

So $Z=T\int\mathbb{e}^{\mathbb{i}L(\text{knot})}dp_1\dots dp_7dx_1\dots dx_7$, where $L(\text{knot})=$

$$\sum_{i=1}^7 x_i(p_{i+1}-p_i)+(T-1)(x_1(p_2-p_6)+x_6(p_7-p_3)+x_3(p_4-p_8))$$

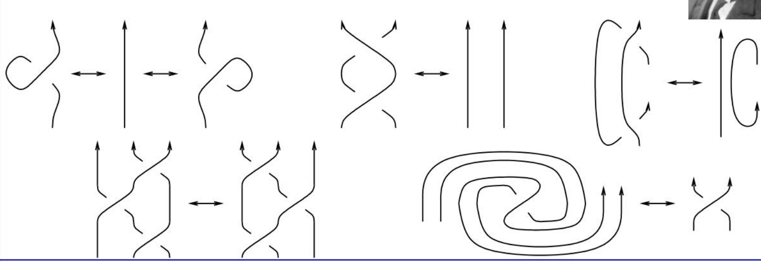
$$+\frac{\epsilon}{2}\left(\begin{array}{c}x_1(p_1-p_5)((T^s-1)x_1p_5+2(1-x_5p_5))-1\\+x_6(p_6-p_2)((T^s-1)x_6p_2+2(1-x_2p_2))-1\\+x_3(p_3-p_7)((T^s-1)x_3p_7+2(1-x_7p_7))-1\\+2x_4p_4-1\end{array}\right),$$

and so $Z=(T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right)=\Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^{-2}}\right)$. Here Δ is the Alexander polynomial and ρ_1 is the Rozansky-Overbay polynomial [Ro, Ov, BV1, BV2].

Theorem. Z is a knot invariant.

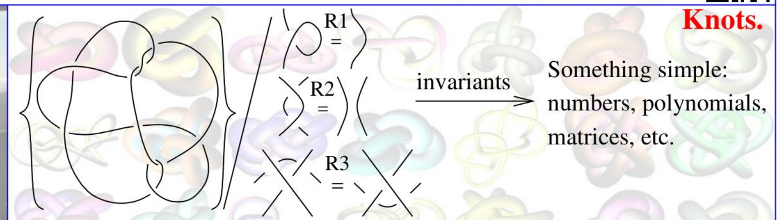
Guido Fubini

Proof. Use Fubini (details later).



To Do. • Human-hard but computer-very-easy (poly time!).
• Strong! • Details of the proof. • Where is it coming from?
• A philosophical point: "Universal invariants" are qualitatively better than representation theory ones.

Knots.

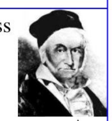


The Good. 1. At the centre of low dimensional topology.
2. "Invariants" connect to pretty much all of algebra.

The Agony. 1&2 don't talk to each other.
• Not enough topological applications for all these invariants.
• The fancy algebra doesn't arise naturally within topology.
$\Longrightarrow$ We're still missing something about the relationship between knots and algebra.

(Alternative) Gaussian Integration.

Gauss



Goal. Compute $\displaystyle\int_{\mathbb{R}^n}dx\exp\left(-\frac12a^{ij}x_ix_j+V(x)\right)$.

Solution. Set $\displaystyle\mathcal{Z}_\lambda(x):=\int_{\mathbb{R}^n}dy\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$.

Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x)=(\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y=0$,

$$\frac12g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)=\frac12\int_{\mathbb{R}^n}dy\,g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$

$$=\frac{1}{2\lambda^2}\int_{\mathbb{R}^n}dy\,a^{ij}y_iy_j\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)=\partial_\lambda\mathcal{Z}_\lambda(x).$$

Hence
$$(*)\qquad\partial_\lambda\mathcal{Z}_\lambda(x)=\frac12g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x),$$

and therefore
$$\mathcal{Z}_\lambda(x)=(\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$

We've just witnessed the birth of "Feynman Diagrams".

Feynman

Even better. With $Z:=\log(\sqrt{\det A}\mathcal{Z})$, by a simple substitution into (*), we get the "Synthesis Equation":

$$Z_0=V,\qquad\partial_\lambda Z_\lambda=\frac12\sum_{i,j=1}^n g_{ij}\left(\partial_{x_i,x_j}Z_\lambda+(\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right),$$

an ODE (in λ) whose solution is pure algebra.

Picard Iteration (used to prove the existance and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda=F(f_\lambda)$ with a given f_0, start with f_0, iterate $f\mapsto f_0+\int_0^\lambda F(f_\lambda)d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!

Picard

Strong. A faster program to compute ρ_1 is available at [BV2]. With it we find that the pair (Δ,ρ_1) attains 53,684 distinct values on the 59,937 prime knots with up to 14 crossings (a deficit of 6,253), whereas the pair ~~$(H,Kh)=$~~ (HOMFLYPT polynomial, Khovanov Homology) attains only 49,149 distinct values on the same knots (a deficit of 10,788).

In as much as we know the pair (Δ,ρ_1) is the strongest knot invariant that can be computed in polynomial time (and hence, even for very large knots).

Acknowledgement. This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Disclaimer. It's fun, but not fully ready.

Preliminaries

This is IType.nb of https://drorbn.net/g24/ap.

```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from
  http://drorbn.net/AP/Talks/Groningen-240530
  to compute rotation numbers.
```

```
CF[ω_. ℰ_𝔼] := CF[ω] × CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
   vs = Cases[ℰ, (x | p | ξ | π)__, ∞] ⋃ {x, p, ϵ};
   Total[CoefficientRules[Expand[ℰ], vs] /.
     (ps_ → c_) :→ Factor[c] (Times @@ vs^ps)] ];
```

Integration

Using Picard Iteration!

```
𝔼 /: 𝔼[A_] × 𝔼[B_] := 𝔼[A + B];
```

```
$π = Identity; (* hacks in pink *)
```

```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) :=
  Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, FZ, a, b},
   n = Length@vs; L0 = L /. ϵ → 0;
   Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /.
      (p | x)__ → 0, {a, n}, {b, n}];
   If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
   Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
   DZ_a_ := ∂_{vs⟦a⟧} Z; DZ_{a_,b_} := ∂_{vs⟦b⟧} DZ_a;
   FZ := CF@$π[1/2 Σ_{a=1}^n Σ_{b=1}^n G⟦a, b⟧ (DZ_{a,b} + DZ_a DZ_b)];
   FixedPoint[(Z = Z0 + ∫_0^λ FZ ⅆλ) &, Z];
   PowerExpand@Factor[ω Δ^(-1/2)] ×
    𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

```
∫𝔼[-μ x²/2 + ⅈ ξ x] ⅆ{x}
```

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

```
L = -1/2 {x₁, x₂}.(a b; b c).{x₁, x₂} + {ξ₁, ξ₂}.{x₁, x₂};
Z12 = ∫𝔼[L] ⅆ{x₁, x₂}
```

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\,(-b^2+a\,c)}+\frac{b\,\xi_1\,\xi_2}{b^2-a\,c}+\frac{a\,\xi_2^2}{2\,(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$$

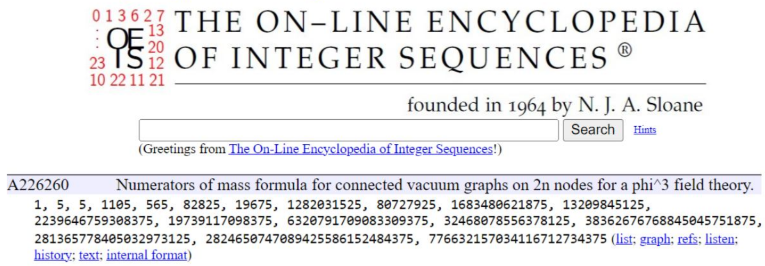
```
{Z1 = ∫𝔼[L] ⅆ{x₁}, Z12 == ∫Z1 ⅆ{x₂}}
```

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\,x_2^2}{2\,a}-\frac{b\,x_2\,\xi_1}{a}+\frac{\xi_1^2}{2\,a}+x_2\,\xi_2\right]}{\sqrt{a}},\ \mathrm{True}\right\}$$

```
$π = Normal[# + O[ϵ]^13] &; ∫𝔼[-ϕ²/2 + ϵ ϕ³/6] ⅆ{ϕ}
```

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24}+\frac{5\,\epsilon^4}{16}+\frac{1105\,\epsilon^6}{1152}+\frac{565\,\epsilon^8}{128}+\frac{82\,825\,\epsilon^{10}}{3072}+\frac{19\,675\,\epsilon^{12}}{96}\right]$$

From https://oeis.org/A226260:


THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.

1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 38362676768845045751875, 281365778405032973125, 2824650747089425586152484375, 776632157034116712734375 (list; graph; refs; listen; history; text; internal format)

The Right-Handed Trefoil

```
K = Mirror@Knot[3, 1]; Features[K]
```

```
Features[7, C₄[-1] X₁,₅[1] X₃,₇[1] X₆,₂[1]]
```

```
ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[
   x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) +
    (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
    (ϵ s / 2) ×
     (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
ℒ[C_{i_}[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧]
vs[K_] :=
  Join @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}]
```

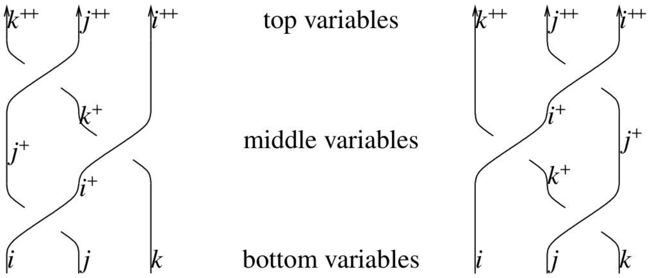
```
{vs[K], ℒ[K]}
```

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$\mathrm{T}\,\mathbb{E}\Big[-2\,\epsilon-p_1\,x_1+\epsilon\,p_1\,x_1+\mathrm{T}\,p_2\,x_1-\epsilon\,p_5\,x_1+(1-\mathrm{T})\,p_6\,x_1+\frac{1}{2}\,(-1+\mathrm{T})\,\epsilon\,p_1\,p_5\,x_1^2+\frac{1}{2}\,(1-\mathrm{T})\,\epsilon\,p_5^2\,x_1^2-p_2\,x_2+p_3\,x_2-p_3\,x_3+\epsilon\,p_3\,x_3+\mathrm{T}\,p_4\,x_3-\epsilon\,p_7\,x_3+(1-\mathrm{T})\,p_8\,x_3+\frac{1}{2}\,(-1+\mathrm{T})\,\epsilon\,p_3\,p_7\,x_3^2+\frac{1}{2}\,(1-\mathrm{T})\,\epsilon\,p_7^2\,x_3^2-p_4\,x_4+\epsilon\,p_4\,x_4+p_5\,x_4-p_5\,x_5+p_6\,x_5-\epsilon\,p_1\,p_5\,x_1\,x_5+\epsilon\,p_5^2\,x_1\,x_5-\epsilon\,p_2\,x_6+(1-\mathrm{T})\,p_3\,x_6-p_6\,x_6+\epsilon\,p_6\,x_6+\mathrm{T}\,p_7\,x_6+\epsilon\,p_2^2\,x_2\,x_6-\epsilon\,p_2\,p_6\,x_2\,x_6+\frac{1}{2}\,(1-\mathrm{T})\,\epsilon\,p_2^2\,x_6^2+\frac{1}{2}\,(-1+\mathrm{T})\,\epsilon\,p_2\,p_6\,x_6^2-p_7\,x_7+p_8\,x_7-\epsilon\,p_3\,p_7\,x_3\,x_7+\epsilon\,p_7^2\,x_3\,x_7\Big]\Big\}$$

```
$π = Normal[# + O[ϵ]^2] &; ∫ℒ[K] ⅆ(vs@K)
```

$$-\frac{\mathrm{i}\,\mathrm{T}\,\mathbb{E}\left[-\frac{(-1+\mathrm{T})^2\,(1+\mathrm{T}^2)\,\epsilon}{(1-\mathrm{T}+\mathrm{T}^2)^2}\right]}{1-\mathrm{T}+\mathrm{T}^2}$$

Invariance Under Reidemeister 3



☺ $\texttt{lhs} = \int (\mathcal{L} \,/@\, (X_{i,j}[1]\; X_{i+1,k}[1]\; X_{j+1,k+1}[1]))\, d\{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$

$\texttt{rhs} = \int (\mathcal{L} \,/@\, (X_{j,k}[1]\; X_{i,k+1}[1]\; X_{i+1,j+1}[1]))\, d\{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$

lhs === rhs

💻 False

Invariance Under Reidemeister 3, Take 2

☺ $\texttt{lhs} = \int (\mathcal{L} \,/@\, (X_{i,j}[1]\; X_{i+1,k}[1]\; X_{j+1,k+1}[1]))\, d\{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$

$\texttt{rhs} = \int (\mathcal{L} \,/@\, (X_{j,k}[1]\; X_{i,k+1}[1]\; X_{i+1,j+1}[1]))\, d\{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$

lhs === rhs

💻 True

☺ lhs

💻 Degenerate Q!

Invariance Under Reidemeister 3, Take 3

☺ $\texttt{lhs} =$

$\int (\mathbb{E}[\mathbb{i}\,\pi_i\, p_i + \mathbb{i}\,\pi_j\, p_j + \mathbb{i}\,\pi_k\, p_k] \times \mathcal{L} \,/@\, (X_{i,j}[1]\; X_{i+1,k}[1]\; X_{j+1,k+1}[1]))$
$d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$

$\texttt{rhs} =$

$\int (\mathbb{E}[\mathbb{i}\,\pi_i\, p_i + \mathbb{i}\,\pi_j\, p_j + \mathbb{i}\,\pi_k\, p_k] \times \mathcal{L} \,/@\, (X_{j,k}[1]\; X_{i,k+1}[1]\; X_{i+1,j+1}[1]))$
$d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$

lhs == rhs

💻 True

☺ lhs

$$
\begin{aligned}
&T^{3/2}\, \mathbb{E}\Big[-\frac{3\epsilon}{2} + \mathbb{i}\, T^2 p_{2+i}\pi_i - \mathbb{i}\,(-1+T)\, T\, p_{2+j}\pi_i + \\
&\mathbb{i}\, T^2 \epsilon\, p_{2+j}\pi_i - \mathbb{i}\,(-1+T)\, p_{2+k}\pi_i + \\
&\mathbb{i}\, T\epsilon\, p_{2+k}\pi_i - \frac{1}{2}(-1+T)\, T^3 \epsilon\, p_{2+i}\, p_{2+j}\pi_i^2 + \\
&\frac{1}{2}(-1+T)\, T^3\epsilon\, p_{2+j}^2\pi_i^2 - \frac{1}{2}(-1+T)\, T^2\epsilon\, p_{2+i}\, p_{2+k}\pi_i^2 + \\
&\frac{1}{2}(-1+T)^2\, T\epsilon\, p_{2+j}\, p_{2+k}\pi_i^2 + \frac{1}{2}(-1+T)\, T\epsilon\, p_{2+k}^2\pi_i^2 + \\
&\mathbb{i}\, T\, p_{2+j}\pi_j - \mathbb{i}\, T\epsilon\, p_{2+j}\pi_j - \mathbb{i}\,(-1+T)\, p_{2+k}\pi_j + \\
&\mathbb{i}\,(-1+2T)\,\epsilon\, p_{2+k}\pi_j + T^3\epsilon\, p_{2+i}\, p_{2+j}\pi_i\pi_j - \\
&T^3\epsilon\, p_{2+j}^2\pi_i\pi_j - (-1+T)\, T^2\epsilon\, p_{2+i}\, p_{2+k}\pi_i\pi_j + \\
&(-1+T)^2\, T\epsilon\, p_{2+j}\, p_{2+k}\pi_i\pi_j + (-1+T)\, T\epsilon\, p_{2+k}^2\pi_i\pi_j - \\
&\frac{1}{2}(-1+T)\, T\epsilon\, p_{2+j}\, p_{2+k}\pi_j^2 + \frac{1}{2}(-1+T)\, T\epsilon\, p_{2+k}^2\pi_j^2 + \\
&\mathbb{i}\, p_{2+k}\pi_k - 2\,\mathbb{i}\,\epsilon\, p_{2+k}\pi_k + T^2\epsilon\, p_{2+i}\, p_{2+k}\pi_i\pi_k - \\
&(-1+T)\, T\epsilon\, p_{2+j}\, p_{2+k}\pi_i\pi_k - T\epsilon\, p_{2+k}^2\pi_i\pi_k + \\
&T\epsilon\, p_{2+j}\, p_{2+k}\pi_j\pi_k - T\epsilon\, p_{2+k}^2\pi_j\pi_k\Big]
\end{aligned}
$$

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at https://drorbn.net/g24/ap.

Where is it coming from? The most honest answer is "we don't know". The second most, "undetermined coefficients for an ansatz that made sense". The ansatz comes from the following principles / earlier work:

Morphisms have generating functions. Indeed, there is an isomorphism

$$\mathcal{G}\colon\ \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j]) \to \mathbb{Q}[\![\xi_i]\!][y_j],$$

and by PBW, many relevant spaces are polynomial rings, though only as vector spaces.

Composition is integration. Indeed, if $f \in \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j])$ and $g \in \mathrm{Hom}(\mathbb{Q}[y_j], \mathbb{Q}[z_k])$, then

$$\mathcal{G}(g \circ f) = \int \mathbb{e}^{-y\cdot\eta} f g\, dy\, d\eta.$$

Use universal invariants. These take values in a universal enveloping algebra (misschien kwantized), and thus they are expressible as long compositions of generating functions.

"Solvable approximation" $\rightsquigarrow$ perturbed Gaussians. Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{b}^{\pm}$ be its upper and lower Borel subalgebras. Then $\mathfrak{b}^{+}$ has a bracket β, and as the dual of $\mathfrak{b}^{-}$ it also has a cobracket δ, and in fact, $\mathfrak{g} \oplus \mathfrak{h} \equiv \mathrm{Double}(\mathfrak{b}^{+}, \beta, \delta)$. Let $\mathfrak{g}_\epsilon := \mathrm{Double}(\mathfrak{b}^{+}, \beta, \epsilon\delta)$ (mod ϵ^{d+1} it is solvable for any d). Then by [BV3, BN] (in the case of $\mathfrak{g} = sl_2$) all the interesting tensors of $\mathcal{U}(\mathfrak{g})$ (quantized or not) are perturbed Gaussian with perturbation parameter ϵ with with understood bounds on the degrees of the perturbations.

References.

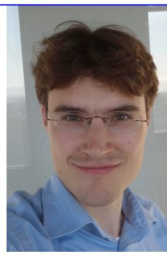
[BN] D. Bar-Natan, *Everything around sl_{2+}^{ϵ} is DoPeGDO. So what?*, talk given in "Quantum Topology and Hyperbolic Geometry Conference", Da Nang, Vietnam, May 2019. Handout and video at ωεβ/DPG.

Thanks for having me in Groningen!

Knot Invariants from Finite Dimensional Integration

ωεβ:=http://drorbn.net/g24

Abstract. For the purpose of today, an "I-Type Knot Invariant" is a knot invariant computed from a knot diagram by integrating the exponential of a Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.



joint with R. van der Veen

Q. Are there any such things? **A.** Yes.

Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.

Q. Didn't Witten do that back in 1988 with path integrals?

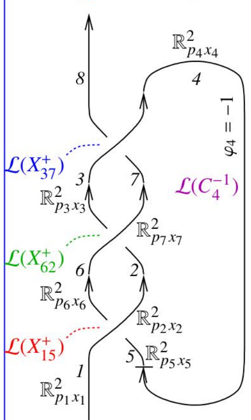
A. No. His constructions are infinite dimensional and far from rigorous.

Q. But integrals belong in analysis!

A. Ours only use squeaky-clean algebra.

The $sl_2^{/\epsilon^2}$ Example.

With T an indeterminate and with $\epsilon^2=0$:



$$\longrightarrow Z=\int_{\mathbb{R}^{14}_{p_ix_i}}\mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})\qquad\text{measure on }\mathbb{R}\text{ is }(2\pi)^{-1/2}\cdot\textit{standard}$$

where $\mathcal{L}(X^s_{ij})=T^{s/2}\mathbb{e}^{\mathbb{i}L(X^s_{ij})}$ and $\mathcal{L}(C^\varphi_i)=T^{\varphi/2}\mathbb{e}^{\mathbb{i}L(C^\varphi_i)}$ ($\mathbb{i}=\sqrt{-1}$ is optional), and

$$L(X^s_{ij})=x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j)+(T^s-1)x_i(p_{i+1}-p_{j+1})+\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\+2(1-x_jp_j)\end{pmatrix}-1\right)$$

$$L(C^\varphi_i)=x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

So $Z=T\int\mathbb{e}^{\mathbb{i}L(\circledcirc)}dp_1\dots dp_7dx_1\dots dx_7$, where $L(\circledcirc)=$

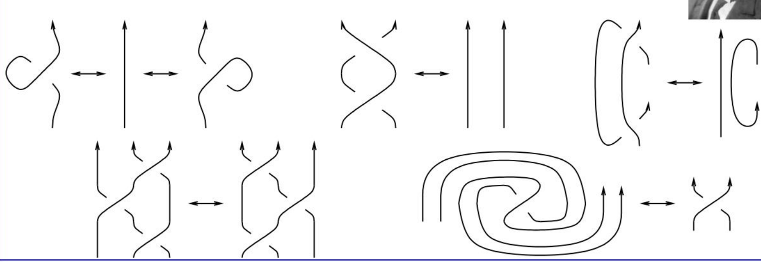
$$\sum_{i=1}^7x_i(p_{i+1}-p_i)+(T-1)(x_1(p_2-p_6)+x_6(p_7-p_3)+x_3(p_4-p_8))+\frac{\epsilon}{2}\begin{pmatrix}x_1(p_1-p_5)((T^s-1)x_1p_5+2(1-x_5p_5))-1\\+x_6(p_6-p_2)((T^s-1)x_6p_2+2(1-x_2p_2))-1\\+x_3(p_3-p_7)((T^s-1)x_3p_7+2(1-x_7p_7))-1\\+2x_4p_4-1\end{pmatrix},$$

and so $Z=(T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right)=\Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^{-2}}\right)$. Here Δ is the Alexander polynomial and ρ_1 is the Rozansky-Overbay polynomial [Ro, Ov, BV1, BV2].

Theorem. Z is a knot invariant.

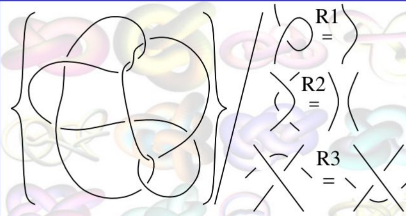
Proof. Use Fubini (details later).

Guido Fubini



To Do. • Human-hard but computer-very-easy (poly time!). • Strong! • Details of the proof. • Where is it coming from? • A philosophical point: "Universal invariants" are qualitatively better than representation theory ones.

Knots.



$\xrightarrow{\text{invariants}}$ Something simple: numbers, polynomials, matrices, etc.

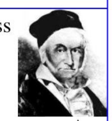
The Good. 1. At the centre of low dimensional topology. 2. "Invariants" connect to pretty much all of algebra.

The Agony. 1&2 don't talk to each other.

- Not enough topological applications for all these invariants.
- The fancy algebra doesn't arise naturally within topology.

$\Longrightarrow$ We're still missing something about the relationship between knots and algebra.

(Alternative) Gaussian Integration.



Gauss

Goal. Compute $\displaystyle\int_{\mathbb{R}^n}dx\exp\left(-\frac12a^{ij}x_ix_j+V(x)\right)$.

Solution. Set $\displaystyle\mathcal{Z}_\lambda(x):=\int_{\mathbb{R}^n}dy\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$.

Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x)=(\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y=0$,

$$\frac12g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)=\frac12\int_{\mathbb{R}^n}dy\,g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)=\frac{1}{2\lambda^2}\int_{\mathbb{R}^n}dy\,a^{ij}y_iy_j\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)=\partial_\lambda\mathcal{Z}_\lambda(x).$$

Hence
$$(*)\qquad\partial_\lambda\mathcal{Z}_\lambda(x)=\frac12g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x),$$
and therefore
$$\mathcal{Z}_\lambda(x)=(\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$

We've just witnessed the birth of "Feynman Diagrams".

Feynman

Even better. With $Z:=\log(\sqrt{\det A}\mathcal{Z})$, by a simple substitution into (*), we get the "Synthesis Equation":

$$Z_0=V,\qquad\partial_\lambda Z_\lambda=\frac12\sum_{i,j=1}^ng_{ij}\left(\partial_{x_i,x_j}Z_\lambda+(\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right),$$

an ODE (in λ) whose solution is pure algebra.

Picard Iteration (used to prove the existance and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda=F(f_\lambda)$ with a given f_0, start with f_0, iterate $f\mapsto f_0+\int_0^\lambda F(f_\lambda)d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!

Picard

Strong. A faster program to compute ρ_1 is available at [BV2]. With it we find that the pair (Δ,ρ_1) attains 53,684 distinct values on the 59,937 prime knots with up to 14 crossings (a deficit of 6,253), whereas the pair (HOMFLYPT polynomial, Khovanov Homology) attains only 49,149 distinct values on the same knots (a deficit of 10,788).

In as much as we know the pair (Δ,ρ_1) is the strongest knot invariant that can be computed in polynomial time (and hence, even for very large knots).

Acknowledgement. This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Disclaimer. It's fun, but not fully ready.

Preliminaries

This is IType.nb of https://drorbn.net/g24/ap.

```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from
  http://drorbn.net/AP/Talks/Groningen-240530
  to compute rotation numbers.
```

```
CF[ω_. ℰ_𝔼] := CF[ω] × CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
   vs = Cases[ℰ, (x | p | ξ | π)__, ∞] ⋃ {x, p, ϵ};
   Total[CoefficientRules[Expand[ℰ], vs] /.
     (ps_ → c_) ⧴ Factor[c] (Times @@ vs^ps)] ];
```

Integration

Using Picard Iteration!

```
𝔼 /: 𝔼[A_] × 𝔼[B_] := 𝔼[A + B];
```

```
$π = Identity; (* hacks in pink *)
```

```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) :=
  Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, FZ, a, b},
   n = Length@vs; L0 = L /. ϵ → 0;
   Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /.
      (p | x)__ → 0, {a, n}, {b, n}];
   If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
   Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
   DZ_a_ := ∂_{vs⟦a⟧} Z; DZ_{a_,b_} := ∂_{vs⟦b⟧} DZ_a;
   FZ := CF@$π[1/2 Σ_{a=1}^{n} Σ_{b=1}^{n} G⟦a, b⟧ (DZ_{a,b} + DZ_a DZ_b)];
   FixedPoint[(Z = Z0 + ∫_0^λ FZ ⅆλ) &, Z];
   PowerExpand@Factor[ω Δ^{-1/2}] ×
    𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

```
∫𝔼[-μ x^2/2 + ⅈ ξ x] ⅆ{x}
```

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

```
L = -1/2 {x1, x2}.(a b
                   b c).{x1, x2} + {ξ1, ξ2}.{x1, x2};
Z12 = ∫𝔼[L] ⅆ{x1, x2}
```

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\left(-b^2+a\,c\right)}+\frac{b\,\xi_1\,\xi_2}{b^2-a\,c}+\frac{a\,\xi_2^2}{2\left(-b^2+a\,c\right)}\right]}{\sqrt{-b^2+a\,c}}$$

```
{Z1 = ∫𝔼[L] ⅆ{x1}, Z12 == ∫Z1 ⅆ{x2}}
```

$$\left\{\frac{\mathbb{E}\left[-\frac{\left(-b^2+a\,c\right)x_2^2}{2\,a}-\frac{b\,x_2\,\xi_1}{a}+\frac{\xi_1^2}{2\,a}+x_2\,\xi_2\right]}{\sqrt{a}},\ \mathtt{True}\right\}$$

```
$π = Normal[# + O[ϵ]^13] &; ∫𝔼[-ϕ^2/2 + ϵ ϕ^3/6] ⅆ{ϕ}
```

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24}+\frac{5\,\epsilon^4}{16}+\frac{1105\,\epsilon^6}{1152}+\frac{565\,\epsilon^8}{128}+\frac{82\,825\,\epsilon^{10}}{3072}+\frac{19\,675\,\epsilon^{12}}{96}\right]$$

From https://oeis.org/A226260:

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.

1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 3836267676884504575187 5, 2813657784050329731 25, 2824650747089425586152484375, 77663215703411671273437 5 (list; graph; refs; listen; history; text; internal format)

The Right-Handed Trefoil

```
K = Mirror@Knot[3, 1]; Features[K]
```

$\mathtt{Features}[7, C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]]$

```
ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[
   x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) +
    (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
    (ϵ s / 2) ×
     (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
ℒ[C_i_[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧]
vs[K_] :=
  Join @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}]
```

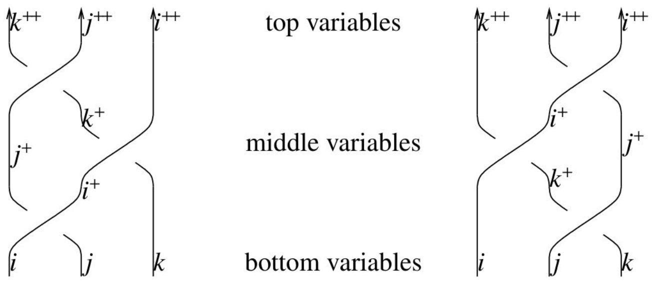
```
{vs[K], ℒ[K]}
```

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-2\,\epsilon-p_1\,x_1+\epsilon\,p_1\,x_1+T\,p_2\,x_1-\epsilon\,p_5\,x_1+(1-T)\,p_6\,x_1+\frac{1}{2}\,(-1+T)\,\epsilon\,p_1\,p_5\,x_1^2+\frac{1}{2}\,(1-T)\,\epsilon\,p_5^2\,x_1^2-p_2\,x_2+p_3\,x_2-p_3\,x_3+\epsilon\,p_3\,x_3+T\,p_4\,x_3-\epsilon\,p_7\,x_3+(1-T)\,p_8\,x_3+\frac{1}{2}\,(-1+T)\,\epsilon\,p_3\,p_7\,x_3^2+\frac{1}{2}\,(1-T)\,\epsilon\,p_7^2\,x_3^2-p_4\,x_4+\epsilon\,p_4\,x_4+p_5\,x_4-p_5\,x_5+p_6\,x_5-\epsilon\,p_1\,p_5\,x_1\,x_5+\epsilon\,p_5^2\,x_1\,x_5-\epsilon\,p_2\,x_6+(1-T)\,p_3\,x_6-p_6\,x_6+\epsilon\,p_6\,x_6+T\,p_7\,x_6+\epsilon\,p_2^2\,x_2\,x_6-\epsilon\,p_2\,p_6\,x_2\,x_6+\frac{1}{2}\,(1-T)\,\epsilon\,p_2^2\,x_6^2+\frac{1}{2}\,(-1+T)\,\epsilon\,p_2\,p_6\,x_6^2-p_7\,x_7+p_8\,x_7-\epsilon\,p_3\,p_7\,x_3\,x_7+\epsilon\,p_7^2\,x_3\,x_7\Big]\Big\}$$

```
$π = Normal[# + O[ϵ]^2] &; ∫ℒ[K] ⅆ(vs@K)
```

$$-\frac{ⅈ\,T\,\mathbb{E}\left[-\frac{(-1+T)^2\left(1+T^2\right)\epsilon}{\left(1-T+T^2\right)^2}\right]}{1-T+T^2}$$

Invariance Under Reidemeister 3



☺ $\mathbf{lhs} = \displaystyle\int (\mathcal{L} \,/@\, (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))$
$\quad d\{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$
$\mathbf{rhs} = \displaystyle\int (\mathcal{L} \,/@\, (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))$
$\quad d\{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$
lhs === rhs

💻 False

Invariance Under Reidemeister 3, Take 2

☺ $\mathbf{lhs} = \displaystyle\int (\mathcal{L} \,/@\, (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))$
$\quad d\{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$
$\mathbf{rhs} = \displaystyle\int (\mathcal{L} \,/@\, (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))$
$\quad d\{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$
lhs === rhs

💻 True

☺ **lhs**

💻 Degenerate Q!

Invariance Under Reidemeister 3, Take 3

☺ $\mathbf{lhs} =$
$\quad \displaystyle\int (\mathbb{E}[i\,\pi_i\, p_i + i\,\pi_j\, p_j + i\,\pi_k\, p_k] \times$
$\quad\quad \mathcal{L} \,/@\, (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))$
$\quad d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$
$\mathbf{rhs} =$
$\quad \displaystyle\int (\mathbb{E}[i\,\pi_i\, p_i + i\,\pi_j\, p_j + i\,\pi_k\, p_k] \times$
$\quad\quad \mathcal{L} \,/@\, (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))$
$\quad d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$
lhs == rhs

💻 True

☺ **lhs**

💻
$$
\begin{aligned}
T^{3/2}\,\mathbb{E}\Big[&-\frac{3\epsilon}{2} + i\,T^2 p_{2+i}\pi_i - i\,(-1+T)\,T\,p_{2+j}\pi_i + {}\\
&i\,T^2\epsilon\,p_{2+j}\pi_i - i\,(-1+T)\,p_{2+k}\pi_i + {}\\
&i\,T\epsilon\,p_{2+k}\pi_i - \frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i^2 + {}\\
&\frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+j}^2\,\pi_i^2 - \frac{1}{2}(-1+T)\,T^2\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i^2 + {}\\
&\frac{1}{2}(-1+T)^2\,T\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i^2 + \frac{1}{2}(-1+T)\,T\epsilon\,p_{2+k}^2\,\pi_i^2 + {}\\
&i\,T\,p_{2+j}\pi_j - i\,T\epsilon\,p_{2+j}\pi_j - i\,(-1+T)\,p_{2+k}\pi_j + {}\\
&i\,(-1+2T)\,\epsilon\,p_{2+k}\pi_j + T^3\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\pi_j - {}\\
&T^3\epsilon\,p_{2+j}^2\,\pi_i\pi_j - (-1+T)\,T^2\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\pi_j + {}\\
&(-1+T)^2\,T\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\pi_j + (-1+T)\,T\epsilon\,p_{2+k}^2\,\pi_i\pi_j - {}\\
&\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j^2 + \frac{1}{2}(-1+T)\,T\epsilon\,p_{2+k}^2\,\pi_j^2 + {}\\
&i\,p_{2+k}\pi_k - 2\,i\,\epsilon\,p_{2+k}\pi_k + T^2\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\pi_k - {}\\
&(-1+T)\,T\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\pi_k - T\epsilon\,p_{2+k}^2\,\pi_i\pi_k + {}\\
&T\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j\pi_k - T\epsilon\,p_{2+k}^2\,\pi_j\pi_k\Big]
\end{aligned}
$$

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at https://drorbn.net/g24/ap.

Where is it coming from? The most honest answer is "we don't know". The second most, "undetermined coefficients for an ansatz that made sense". The ansatz comes from the following principles / earlier work:

Morphisms have generating functions. Indeed, there is an isomorphism

$$\mathcal{G}\colon\ \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j]) \to \mathbb{Q}[\![\xi_i]\!][y_j],$$

and by PBW, many relevant spaces are polynomial rings, though only as vector spaces.

Composition is integration. Indeed, if $f \in \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j])$ and $g \in \mathrm{Hom}(\mathbb{Q}[y_j], \mathbb{Q}[z_k])$, then

$$\mathcal{G}(g \circ f) = \int e^{-y\cdot\eta} f g \, dy\, d\eta$$

Use universal invariants. These take values in a universal enveloping algebra (misschien kwantized), and thus they are expressible as long compositions of generating functions.

"Solvable approximation" $\rightsquigarrow$ perturbed Gaussians. Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{b}^u$ and $\mathfrak{b}^l$ be its upper and lower Borel subalgebras. Then $\mathfrak{b}^u$ has a bracket β, and as the dual of $\mathfrak{b}^l$ it also has a cobracket δ, and in fact, $\mathfrak{g} \oplus \mathfrak{h} \equiv \mathrm{Double}(\mathfrak{b}^u, \beta, \delta)$. Let $\mathfrak{g}^+_\epsilon := \mathrm{Double}(\mathfrak{b}^u, \beta, \epsilon\delta)$ (mod ϵ^{d+1} it is solvable for any d). Then by [BV3, BN] (in the case of $\mathfrak{g} = sl_2$) all the interesting tensors of $\mathcal{U}(\mathfrak{g}^+_\epsilon)$ (quantized or not) are perturbed Gaussian with perturbation parameter ϵ with with understood bounds on the degrees of the perturbations.

References.

[BN] D. Bar-Natan, *Everything around sl_{2+}^ϵ is DoPeGDO. So what?,* talk given in "Quantum Topology and Hyperbolic Geometry Conference", Da Nang, Vietnam, May 2019. Handout and video at ωεβ/DPG.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* Proc. Amer. Math. Soc. **147** (2019) 377–397, arXiv:1708.04853.

[BV2] D. Bar-Natan and R. van der Veen, *A Perturbed-Alexander Invariant,* to appear in Quantum Topology, ωεβ/APAI.

[BV3] D. Bar-Natan and R. van der Veen, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* arXiv:2109.02057.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.

[Ro] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061; *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005; *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

There's more where this came from

Universal is better than rep. theoretical

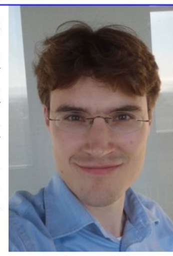
Sort out the i factors

spellcheck?

Activate links.

Knot Invariants from Finite Dimensional Integration

Abstract. For the purpose of today, an "I-Type Knot Invariant" is a knot invariant computed from a knot diagram by integrating the exponential of a Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.

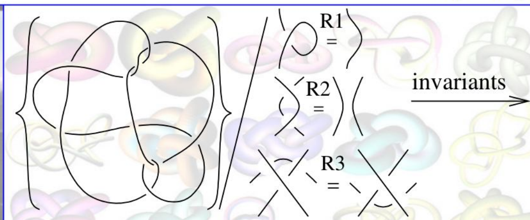


joint with R. van der Veen

Q. Are there any such things? **A.** Yes.
Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.
Q. Didn't Witten do that back in 1988 with path integrals?
A. No. His constructions are infinite dimensional and far from rigorous.
Q. But integrals belong in analysis!
A. Ours only use squeaky-clean algebra.

Continues  Rozansky  Overbay

Knots.



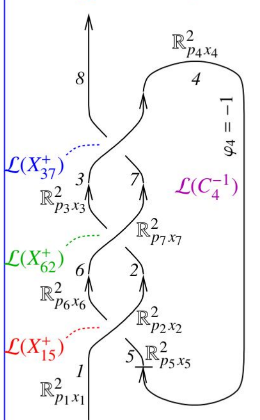
Something simple: numbers, polynomials, matrices, etc.

The Good. 1. At the centre of low dimensional topology.
2. "Invariants" connect to pretty much all of algebra.
The Agony. 1&2 don't talk to each other.
- Not enough topological applications for all these invariants.
- The fancy algebra doesn't arise naturally within topology.

$\Longrightarrow$ We're still missing something about the relationship between knots and algebra.

The $sl_2^{/\epsilon^2}$ Example.

With T an indeterminate and with $\epsilon^2=0$:



$$\longrightarrow Z=\oint_{\mathbb{R}^{14}_{p_ix_i}}\mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

measure on $\mathbb{R}$ is $(2\pi)^{-1/2}\cdot$*standard*

where $\mathcal{L}(X^s_{ij})=T^{s/2}\mathbb{e}^{L(X^s_{ij})}$ and $\mathcal{L}(C^\varphi_i)=T^{\varphi/2}\mathbb{e}^{L(C^\varphi_i)}$, and

$$L(X^s_{ij})=x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j)+(T^s-1)x_i(p_{i+1}-p_{j+1})+\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\+2(1-x_jp_j)\end{pmatrix}-1\right)$$

$$L(C^\varphi_i)=x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

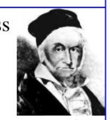
So $Z=T\oint\mathbb{e}^{L(\circledast)}dp_1\dots dp_7dx_1\dots dx_7$, where $L(\circledast)=$

$$\sum_{i=1}^7x_i(p_{i+1}-p_i)+(T-1)(x_1(p_2-p_6)+x_6(p_7-p_3)+x_3(p_4-p_8))$$
$$+\frac{\epsilon}{2}\begin{pmatrix}x_1(p_1-p_5)\left((T-1)x_1p_5+2(1-x_5p_5)\right)-1\\+x_6(p_6-p_2)\left((T-1)x_6p_2+2(1-x_2p_2)\right)-1\\+x_3(p_3-p_7)\left((T-1)x_3p_7+2(1-x_7p_7)\right)-1\\+2x_4p_4-1\end{pmatrix},$$

and so $Z=(T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right)=\Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^{-2}}\right)$. Here Δ is Alexander's polynomial and ρ_1 is Rozansky-Overbay's polynomial [R1]–[R3], [Ov, BV1, BV2].

(Alternative) Gaussian Integration.

Gauss



Goal. Compute (if convergent) $$\int_{\mathbb{R}^n}dx\exp\left(-\frac12a^{ij}x_ix_j+V(x)\right).$$

Solution. Set $$\mathcal{Z}_\lambda(x):=\int_{\mathbb{R}^n}dy\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right).$$

Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x)=(\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y=0$,

$$\frac12g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)=\frac12\int_{\mathbb{R}^n}dy\,g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\frac{1}{2\lambda^2}\int_{\mathbb{R}^n}dy\,a^{ij}y_iy_j\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)=\partial_\lambda\mathcal{Z}_\lambda(x).$$

Hence $$(*)\quad\partial_\lambda\mathcal{Z}_\lambda(x)=\frac12g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x),$$

and therefore $$\mathcal{Z}_\lambda(x)=(\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$

We've just witnessed the birth of "Feynman Diagrams".

Feynman

Even better. With $Z:=\log(\sqrt{\det A}\mathcal{Z})$, by a simple substitution into (*), we get the "Synthesis Equation":

$$Z_0=V,\qquad\partial_\lambda Z_\lambda=\frac12\sum_{i,j=1}^ng_{ij}\left(\partial_{x_i,x_j}Z_\lambda+(\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right),$$

an ODE (in λ) whose solution is pure algebra.

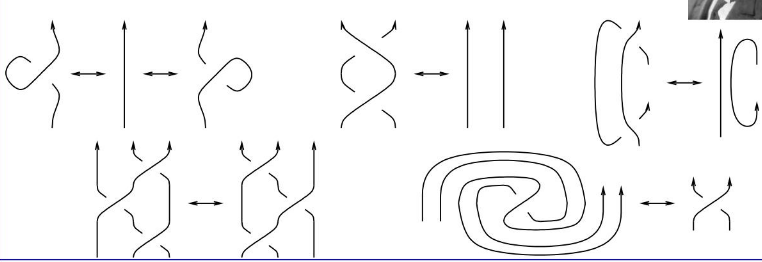
Picard Iteration (used to prove the existence and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda=F(f_\lambda)$ with a given f_0, start with f_0, iterate $f\mapsto f_0+\int_0^\lambda F(f_\lambda)d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!

Picard

Definition. $\oint$: The result of this process, ignoring $\int$ convergence.

Theorem. Z is a knot invariant.
Proof. Use Fubini (details later).

Guido Fubini



Strong. A faster program to compute ρ_1 is available at [BV2]. With it we find that the pair (Δ,ρ_1) attains 53,684 distinct values on the 59,937 prime knots with up to 14 crossings (a deficit of 6,253), whereas the pair (HOMFLYPT polynomial, Khovanov Homology) attains only 49,149 distinct values on the same knots (a deficit of 10,788).
In as much as we know the pair (Δ,ρ_1) is the strongest knot invariant that can be computed in polynomial time (and hence, even for very large knots).

To Do. • Human-hard but computer-very-easy (poly time!).
• Strong! • Details of the proof. • Where is it coming from?
• A philosophical point: "Universal invariants" are qualitatively better than representation theory ones.

Acknowledgement. This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Disclaimer. It's fun, but not fully ready.

Marton W.

Preliminaries

This is IType.nb of ωεβ/ap.

```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from
  http://drorbn.net/AP/Talks/Groningen-240530
  to compute rotation numbers.
```

```
CF[ω_. ℰ_𝔼] := CF[ω] × CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
  vs = Cases[ℰ, (x | p | ξ | π)__, ∞] ⋃ {x, p, ϵ};
  Total[CoefficientRules[Expand[ℰ], vs] /.
    (ps_ → c_) ↦ Factor[c] (Times @@ vs^ps)] ];
```

Integration

Using Picard Iteration!

```
𝔼 /: 𝔼[A_] × 𝔼[B_] := 𝔼[A + B];
```

```
$π = Identity; (* hacks in pink *)
```

```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) :=
 Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, FZ, a, b},
  n = Length@vs; L0 = L /. ϵ → 0;
  Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /.
     (p | x)__ → 0, {a, n}, {b, n}];
  If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
  Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
  DZ_a_ := ∂_{vs⟦a⟧} Z; DZ_{a_,b_} := ∂_{vs⟦b⟧} DZ_a;
  FZ := CF@$π[1/2 Σ_{a=1}^{n} Σ_{b=1}^{n} G⟦a, b⟧ (DZ_{a,b} + DZ_a DZ_b)];
  FixedPoint[(Z = Z0 + ∫_0^λ FZ ⅆλ) &, Z];
  PowerExpand@Factor[ω Δ^(-1/2)] ×
   𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

```
∫𝔼[- μ x^2 / 2 + ⅈ ξ x] ⅆ{x}
```

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

```
L = -1/2 {x1, x2}.(a b; b c).{x1, x2} + {ξ1, ξ2}.{x1, x2};
Z12 = ∫𝔼[L] ⅆ{x1, x2}
```

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\left(-b^2+a c\right)}+\frac{b\,\xi_1\,\xi_2}{b^2-a c}+\frac{a\,\xi_2^2}{2\left(-b^2+a c\right)}\right]}{\sqrt{-b^2+a c}}$$

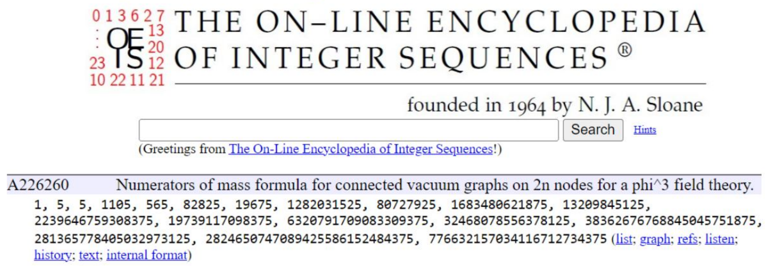
```
{Z1 = ∫𝔼[L] ⅆ{x1}, Z12 == ∫Z1 ⅆ{x2}}
```

$$\left\{\frac{\mathbb{E}\left[-\frac{\left(-b^2+a c\right) x_2^2}{2 a}-\frac{b\, x_2\, \xi_1}{a}+\frac{\xi_1^2}{2 a}+x_2\, \xi_2\right]}{\sqrt{a}},\ \text{True}\right\}$$

```
$π = Normal[# + O[ϵ]^13] &; ∫𝔼[-ϕ^2 / 2 + ϵ ϕ^3 / 6] ⅆ{ϕ}
```

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24}+\frac{5\,\epsilon^4}{16}+\frac{1105\,\epsilon^6}{1152}+\frac{565\,\epsilon^8}{128}+\frac{82825\,\epsilon^{10}}{3072}+\frac{19675\,\epsilon^{12}}{96}\right]$$

From https://oeis.org/A226260:


THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.

1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 38362676768845045751875, 281365778405032973125, 2824650747089425586152484375, 776632157034116712734375 (list; graph; refs; listen; history; text; internal format)

The Right-Handed Trefoil

```
K = Mirror@Knot[3, 1]; Features[K]
```

```
Features[7, C4[-1] X1,5[1] X3,7[1] X6,2[1]]
```

```
ℒ[X_{i_,j_}[s_]] := T^(s/2) 𝔼[
   x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) +
    (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
    (ϵ s / 2) ×
     (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
ℒ[C_{i_}[φ_]] := T^(φ/2) 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧]
vs[K_] :=
 Join @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}]
```

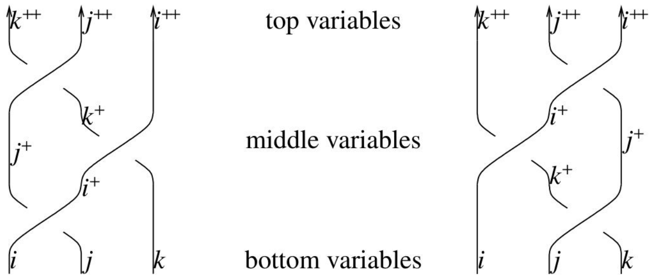
```
{vs[K], ℒ[K]}
```

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-2\epsilon - p_1 x_1 + \epsilon p_1 x_1 + T p_2 x_1 - \epsilon p_5 x_1 + (1-T) p_6 x_1 + \frac{1}{2}(-1+T)\,\epsilon\, p_1 p_5 x_1^2 + \frac{1}{2}(1-T)\,\epsilon\, p_5^2 x_1^2 - p_2 x_2 +$$
$$p_3 x_2 - p_3 x_3 + \epsilon p_3 x_3 + T p_4 x_3 - \epsilon p_7 x_3 + (1-T) p_8 x_3 + \frac{1}{2}(-1+T)\,\epsilon\, p_3 p_7 x_3^2 + \frac{1}{2}(1-T)\,\epsilon\, p_7^2 x_3^2 - p_4 x_4 +$$
$$\epsilon p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \epsilon p_1 p_5 x_1 x_5 + \epsilon p_5^2 x_1 x_5 - \epsilon p_2 x_6 + (1-T) p_3 x_6 - p_6 x_6 +$$
$$\epsilon p_6 x_6 + T p_7 x_6 + \epsilon p_2^2 x_2 x_6 - \epsilon p_2 p_6 x_2 x_6 + \frac{1}{2}(1-T)\,\epsilon\, p_2^2 x_6^2 + \frac{1}{2}(-1+T)\,\epsilon\, p_2 p_6 x_6^2 -$$
$$p_7 x_7 + p_8 x_7 - \epsilon p_3 p_7 x_3 x_7 + \epsilon p_7^2 x_3 x_7\Big]\Big\}$$

```
$π = Normal[# + O[ϵ]^2] &; ∫ℒ[K] ⅆ(vs@K)
```

$$-\frac{ⅈ\, T\, \mathbb{E}\left[-\frac{(-1+T)^2 \left(1+T^2\right) \epsilon}{\left(1-T+T^2\right)^2}\right]}{1-T+T^2}$$

Invariance Under Reidemeister 3




```
☺ lhs = ∫(ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
      ⅆ{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
  rhs = ∫(ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
      ⅆ{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}};
  lhs === rhs
```

```
False
```

Invariance Under Reidemeister 3, Take 2

```
☺ lhs = ∫(ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
      ⅆ{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
  rhs = ∫(ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
      ⅆ{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}};
  lhs === rhs
```

```
True
```

```
☺ lhs
```

```
Degenerate Q!
```

Invariance Under Reidemeister 3, Take 3

```
☺ lhs =
    ∫(𝔼[ⅈ π_i p_i + ⅈ π_j p_j + ⅈ π_k p_k] ×
        ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
      ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1},
        x_{j+1}, x_{k+1}};
  rhs =
    ∫(𝔼[ⅈ π_i p_i + ⅈ π_j p_j + ⅈ π_k p_k] ×
        ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
      ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1},
        x_{j+1}, x_{k+1}};
  lhs == rhs
```

```
True
```

```
☺ lhs
```

$$
\begin{aligned}
&T^{3/2}\,\mathbb{E}\Big[-\frac{3\,\epsilon}{2} + i\,T^2\,p_{2+i}\,\pi_i - i\,(-1+T)\,T\,p_{2+j}\,\pi_i + \\
&\quad i\,T^2\,\epsilon\,p_{2+j}\,\pi_i - i\,(-1+T)\,p_{2+k}\,\pi_i + \\
&\quad i\,T\,\epsilon\,p_{2+k}\,\pi_i - \frac{1}{2}\,(-1+T)\,T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i^2 + \\
&\quad \frac{1}{2}\,(-1+T)\,T^3\,\epsilon\,p_{2+j}^2\,\pi_i^2 - \frac{1}{2}\,(-1+T)\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i^2 + \\
&\quad \frac{1}{2}\,(-1+T)^2\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i^2 + \frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_i^2 + \\
&\quad i\,T\,p_{2+j}\,\pi_j - i\,T\,\epsilon\,p_{2+j}\,\pi_j - i\,(-1+T)\,p_{2+k}\,\pi_j + \\
&\quad i\,(-1+2\,T)\,\epsilon\,p_{2+k}\,\pi_j + T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,\pi_j - \\
&\quad T^3\,\epsilon\,p_{2+j}^2\,\pi_i\,\pi_j - (-1+T)\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,\pi_j + \\
&\quad (-1+T)^2\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_j + (-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_i\,\pi_j - \\
&\quad \frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j^2 + \frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_j^2 + \\
&\quad i\,p_{2+k}\,\pi_k - 2\,i\,\epsilon\,p_{2+k}\,\pi_k + T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,\pi_k - \\
&\quad (-1+T)\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_k - T\,\epsilon\,p_{2+k}^2\,\pi_i\,\pi_k + \\
&\quad T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j\,\pi_k - T\,\epsilon\,p_{2+k}^2\,\pi_j\,\pi_k\Big]
\end{aligned}
$$

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at ωεβ/ap.

Where is it coming from? The most honest answer is "we don't know". The second most, "undetermined coefficients for an ansatz that made sense". The ansatz comes from the following principles / earlier work:

Morphisms have generating functions. Indeed, there is an isomorphism

$$\mathcal{G}\colon\ \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j]) \to \mathbb{Q}[\![\xi_i]\!][y_j],$$

and by PBW, many relevant spaces are polynomial rings, though only as vector spaces.

Composition is integration. Indeed, if $f \in \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j])$ and $g \in \mathrm{Hom}(\mathbb{Q}[y_j], \mathbb{Q}[z_k])$, then

$$\mathcal{G}(g \circ f) = \int \mathbb{e}^{-y\cdot\eta} f g \, dy \, d\eta$$

Use universal invariants. These take values in a universal enveloping algebra (misschien kwantized), and thus they are expressible as long compositions of generating functions. See [La, Oh].

"Solvable approximation" $\rightsquigarrow$ perturbed Gaussians. Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{b}^u$ and $\mathfrak{b}^l$ be its upper and lower Borel subalgebras. Then $\mathfrak{b}^u$ has a bracket β, and as the dual of $\mathfrak{b}^l$ it also has a cobracket δ, and in fact, $\mathfrak{g} \oplus \mathfrak{h} \equiv \mathrm{Double}(\mathfrak{b}^u, \beta, \delta)$. Let $\mathfrak{g}^+_\epsilon := \mathrm{Double}(\mathfrak{b}^u, \beta, \epsilon\delta)$ (mod ϵ^{d+1} it is solvable for any d). Then by [BV3, BN1] (in the case of $\mathfrak{g} = sl_2$) all the interesting tensors of $\mathcal{U}(\mathfrak{g}^+_\epsilon)$ (quantized or not) are perturbed Gaussian with perturbation parameter ϵ with with understood bounds on the degrees of the perturbations.

The Philosophy Corner. "Universal invariants", valued in universal enveloping algebra (possibly quantized) rather than in representations thereof, are a priori better than the representation theoretic ones. They are compatible with strand doubling (the Hopf coproduct), and as the knot genus and the ribbon property for knots are expressible in terms of strand doubling, universal invariants stand a chance to say something about these properties. Indeed, they sometimes do! See e.g. [BN2, GK, LV, BG]. Representation theoretic invariants don't do that!

There's more! To get sl_2 inveriants mod ϵ^3, add the following to $L(X_{ij}^+)$, $L(X_{ij}^-)$, and $L(C_i^\varphi)$, respectively (and see More.nb at ωεβ/ap for the verifications):

☺ $\epsilon^2\, r_2[1, i, j]$

$$\frac{1}{12}\epsilon^2 \left(-6 p_i x_i + 6 p_j x_i - 3(-1+3T) p_i p_j x_i^2 + 3(-1+3T) p_j^2 x_i^2 + 4(-1+T) p_i^2 p_j x_i^3 - 2(-1+T)(5+T) p_i p_j^2 x_i^3 + 2(-1+T)(3+T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j - 6 p_i^2 p_j x_i^2 x_j + 6(2+T) p_i p_j^2 x_i^2 x_j - 6(1+T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2\right)$$

☺ $\epsilon^2\, r_2[-1, i, j]$

$$\frac{1}{12T^2}\epsilon^2 \left(-6T^2 p_i x_i + 6T^2 p_j x_i + 3(-3+T) T p_i p_j x_i^2 - 3(-3+T) T p_j^2 x_i^2 - 4(-1+T) T p_i^2 p_j x_i^3 + 2(-1+T)(1+5T) p_i p_j^2 x_i^3 - 2(-1+T)(1+3T) p_j^3 x_i^3 + 18T^2 p_i p_j x_i x_j - 18T^2 p_j^2 x_i x_j - 6T^2 p_i^2 p_j x_i^2 x_j + 6T(1+2T) p_i p_j^2 x_i^2 x_j - 6T(1+T) p_j^3 x_i^2 x_j - 6T^2 p_i p_j^2 x_i x_j^2 + 6T^2 p_j^3 x_i x_j^2\right)$$

☺ $\epsilon^2\, \gamma_2[\varphi, i]$

$$-\frac{1}{2}\epsilon^2 \varphi^2 p_i x_i$$

The sl_2 formulas mod ϵ^4 are in the last page of the handout of [BN3].

We are very close to having some sl_3 formulas, but they are certainly not ready for prime time.

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[R2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[R3] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

spellcheck?

Activate links.

Knot Invariants from Finite Dimensional Integration

Thanks for having me in Groningen!

ωεβ:=http://drorbn.net/g24

Abstract. For the purpose of today, an "I-Type Knot Invariant" is a knot invariant computed from a knot diagram by integrating the exponential of a Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.

joint with R. van der Veen

Q. Are there any such things? **A.** Yes.

Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.

Q. Didn't Witten do that back in 1988 with path integrals?

A. No. His constructions are infinite dimensional and far from rigorous.

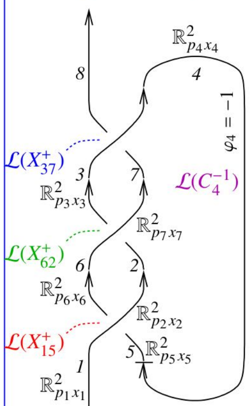
Q. But integrals belong in analysis!

A. Ours only use squeaky-clean algebra.

Continues Rozansky Overbay

The $sl_2^{\epsilon/\epsilon^2}$ Example.

With T an indeterminate and with $\epsilon^2=0$:



$$\longrightarrow Z=\oint_{\mathbb{R}^{14}_{p_ix_i}}\mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

measure on $\mathbb{R}$ is $(2\pi)^{-1/2}\cdot$*standard*

where $\mathcal{L}(X^s_{ij})=T^{s/2}\mathbb{e}^{L(X^s_{ij})}$ and $\mathcal{L}(C_i^\varphi)=T^{\varphi/2}\mathbb{e}^{L(C_i^\varphi)}$, and

$$L(X^s_{ij})=x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j)+(T^s-1)x_i(p_{i+1}-p_{j+1})+\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\+2(1-x_jp_j)\end{pmatrix}-1\right)$$

$$L(C_i^\varphi)=x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

So $Z=T\oint\mathbb{e}^{L(\circledcirc)}dp_1\dots dp_7dx_1\dots dx_7$, where $L(\circledcirc)=$

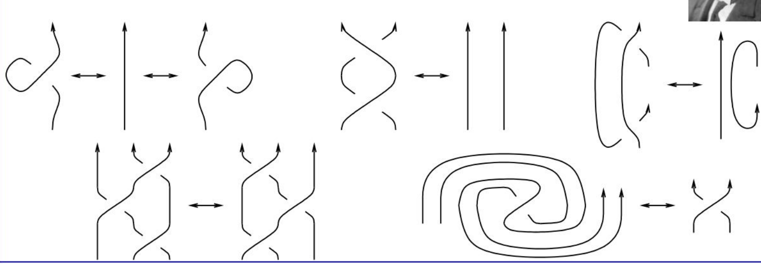
$$\sum_{i=1}^7x_i(p_{i+1}-p_i)+(T-1)(x_1(p_2-p_6)+x_6(p_7-p_3)+x_3(p_4-p_8))+\frac{\epsilon}{2}\begin{pmatrix}x_1(p_1-p_5)((T-1)x_1p_5+2(1-x_5p_5))-1\\+x_6(p_6-p_2)((T-1)x_6p_2+2(1-x_2p_2))-1\\+x_3(p_3-p_7)((T-1)x_3p_7+2(1-x_7p_7))-1\\+2x_4p_4-1\end{pmatrix},$$

and so $Z=(T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right)=\Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^{-2}}\right)$. Here Δ is Alexander's polynomial and ρ_1 is Rozansky-Overbay's polynomial [R1]–[R3], [Ov, BV1, BV2].

Theorem. Z is a knot invariant.

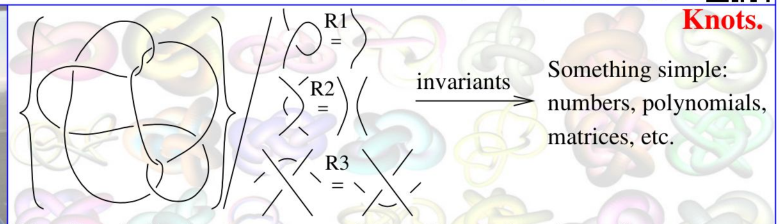
Proof. Use Fubini (details later).

Guido Fubini



To Do. • Human-hard but computer-very-easy (poly time!). • Strong! • Details of the proof. • Where is it coming from? • A philosophical point: "Universal invariants" are qualitatively better than representation theory ones.

Knots.



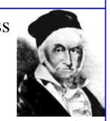
The Good. 1. At the centre of low dimensional topology. 2. "Invariants" connect to pretty much all of algebra.

The Agony. 1&2 don't talk to each other.

- Not enough topological applications for all these invariants.
- The fancy algebra doesn't arise naturally within topology.

$\Longrightarrow$ We're still missing something about the relationship between knots and algebra.

(Alternative) Gaussian Integration.



Gauss

Goal. Compute (if convergent) $\displaystyle\int_{\mathbb{R}^n}dx\exp\left(-\frac{1}{2}a^{ij}x_ix_j+V(x)\right)$.

Solution. Set $\displaystyle\mathcal{Z}_\lambda(x):=\lambda^{n/2}\int_{\mathbb{R}^n}dy\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$.

Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x)=(\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y=0$,

$$\frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)$$
$$=\frac{1}{2}\int_{\mathbb{R}^n}dy\,g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\frac{1}{2\lambda^2}\int_{\mathbb{R}^n}dy\left(g_{ij}a^{ii'}a^{jj'}y_{i'}y_{j'}+\lambda g_{ij}a^{ji}\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\frac{1}{2\lambda^2}\int_{\mathbb{R}^n}dy\left(a^{ij}y_iy_j+\lambda n\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\partial_\lambda\mathcal{Z}_\lambda(x).$$

Hence
$$(*)\qquad\partial_\lambda\mathcal{Z}_\lambda(x)=\frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x),$$
and therefore
$$\mathcal{Z}_\lambda(x)=(\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$

We've just witnessed the birth of "Feynman Diagrams".

Feynman

Even better. With $Z_\lambda:=\log(\sqrt{\det A}\mathcal{Z}_\lambda)$, by a simple substitution into (*), we get the "Synthesis Equation":

$$Z_0=V,\qquad\partial_\lambda Z_\lambda=\frac{1}{2}\sum_{i,j=1}^ng_{ij}\left(\partial_{x_i,x_j}Z_\lambda+(\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right),$$

an ODE (in λ) whose solution is pure algebra.

Picard Iteration (used to prove the existence and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda=F(f_\lambda)$ with a given f_0, start with f_0, iterate $f\mapsto f_0+\int_0^\lambda F(f_\lambda)d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!

Picard

Definition. $\oint$: The result of this process, ignoring the convergence of the actual integral.

Strong. A faster program to compute ρ_1 is available at [BV2]. With it we find that the pair (Δ,ρ_1) attains 53,684 distinct values on the 59,937 prime knots with up to 14 crossings (a deficit of 6,253), whereas the pair (HOMFLYPT polynomial, Khovanov Homology) attains only 49,149 distinct values on the same knots (a deficit of 10,788).

In as much as we know the pair (Δ,ρ_1) is the strongest knot invariant that can be computed in polynomial time (and hence, even for very large knots).

Preliminaries

This is IType.nb of ωεβ/ap.

```
☺ Once[<< KnotTheory`; << Rot.m];
```

```
💻 Loading KnotTheory` version
     of February 2, 2020, 10:53:45.2097.
   Read more at http://katlas.org/wiki/KnotTheory.
```

```
💻 Loading Rot.m from
     http://drorbn.net/AP/Talks/Groningen-240530
     to compute rotation numbers.
```

```
☺ CF[ω_. ℰ_𝔼] := CF[ω] × CF /@ ℰ;
  CF[ℰ_List] := CF /@ ℰ;
  CF[ℰ_] := Module[{vs, ps, c},
     vs = Cases[ℰ, (x | p | ξ | π)__, ∞] ⋃ {x, p, ϵ};
     Total[CoefficientRules[Expand[ℰ], vs] /.
       (ps_ → c_) ⧴ Factor[c] (Times @@ vs^ps)]];
```

Integration

Using Picard Iteration!

```
☺ 𝔼 /: 𝔼[A_] × 𝔼[B_] := 𝔼[A + B];
```

```
☺ $π = Identity; (* hacks in pink *)
```

```
☺ Unprotect[Integrate];
  ∫ω_. 𝔼[L_] ⅆ(vs_List) :=
   Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, FZ, a, b},
    n = Length@vs; L0 = L /. ϵ → 0;
    Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /.
       (p | x)__ → 0, {a, n}, {b, n}];
    If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
    Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
    DZ_a_ := ∂_{vs⟦a⟧} Z; DZ_a_,b_ := ∂_{vs⟦b⟧} DZ_a;
    FZ := CF@$π[1/2 Σ_{a=1}^{n} Σ_{b=1}^{n} G⟦a, b⟧ (DZ_{a,b} + DZ_a DZ_b)];
    FixedPoint[(Z = Z0 + ∫_0^λ FZ ⅆλ) &, Z];
    PowerExpand@Factor[ω Δ^{-1/2}] ×
     𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
  Protect[Integrate];
```

☺ $\int \mathbb{E}\left[-\mu x^2/2 + i\,\xi\,x\right] ⅆ\{x\}$

💻 $\dfrac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$

☺ $L = -\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c\end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$

$Z12 = \int \mathbb{E}[L]\, ⅆ\{x_1, x_2\}$

💻 $\dfrac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2(-b^2+a\,c)} + \frac{b\,\xi_1\,\xi_2}{b^2-a\,c} + \frac{a\,\xi_2^2}{2(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$

☺ $\left\{Z1 = \int \mathbb{E}[L]\, ⅆ\{x_1\},\ Z12 == \int Z1\, ⅆ\{x_2\}\right\}$

💻 $\left\{\dfrac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\,x_2^2}{2a} - \frac{b\,x_2\,\xi_1}{a} + \frac{\xi_1^2}{2a} + x_2\,\xi_2\right]}{\sqrt{a}},\ \text{True}\right\}$

☺ $\$\pi = \text{Normal}[\# + O[\epsilon]^{13}]\ \&;\ \int \mathbb{E}\left[-\phi^2/2 + \epsilon\,\phi^3/6\right] ⅆ\{\phi\}$

💻 $\mathbb{E}\left[\frac{5\,\epsilon^2}{24} + \frac{5\,\epsilon^4}{16} + \frac{1105\,\epsilon^6}{1152} + \frac{565\,\epsilon^8}{128} + \frac{82\,825\,\epsilon^{10}}{3072} + \frac{19\,675\,\epsilon^{12}}{96}\right]$

From https://oeis.org/A226260:


THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.

1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 38362676768845045751875, 281365778405032973125, 2824650747089425586152484375, 776632157034116712734375 (list; graph; refs; listen; history; text; internal format)

The Right-Handed Trefoil

```
☺ K = Mirror@Knot[3, 1]; Features[K]
```

💻 $\text{Features}[7, C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]]$

```
☺ ℒ[X_i_,j_[s_]] := T^{s/2} 𝔼[
     x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) +
      (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
      (ϵ s / 2) ×
       (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
  ℒ[C_i_[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
  ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧]
  vs[K_] :=
   Join @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}]
```

```
☺ {vs[K], ℒ[K]}
```

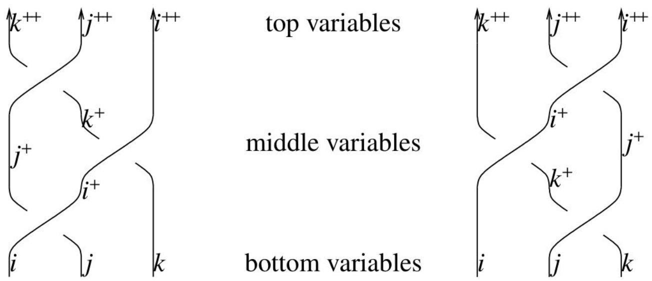
💻 $\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$
$T\,\mathbb{E}\Big[-2\epsilon - p_1 x_1 + \epsilon\, p_1 x_1 + T\, p_2 x_1 - \epsilon\, p_5 x_1 + (1-T)\, p_6 x_1 + \frac{1}{2}(-1+T)\,\epsilon\, p_1 p_5 x_1^2 + \frac{1}{2}(1-T)\,\epsilon\, p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 - p_3 x_3 + \epsilon\, p_3 x_3 + T\, p_4 x_3 - \epsilon\, p_7 x_3 + (1-T)\, p_8 x_3 + \frac{1}{2}(-1+T)\,\epsilon\, p_3 p_7 x_3^2 + \frac{1}{2}(1-T)\,\epsilon\, p_7^2 x_3^2 - p_4 x_4 + \epsilon\, p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \epsilon\, p_1 p_5 x_1 x_5 + \epsilon\, p_5^2 x_1 x_5 - \epsilon\, p_2 x_6 + (1-T)\, p_3 x_6 - p_6 x_6 + \epsilon\, p_6 x_6 + T\, p_7 x_6 + \epsilon\, p_2^2 x_2 x_6 - \epsilon\, p_2 p_6 x_2 x_6 + \frac{1}{2}(1-T)\,\epsilon\, p_2^2 x_6^2 + \frac{1}{2}(-1+T)\,\epsilon\, p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \epsilon\, p_3 p_7 x_3 x_7 + \epsilon\, p_7^2 x_3 x_7\Big]\Big\}$

☺ $\$\pi = \text{Normal}[\# + O[\epsilon]^2]\ \&;\ \int \mathcal{L}[K]\, ⅆ(vs@K)$

💻 $-\dfrac{i\, T\, \mathbb{E}\left[-\frac{(-1+T)^2 (1+T^2)\,\epsilon}{(1-T+T^2)^2}\right]}{1-T+T^2}$

?

Invariance Under Reidemeister 3



☺
```
lhs = ∫(ℒ /@ (X[i,j][1] X[i+1,k][1] X[j+1,k+1][1]))
    d{p[i+1], p[j+1], p[k+1], x[i+1], x[j+1], x[k+1]};
rhs = ∫(ℒ /@ (X[j,k][1] X[i,k+1][1] X[i+1,j+1][1]))
    d{x[i+1], p[i+1], p[j+1], p[k+1], x[j+1], x[k+1]};
lhs === rhs
```

💻 False

Invariance Under Reidemeister 3, Take 2

☺
```
lhs = ∫(ℒ /@ (X[i,j][1] X[i+1,k][1] X[j+1,k+1][1]))
    d{x[i], x[j], x[k], p[i+1], p[j+1], p[k+1], x[i+1], x[j+1], x[k+1]};
rhs = ∫(ℒ /@ (X[j,k][1] X[i,k+1][1] X[i+1,j+1][1]))
    d{x[i], x[j], x[k], x[i+1], p[i+1], p[j+1], p[k+1], x[j+1], x[k+1]};
lhs === rhs
```

💻 True

☺ `lhs`

💻 Degenerate Q!

Invariance Under Reidemeister 3, Take 3

☺
```
lhs =
  ∫(𝔼[i π[i] p[i] + i π[j] p[j] + i π[k] p[k]] ×
      ℒ /@ (X[i,j][1] X[i+1,k][1] X[j+1,k+1][1]))
    d{p[i], p[j], p[k], x[i], x[j], x[k], p[i+1], p[j+1], p[k+1], x[i+1],
      x[j+1], x[k+1]};
rhs =
  ∫(𝔼[i π[i] p[i] + i π[j] p[j] + i π[k] p[k]] ×
      ℒ /@ (X[j,k][1] X[i,k+1][1] X[i+1,j+1][1]))
    d{p[i], p[j], p[k], x[i], x[j], x[k], p[i+1], p[j+1], p[k+1], x[i+1],
      x[j+1], x[k+1]};
lhs == rhs
```

💻 True

☺ `lhs`

💻
$$\begin{aligned}
T^{3/2}\,\mathbb{E}\Big[& -\frac{3\epsilon}{2} + i\,T^2 p_{2+i}\pi_i - i\,(-1+T)\,T\,p_{2+j}\pi_i + \\
& i\,T^2\epsilon\,p_{2+j}\pi_i - i\,(-1+T)\,p_{2+k}\pi_i + \\
& i\,T\epsilon\,p_{2+k}\pi_i - \frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+i}p_{2+j}\pi_i^2 + \\
& \frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+j}^2\pi_i^2 - \frac{1}{2}(-1+T)\,T^2\epsilon\,p_{2+i}p_{2+k}\pi_i^2 + \\
& \frac{1}{2}(-1+T)^2\,T\epsilon\,p_{2+j}p_{2+k}\pi_i^2 + \frac{1}{2}(-1+T)\,T\epsilon\,p_{2+k}^2\pi_i^2 + \\
& i\,T\,p_{2+j}\pi_j - i\,T\epsilon\,p_{2+j}\pi_j - i\,(-1+T)\,p_{2+k}\pi_j + \\
& i\,(-1+2T)\,\epsilon\,p_{2+k}\pi_j + T^3\epsilon\,p_{2+i}p_{2+j}\pi_i\pi_j - \\
& T^3\epsilon\,p_{2+j}^2\pi_i\pi_j - (-1+T)\,T^2\epsilon\,p_{2+i}p_{2+k}\pi_i\pi_j + \\
& (-1+T)^2\,T\epsilon\,p_{2+j}p_{2+k}\pi_i\pi_j + (-1+T)\,T\epsilon\,p_{2+k}^2\pi_i\pi_j - \\
& \frac{1}{2}(-1+T)\,T\epsilon\,p_{2+j}p_{2+k}\pi_j^2 + \frac{1}{2}(-1+T)\,T\epsilon\,p_{2+k}^2\pi_j^2 + \\
& i\,p_{2+k}\pi_k - 2\,i\,\epsilon\,p_{2+k}\pi_k + T^2\epsilon\,p_{2+i}p_{2+k}\pi_i\pi_k - \\
& (-1+T)\,T\epsilon\,p_{2+j}p_{2+k}\pi_i\pi_k - T\epsilon\,p_{2+k}^2\pi_i\pi_k + \\
& T\epsilon\,p_{2+j}p_{2+k}\pi_j\pi_k - T\epsilon\,p_{2+k}^2\pi_j\pi_k\Big]
\end{aligned}$$

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at ωεβ/ap.

Where is it coming from? The most honest answer is "we don't know". The second most, "undetermined coefficients for an ansatz that made sense". The ansatz comes from the following principles / earlier work:

Morphisms have generating functions. Indeed, there is an isomorphism

$$\mathcal{G}\colon\ \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j]) \to \mathbb{Q}[\![\xi_i]\!][y_j],$$

and by PBW, many relevant spaces are polynomial rings, though only as vector spaces.

Composition is integration. Indeed, if $f \in \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j])$ and $g \in \mathrm{Hom}(\mathbb{Q}[y_j], \mathbb{Q}[z_k])$, then

$$\mathcal{G}(g \circ f) = \int \mathbb{e}^{-y\cdot\eta} f g\, dy\, d\eta$$

Use universal invariants. These take values in a universal enveloping algebra (misschien kwantized), and thus they are expressible as long compositions of generating functions. See [La, Oh].

"Solvable approximation" ⇝ perturbed Gaussians. Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{b}^u$ and $\mathfrak{b}^l$ be its upper and lower Borel subalgebras. Then $\mathfrak{b}^u$ has a bracket β, and as the dual of $\mathfrak{b}^l$ it also has a cobracket δ, and in fact, $\mathfrak{g} \oplus \mathfrak{h} \equiv \mathrm{Double}(\mathfrak{b}^u, \beta, \delta)$. Let $\mathfrak{g}^+_\epsilon := \mathrm{Double}(\mathfrak{b}^u, \beta, \epsilon\delta)$ (mod ϵ^{d+1} it is solvable for any d). Then by [BV3, BN1] (in the case of $\mathfrak{g} = sl_2$) all the interesting tensors of $\mathcal{U}(\mathfrak{g}^+_\epsilon)$ (quantized or not) are perturbed Gaussian with perturbation parameter ϵ with with understood bounds on the degrees of the perturbations.

The Philosophy Corner. "Universal invariants", valued in universal enveloping algebra (possibly quantized) rather than in representations thereof, are a priori better than the representation theoretic ones. They are compatible with strand doubling (the Hopf coproduct), and as the knot genus and the ribbon property for knots are expressible in terms of strand doubling, universal invariants stand a chance to say something about these properties. Indeed, they sometimes do! See e.g. [BN2, GK, LV, BG]. Representation theoretic invariants don't do that!

There's more! To get sl_2 invariants mod ϵ^3, add the following to $L(X^+_{ij})$, $L(X^-_{ij})$, and $L(C^\varphi_i)$, respectively (and see More.nb at ωεβ/ap for the verifications):

☺ `ε^2 r2[1, i, j]`

$$\frac{1}{12}\epsilon^2\left(-6 p_i x_i + 6 p_j x_i - 3(-1+3T)p_i p_j x_i^2 + 3(-1+3T)p_j^2 x_i^2 + 4(-1+T)p_i^2 p_j x_i^3 - 2(-1+T)(5+T)p_i p_j^2 x_i^3 + 2(-1+T)(3+T)p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j - 6 p_i^2 p_j x_i^2 x_j + 6(2+T)p_i p_j^2 x_i^2 x_j - 6(1+T)p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2\right)$$

☺ `ε^2 r2[-1, i, j]`

$$\frac{1}{12 T^2}\epsilon^2\left(-6T^2 p_i x_i + 6T^2 p_j x_i + 3(-3+T)T p_i p_j x_i^2 - 3(-3+T)T p_j^2 x_i^2 - 4(-1+T)T p_i^2 p_j x_i^3 + 2(-1+T)(1+5T)p_i p_j^2 x_i^3 - 2(-1+T)(1+3T)p_j^3 x_i^3 + 18T^2 p_i p_j x_i x_j - 18T^2 p_j^2 x_i x_j - 6T^2 p_i^2 p_j x_i^2 x_j + 6T(1+2T)p_i p_j^2 x_i^2 x_j - 6T(1+T)p_j^3 x_i^2 x_j - 6T^2 p_i p_j^2 x_i x_j^2 + 6T^2 p_j^3 x_i x_j^2\right)$$

☺ `ε^2 γ2[φ, i]`

$$-\frac{1}{2}\epsilon^2\varphi^2 p_i x_i$$

The sl_2 formulas mod ϵ^4 are in the last page of the handout of [BN3].

We are very close to having some sl_3 formulas, but they are certainly not ready for prime time.

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Disclaimer. It's fun, but not fully ready.