Pensieve header: Proof of invariance of $\rho_2$ using integration techniques; continues pensieve://Talks/Groningen-240530.

## Initialization

In[◦]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Beijing-2407"];
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Beijing-2407 to compute rotation numbers.
```

## Initialization

In[◦]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CCF[ℰ_] := Factor[ℰ];
CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs = Cases[ℰ, (x | p | π)__, ∞] ⋃ {x, p, ϵ}, ps, c},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ CCF[c] (Times @@ vs^ps)]];
```

## The Basic Feynman Ring

In[◦]:=
```
𝒮 = {x, x_, y, z};
q_{x_,y_}[f_] := (∂_{x,y} f) /. Thread[𝒮 → 0];
θ_{x_,y_} := x y;
f_ ≡ 0 := f === 0;
Ev_{vs_List→0}[f_] := CF[f /. Thread[vs → 0]]
```

## The ϵSeries Feynman Ring

```
In[∘]:= 𝓈 = {x, y, z, ϕ, x_, p_, x̲_, p̲_};
q_{x_,y_}[ser_ϵSeries] := (∂_{x,y} ser⟦1⟧) /. Thread[𝓈 → 0];
θ_{x_,y_} := x y;
ϵSeries /: D[ser_ϵSeries, vs___] := D[#, vs] & /@ ser;
ϵSeries /: Plus[ss___ϵSeries] /; Length[{ss}] > 1 := Module[{l = Min[Length /@ {ss}]},
   ϵSeries @@ Total[Take[List @@ #, l] & /@ {ss}]]
ϵSeries /: t_ + ser_ϵSeries := MapAt[(# + t) &, ser, 1];
ϵSeries /: s1_ϵSeries * s2_ϵSeries := ϵSeries @@ Table[
     Sum[s1⟦ii + 1⟧ s2⟦kk - ii + 1⟧, {ii, 0, kk}], {kk, 0, Min[Length@s1, Length@s2] - 1}];
ϵSeries /: c_ * ser_ϵSeries := (c #) & /@ ser;
ser_ϵSeries ≡ 0 := And @@ ((# === 0) & /@ ser);
ϵSeries /: Integrate[ser_ϵSeries, pars__] := ϵSeries @@ (Integrate[#, pars] & /@ ser);
ϵSeries /: Ev_{vs_List→0}[ser_ϵSeries] := ser /. Thread[vs → 0];
CF[ser_ϵSeries] := CF /@ ser;
```

## Integration

Using Picard Iteration!

```
In[∘]:= 𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B]
```

```
In[∘]:= 𝔼[sd_SeriesData] /; (List @@ sd)⟦{1, 2, 4, 6}⟧ === {ϵ, 0, 0, 1} :=
  𝔼[ϵSeries @@ PadRight[sd⟦3⟧, sd⟦5⟧, 0]]
```

Following a program in Projects/FullDoPeGDO/Engine.nb, we write $Z_\lambda = \sum Z[m]\,\lambda^m$.

In[∘]:=
```
Unprotect[Integrate];
Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, Q, Δ, G, a, b, m, m1, $m}, Clear[Z];
  n = Length@vs;
  Q = Table[q_{vs⟦a⟧,vs⟦b⟧}[L], {a, n}, {b, n}];
  If[(Δ = CF@Det[-Q]) == 0, Message[Integrate::sing]; Return[]];
  G = CF[-Inverse[Q] / 2];
  Z[] = Z[0] = CF[L - Sum[Q⟦a, b⟧ θ_{vs⟦a⟧,vs⟦b⟧}, {a, n}, {b, n}] / 2];
  Z[m_, a_] := Z[m, a] = CF@D[Z[m], vs⟦a⟧];
  Z[m_, a_, b_] /; a ≤ b := Z[m, a, b] = CF@D[Z[m, a], vs⟦b⟧];
  Z[m_, a_, b_] /; a > b := Z[m, b, a];
  For[$m = m = 0, m ≤ 2 $m, ++m,
   Z[m + 1] = CF@Sum[Sum[If[G⟦a, b⟧ === 0, 0,
        G⟦a, b⟧/(m + 1) (Z[m, a, b] + Sum[Z[m1, a] Z[m - m1, b], {m1, 0, m}])],
       {a, n}], {b, n}];
   If[! (Z[m + 1] ≡ 0), $m = m + 1; Z[] += Z[m + 1]];
  ];
  PowerExpand@Factor[ω Δ^(-1/2)] 𝔼[CF[Ev_{vs→0}[Z[]]]]
 ];
Protect[Integrate];
```

In[∘]:=
```
∫𝔼[- μ x^2 / 2 + ⅈ ξ x] ⅆ{x}
```

Out[∘]=

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

In[∘]:=
```
L = -1/2 {x1, x2}.(a b; b c).{x1, x2} + {ξ1, ξ2}.{x1, x2};
Z12 = ∫𝔼[L] ⅆ{x1, x2}
```

Out[∘]=

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2-2\,b\,\xi_1\,\xi_2+a\,\xi_2^2}{2\,\left(-b^2+a\,c\right)}\right]}{\sqrt{-b^2+a\,c}}$$

In[∘]:=
```
{Z1 = ∫𝔼[L] ⅆ{x1}, Z12 == ∫Z1 ⅆ{x2}}
```

Out[∘]=

$$\left\{\frac{\mathbb{E}\left[-\frac{\left(-b^2+a\,c\right)\,x_2^2}{2\,a}+\frac{\xi_1^2}{2\,a}+\frac{x_2\,\left(-b\,\xi_1+a\,\xi_2\right)}{a}\right]}{\sqrt{a}},\ \text{True}\right\}$$

## Integration of ϵSeries

In[∘]:=
```
∫𝔼[-x^2/2 + ϵ x^3/6 + O[ϵ]^13] ⅆ{x}
```

Out[∘]=

$$\mathbb{E}\left[\epsilon\text{Series}\left[0,\,0,\,\frac{5}{24},\,0,\,\frac{5}{16},\,0,\,\frac{1105}{1152},\,0,\,\frac{565}{128},\,0,\,\frac{82\,825}{3072},\,0,\,\frac{19\,675}{96}\right]\right]$$

## The $\rho_2$ Integrand

Adopted from pensieve://Talks//Oaxaca-2210/Rho.nb.

In[∘]:=
```
𝒮 = {x_, p_};
q[s_, i_, j_] := x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1});
r1[s_, i_, j_] := s/2 (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - p_j x_j)) - 1);
r2[1, i_, j_] :=
  (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
     2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j -
     6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12;
r2[-1, i_, j_] :=
  (-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 - 4 (-1 + T) T p_i^2 p_j x_i^3 +
     2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j -
     18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j -
     6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2);
γ1[φ_, k_] := φ (1/2 - x_k p_k);
γ2[φ_, k_] := -φ^2 p_k x_k / 2;
ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[q[s, i, j] + ϵ r1[s, i, j] + ϵ^2 r2[s, i, j] + O[ϵ]^3];
ℒ[C_k_[φ_]] := T^{φ/2} 𝔼[-x_k (p_k - p_{k+1}) + ϵ γ1[φ, k] + ϵ^2 γ2[φ, k] + O[ϵ]^3];
ℒ[K_] := (2 π)^{-Features[K]⟦1⟧} CF[ℒ /@ Features[K]⟦2⟧];
vs[K_] := Union @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}]
```

exec
```
nb2tex$PDFWidth *= 1.25;
```

pdf

In[∘]:=
```
ϵ^2 r2[1, i, j]
```

Out[∘]=
pdf

$$\frac{1}{12}\epsilon^2\left(-6\,p_i\,x_i+6\,p_j\,x_i-3\,(-1+3\,T)\,p_i\,p_j\,x_i^2+3\,(-1+3\,T)\,p_j^2\,x_i^2+4\,(-1+T)\,p_i^2\,p_j\,x_i^3-2\,(-1+T)\,(5+T)\,p_i\,p_j^2\,x_i^3+2\,(-1+T)\,(3+T)\,p_j^3\,x_i^3+18\,p_i\,p_j\,x_i\,x_j-18\,p_j^2\,x_i\,x_j-6\,p_i^2\,p_j\,x_i^2\,x_j+6\,(2+T)\,p_i\,p_j^2\,x_i^2\,x_j-6\,(1+T)\,p_j^3\,x_i^2\,x_j-6\,p_i\,p_j^2\,x_i\,x_j^2+6\,p_j^3\,x_i\,x_j^2\right)$$

pdf

In[∘]:= **$\epsilon^2\, r_2[-1, i, j]$**

Out[∘]=
pdf

$$\frac{1}{12 T^2} \epsilon^2 \left(-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3+T) T p_i p_j x_i^2 - 3 (-3+T) T p_j^2 x_i^2 - 4 (-1+T) T p_i^2 p_j x_i^3 + 2 (-1+T) (1+5 T) p_i p_j^2 x_i^3 - 2 (-1+T) (1+3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j - 18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1+2 T) p_i p_j^2 x_i^2 x_j - 6 T (1+T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2\right)$$

pdf

In[∘]:= **$\epsilon^2\, \gamma_2[\varphi, i]$**

Out[∘]=
pdf

$$-\frac{1}{2} \epsilon^2 \varphi^2 p_i x_i$$

exec

```
nb2tex$PDFWidth /= 1.25;
```

In[∘]:= **Features[Knot[3, 1]]**

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

Features[7, $C_4[-1]\, X_{2,6}[-1]\, X_{5,1}[-1]\, X_{7,3}[-1]$]

In[∘]:=
```
ℒ[Knot[3, 1]]
```

Out[∘]=

$$
\begin{aligned}
&\frac{1}{128\,\pi^7\,T^2}\\
&\mathbb{E}\Big[\epsilon\mathrm{Series}\Big[-p_1 x_1+p_2 x_1-p_2 x_2+\frac{p_3 x_2}{T}+\frac{(-1+T)\,p_7 x_2}{T}-p_3 x_3+p_4 x_3-p_4 x_4+p_5 x_4+\frac{(-1+T)\,p_2 x_5}{T}-\\
&\quad p_5 x_5+\frac{p_6 x_5}{T}-p_6 x_6+p_7 x_6+\frac{(-1+T)\,p_4 x_7}{T}-p_7 x_7+\frac{p_8 x_7}{T},\ 1-p_2 x_2+p_6 x_2+\frac{(-1+T)\,p_2 p_6 x_2^2}{2T}-\\
&\quad \frac{(-1+T)\,p_6^2 x_2^2}{2T}+p_4 x_4+p_1 x_5-p_5 x_5-p_1^2 x_1 x_5+p_1 p_5 x_1 x_5-\frac{(-1+T)\,p_1^2 x_5^2}{2T}+\frac{(-1+T)\,p_1 p_5 x_5^2}{2T}+\\
&\quad p_2 p_6 x_2 x_6-p_6^2 x_2 x_6+p_3 x_7-p_7 x_7-p_3^2 x_3 x_7+p_3 p_7 x_3 x_7-\frac{(-1+T)\,p_3^2 x_7^2}{2T}+\frac{(-1+T)\,p_3 p_7 x_7^2}{2T},\\
&\quad -\frac{1}{2}p_2 x_2+\frac{p_6 x_2}{2}+\frac{(-3+T)\,p_2 p_6 x_2^2}{4T}-\frac{(-3+T)\,p_6^2 x_2^2}{4T}-\frac{(-1+T)\,p_2^2 p_6 x_2^3}{3T}+\\
&\quad \frac{(-1+T)\,(1+5T)\,p_2 p_6^2 x_2^3}{6T^2}-\frac{(-1+T)\,(1+3T)\,p_6^3 x_2^3}{6T^2}-\frac{p_4 x_4}{2}+\frac{p_1 x_5}{2}-\frac{p_5 x_5}{2}-\frac{3}{2}p_1^2 x_1 x_5+\\
&\quad \frac{3}{2}p_1 p_5 x_1 x_5+\frac{1}{2}p_1^3 x_1^2 x_5-\frac{1}{2}p_1^2 p_5 x_1^2 x_5-\frac{(-3+T)\,p_1^2 x_5^2}{4T}+\frac{(-3+T)\,p_1 p_5 x_5^2}{4T}-\frac{(1+T)\,p_1^3 x_1 x_5^2}{2T}+\\
&\quad \frac{(1+2T)\,p_1^2 p_5 x_1 x_5^2}{2T}-\frac{1}{2}p_1 p_5^2 x_1 x_5^2-\frac{(-1+T)\,(1+3T)\,p_1^3 x_5^3}{6T^2}+\frac{(-1+T)\,(1+5T)\,p_1^2 p_5 x_5^3}{6T^2}-\\
&\quad \frac{(-1+T)\,p_1 p_5^2 x_5^3}{3T}+\frac{3}{2}p_2 p_6 x_2 x_6-\frac{3}{2}p_6^2 x_2 x_6-\frac{1}{2}p_2^2 p_6 x_2^2 x_6+\frac{(1+2T)\,p_2 p_6^2 x_2^2 x_6}{2T}-\\
&\quad \frac{(1+T)\,p_6^3 x_2^2 x_6}{2T}-\frac{1}{2}p_2 p_6^2 x_2 x_6^2+\frac{1}{2}p_6^3 x_2 x_6^2+\frac{p_3 x_7}{2}-\frac{p_7 x_7}{2}-\frac{3}{2}p_3^2 x_3 x_7+\frac{3}{2}p_3 p_7 x_3 x_7+\frac{1}{2}p_3^3 x_3^2 x_7-\\
&\quad \frac{1}{2}p_3^2 p_7 x_3^2 x_7-\frac{(-3+T)\,p_3^2 x_7^2}{4T}+\frac{(-3+T)\,p_3 p_7 x_7^2}{4T}-\frac{(1+T)\,p_3^3 x_3 x_7^2}{2T}+\frac{(1+2T)\,p_3^2 p_7 x_3 x_7^2}{2T}-\\
&\quad \frac{1}{2}p_3 p_7^2 x_3 x_7^2-\frac{(-1+T)\,(1+3T)\,p_3^3 x_7^3}{6T^2}+\frac{(-1+T)\,(1+5T)\,p_3^2 p_7 x_7^3}{6T^2}-\frac{(-1+T)\,p_3 p_7^2 x_7^3}{3T}\Big]\Big]
\end{aligned}
$$

In[∘]:=
```
vs[Knot[3, 1]]
```

Out[∘]=

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

In[∘]:=
```
K = Knot[3, 1]; ∫ℒ[K] ⅆ(vs@K)
```

Out[∘]=

$$-\frac{i\,T\,\mathbb{E}\Big[\epsilon\mathrm{Series}\Big[0,\ \frac{(-1+T)^2\,(1+T^2)}{(1-T+T^2)^2},\ -\frac{T^2\,(1-4T^2+T^4)}{2\,(1-T+T^2)^4}\Big]\Big]}{128\,\pi^7\,(1-T+T^2)}$$

## Invariance Under Reidemeister 3

In[•]:=

```
lhs = ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
    d{p_i, p_j, p_k, p_{i+1}, p_{j+1}, p_{k+1}, x_i, x_j, x_k, x_{i+1}, x_{j+1}, x_{k+1}}
rhs = ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
    d{p_i, p_j, p_k, p_{i+1}, p_{j+1}, p_{k+1}, x_i, x_j, x_k, x_{i+1}, x_{j+1}, x_{k+1}};
lhs == rhs
```

Out[•]=

$$T^{3/2}\,\mathbb{E}\Big[\epsilon\text{Series}\Big[T^2 p_{2+i}\pi_i - (-1+T)\,T\,p_{2+j}\pi_i + (1-T)\,p_{2+k}\pi_i + T\,p_{2+j}\pi_j + (1-T)\,p_{2+k}\pi_j + p_{2+k}\pi_k,$$

$$-\frac{3}{2} + T^2 p_{2+j}\pi_i + T\,p_{2+k}\pi_i + \frac{1}{2}(-1+T)\,T^3 p_{2+i}p_{2+j}\pi_i^2 - \frac{1}{2}(-1+T)\,T^3 p_{2+j}^2\pi_i^2 +$$

$$\frac{1}{2}(-1+T)\,T^2 p_{2+i}p_{2+k}\pi_i^2 - \frac{1}{2}(-1+T)^2\,T\,p_{2+j}p_{2+k}\pi_i^2 - \frac{1}{2}(-1+T)\,T\,p_{2+k}^2\pi_i^2 - T\,p_{2+j}\pi_j +$$

$$(-1+2T)\,p_{2+k}\pi_j - T^3 p_{2+i}p_{2+j}\pi_i\pi_j + T^3 p_{2+j}^2\pi_i\pi_j + (-1+T)\,T^2 p_{2+i}p_{2+k}\pi_i\pi_j -$$

$$(-1+T)^2\,T\,p_{2+j}p_{2+k}\pi_i\pi_j - (-1+T)\,T\,p_{2+k}^2\pi_i\pi_j + \frac{1}{2}(-1+T)\,T\,p_{2+j}p_{2+k}\pi_j^2 - \frac{1}{2}(-1+T)\,T\,p_{2+k}^2\pi_j^2 -$$

$$2\,p_{2+k}\pi_k - T^2 p_{2+i}p_{2+k}\pi_i\pi_k + (-1+T)\,T\,p_{2+j}p_{2+k}\pi_i\pi_k + T\,p_{2+k}^2\pi_i\pi_k - T\,p_{2+j}p_{2+k}\pi_j\pi_k + T\,p_{2+k}^2\pi_j\pi_k,$$

$$-\frac{1}{2}T^2 p_{2+j}\pi_i - \frac{1}{2}T\,p_{2+k}\pi_i - \frac{1}{4}T^3(-1+3T)\,p_{2+i}p_{2+j}\pi_i^2 + \frac{1}{4}T^3(-3+5T)\,p_{2+j}^2\pi_i^2 -$$

$$\frac{1}{4}T^2(-1+3T)\,p_{2+i}p_{2+k}\pi_i^2 + \frac{1}{4}(-1+T)\,T\,(-1+5T)\,p_{2+j}p_{2+k}\pi_i^2 +$$

$$\frac{1}{4}T\,(-3+5T)\,p_{2+k}^2\pi_i^2 - \frac{1}{6}(-1+T)\,T^5 p_{2+i}^2 p_{2+j}\pi_i^3 + \frac{1}{6}(-1+T)\,T^4(-1+4T)\,p_{2+i}p_{2+j}^2\pi_i^3 -$$

$$\frac{1}{6}(-1+T)\,T^4(-1+3T)\,p_{2+j}^3\pi_i^3 - \frac{1}{6}(-1+T)\,T^4 p_{2+i}^2 p_{2+k}\pi_i^3 + \frac{5}{6}(-1+T)^2\,T^3 p_{2+i}p_{2+j}p_{2+k}\pi_i^3 -$$

$$\frac{1}{6}(-1+T)^2\,T^2(-1+4T)\,p_{2+j}^2 p_{2+k}\pi_i^3 + \frac{1}{6}(-1+T)\,T^2(-1+4T)\,p_{2+i}p_{2+k}^2\pi_i^3 -$$

$$\frac{1}{6}(-1+T)^2\,T\,(-1+4T)\,p_{2+j}p_{2+k}^2\pi_i^3 - \frac{1}{6}(-1+T)\,T\,(-1+3T)\,p_{2+k}^3\pi_i^3 + \frac{1}{2}T\,p_{2+j}\pi_j +$$

$$\frac{1}{2}(1-4T)\,p_{2+k}\pi_j + \frac{3}{2}T^3 p_{2+i}p_{2+j}\pi_i\pi_j - \frac{5}{2}T^3 p_{2+j}^2\pi_i\pi_j - \frac{1}{2}T^2(-3+5T)\,p_{2+i}p_{2+k}\pi_i\pi_j +$$

$$\frac{1}{2}(-1+T)\,T\,(-3+7T)\,p_{2+j}p_{2+k}\pi_i\pi_j + \frac{1}{2}T\,(-5+7T)\,p_{2+k}^2\pi_i\pi_j + \frac{1}{2}T^5 p_{2+i}^2 p_{2+j}\pi_i^2\pi_j -$$

$$\frac{1}{2}T^4(-1+4T)\,p_{2+i}p_{2+j}^2\pi_i^2\pi_j + \frac{1}{2}T^4(-1+3T)\,p_{2+j}^3\pi_i^2\pi_j - \frac{1}{2}(-1+T)\,T^4 p_{2+i}^2 p_{2+k}\pi_i^2\pi_j +$$

$$\frac{1}{2}(-1+T)\,T^3(-5+4T)\,p_{2+i}p_{2+j}p_{2+k}\pi_i^2\pi_j - \frac{1}{2}(-1+T)\,T^2\left(1-5T+3T^2\right)p_{2+j}^2 p_{2+k}\pi_i^2\pi_j +$$

$$\frac{1}{2}(-1+T)\,T^2(-1+4T)\,p_{2+i}p_{2+k}^2\pi_i^2\pi_j - \frac{1}{2}(-1+T)^2\,T\,(-1+4T)\,p_{2+j}p_{2+k}^2\pi_i^2\pi_j -$$

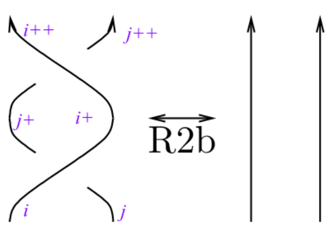
$$\frac{1}{2}(-1+T)\,T\,(-1+3T)\,p_{2+k}^3\pi_i^2\pi_j - \frac{1}{4}T\,(-5+7T)\,p_{2+j}p_{2+k}\pi_j^2 + \frac{1}{4}T\,(-7+9T)\,p_{2+k}^2\pi_j^2 +$$

$$\frac{1}{2} T^4 p_{2+i} p_{2+j}^2 \pi_i \pi_j^2 - \frac{1}{2} T^4 p_{2+j}^3 \pi_i \pi_j^2 - 2 (-1+T) T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i \pi_j^2 +$$
$$\frac{1}{2} (-1+T) T^2 (-1+4T) p_{2+j}^2 p_{2+k} \pi_i \pi_j^2 + \frac{1}{2} (-1+T) T^2 (-1+3T) p_{2+i} p_{2+k}^2 \pi_i \pi_j^2 -$$
$$\frac{1}{2} (-1+T) T \left(1-5T+3T^2\right) p_{2+j} p_{2+k}^2 \pi_i \pi_j^2 - \frac{1}{2} (-1+T) T (-1+3T) p_{2+k}^3 \pi_i \pi_j^2 -$$
$$\frac{1}{6} (-1+T) T^2 p_{2+j}^2 p_{2+k} \pi_j^3 + \frac{1}{6} (-1+T) T (-1+4T) p_{2+j} p_{2+k}^2 \pi_j^3 - \frac{1}{6} (-1+T) T (-1+3T) p_{2+k}^3 \pi_j^3 +$$
$$2 p_{2+k} \pi_k + \frac{5}{2} T^2 p_{2+i} p_{2+k} \pi_i \pi_k - \frac{1}{2} T (-5+7T) p_{2+j} p_{2+k} \pi_i \pi_k - \frac{7}{2} T p_{2+k}^2 \pi_i \pi_k +$$
$$\frac{1}{2} T^4 p_{2+i}^2 p_{2+k} \pi_i^2 \pi_k - 2 (-1+T) T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i^2 \pi_k + \frac{1}{2} (-1+T) T^2 (-1+3T) p_{2+j}^2 p_{2+k} \pi_i^2 \pi_k -$$
$$\frac{1}{2} T^2 (-1+4T) p_{2+i} p_{2+k}^2 \pi_i^2 \pi_k + \frac{1}{2} (-1+T) T (-1+4T) p_{2+j} p_{2+k}^2 \pi_i^2 \pi_k + \frac{1}{2} T (-1+3T) p_{2+k}^3 \pi_i^2 \pi_k +$$
$$\frac{7}{2} T p_{2+j} p_{2+k} \pi_j \pi_k - \frac{9}{2} T p_{2+k}^2 \pi_j \pi_k + 3 T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i \pi_j \pi_k - T^2 (-1+3T) p_{2+j}^2 p_{2+k} \pi_i \pi_j \pi_k -$$
$$T^2 (-1+3T) p_{2+i} p_{2+k}^2 \pi_i \pi_j \pi_k + T \left(1-5T+3T^2\right) p_{2+j} p_{2+k}^2 \pi_i \pi_j \pi_k + T (-1+3T) p_{2+k}^3 \pi_i \pi_j \pi_k +$$
$$\frac{1}{2} T^2 p_{2+j}^2 p_{2+k} \pi_j^2 \pi_k - \frac{1}{2} T (-1+4T) p_{2+j} p_{2+k}^2 \pi_j^2 \pi_k + \frac{1}{2} T (-1+3T) p_{2+k}^3 \pi_j^2 \pi_k + \frac{1}{2} T^2 p_{2+i} p_{2+k}^2 \pi_i \pi_k^2 -$$
$$\frac{1}{2} (-1+T) T p_{2+j} p_{2+k}^2 \pi_i \pi_k^2 - \frac{1}{2} T p_{2+k}^3 \pi_i \pi_k^2 + \frac{1}{2} T p_{2+j} p_{2+k}^2 \pi_j \pi_k^2 - \frac{1}{2} T p_{2+k}^3 \pi_j \pi_k^2 \Big] \Big]$$

Out[•]=

```
True
```

## Invariance Under Reidemeister 2b




Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[•]:=

```
lhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
rhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
lhs == rhs
```

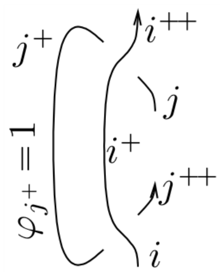
Out[•]=

```
𝔼[∈Series[p_{2+i} π_i + p_{2+j} π_j, 0, 0]]
```

Out[•]=

```
True
```

## Invariance Under R2c




In[∘]:= lhs = $\int \mathbb{E}[\pi_i\, p_i + \pi_j\, p_j]$ ℒ /@ ($X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1]$)
 $d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\}$
rhs = $\int \mathbb{E}[\pi_i\, p_i + \pi_j\, p_j]$ ℒ /@ ($C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0]$)
 $d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\}$;
lhs == rhs

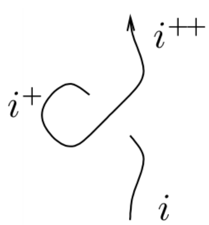
Out[∘]=

$$-i\sqrt{T}\ \mathbb{E}\left[\epsilon\text{Series}\left[p_{2+i}\,\pi_i + p_{3+j}\,\pi_j,\ -\frac{1}{2} - p_{3+j}\,\pi_j,\ \frac{1}{2}\,p_{3+j}\,\pi_j\right]\right]$$

Out[∘]=

True

## Invariance Under R1l




In[∘]:= lhs = $\int \mathbb{E}[\pi_i\, p_i]$ ℒ /@ ($X_{i+2,i}[1]\ C_{i+1}[1]$) $d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$
rhs = $\int \mathbb{E}[\pi_i\, p_i]$ ℒ /@ ($C_i[0]\ C_{i+1}[0]\ C_{i+2}[0]$) $d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$;
lhs == rhs

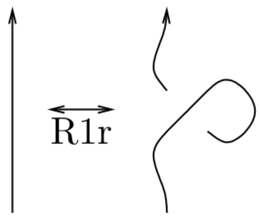
Out[∘]=

$$-i\ \mathbb{E}[\epsilon\text{Series}[p_{3+i}\,\pi_i, 0, 0]]$$

Out[∘]=

True

## Invariance Under R1r

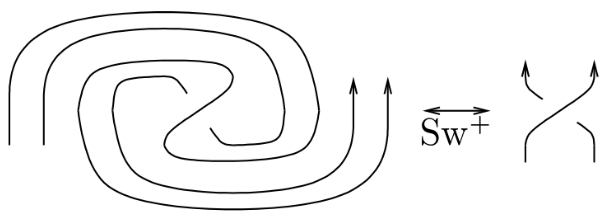



In[•]:=
```
lhs = ∫𝔼[π_i p_i] ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[π_i p_i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs
```

Out[•]=
```
-ⅈ 𝔼[∈Series[p_{3+i} π_i, 0, 0]]
```

Out[•]=
```
True
```

## Invariance Under Sw



In[•]:=

```
lhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}};
lhs == rhs
```

Out[•]=

$$\sqrt{T}\ \mathbb{E}\Big[\epsilon\text{Series}\Big[T\, p_{3+i}\, \pi_i + (1-T)\, p_{3+j}\, \pi_i + p_{3+j}\, \pi_j,\ -\frac{1}{2} + T\, p_{3+j}\, \pi_i + \frac{1}{2}(-1+T)\, T\, p_{3+i}\, p_{3+j}\, \pi_i^2 - \frac{1}{2}(-1+T)\, T\, p_{3+j}^2\, \pi_i^2 - p_{3+j}\, \pi_j - T\, p_{3+i}\, p_{3+j}\, \pi_i\, \pi_j + T\, p_{3+j}^2\, \pi_i\, \pi_j,\ -\frac{1}{2} T\, p_{3+j}\, \pi_i - \frac{1}{4} T(-1+3T)\, p_{3+i}\, p_{3+j}\, \pi_i^2 + \frac{1}{4} T(-3+5T)\, p_{3+j}^2\, \pi_i^2 - \frac{1}{6}(-1+T)\, T^2\, p_{3+i}^2\, p_{3+j}\, \pi_i^3 + \frac{1}{6}(-1+T)\, T(-1+4T)\, p_{3+i}\, p_{3+j}^2\, \pi_i^3 - \frac{1}{6}(-1+T)\, T(-1+3T)\, p_{3+j}^3\, \pi_i^3 + \frac{1}{2} p_{3+j}\, \pi_j + \frac{3}{2} T\, p_{3+i}\, p_{3+j}\, \pi_i\, \pi_j - \frac{5}{2} T\, p_{3+j}^2\, \pi_i\, \pi_j + \frac{1}{2} T^2\, p_{3+i}^2\, p_{3+j}\, \pi_i^2\, \pi_j - \frac{1}{2} T(-1+4T)\, p_{3+i}\, p_{3+j}^2\, \pi_i^2\, \pi_j + \frac{1}{2} T(-1+3T)\, p_{3+j}^3\, \pi_i^2\, \pi_j + \frac{1}{2} T\, p_{3+i}\, p_{3+j}^2\, \pi_i\, \pi_j^2 - \frac{1}{2} T\, p_{3+j}^3\, \pi_i\, \pi_j^2\Big]\Big]$$

Out[•]=

True