



# Shifted Partial Quadratics, their Pushforwards, and Signature Invariants for Tangles

**Abstract.** Following a general discussion of the computation of zombians of unfinished columbaria (with examples), I will tell you about my recent joint work w/ Jessica Liu on what we feel is the “textbook” extension of knot signatures to tangles, which for unknown reasons, is not in any of the textbooks that we know.



Jessica Liu



Columbaria in an East Sydney Cemetery



Jacobian, Hamiltonian, Zombian

**Kashaev's Conjecture** [Ka]

$$\text{For knots, } \sigma_{Kas} = 2\sigma_{TL}.$$

**Liu's Theorem** [Li].

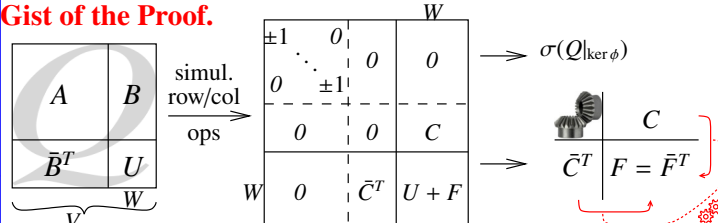
A **Partial Quadratic (PQ)** on  $V$  is a quadratic  $Q$  defined only on a subspace  $\mathcal{D}_Q \subset V$ . We add PQs with  $\mathcal{D}_{Q_1+Q_2} := \mathcal{D}_{Q_1} \cap \mathcal{D}_{Q_2}$ . Given a linear  $\psi: V \rightarrow W$  and a PQ  $Q$  on  $W$ , there is an obvious **pullback**  $\psi^*Q$ , a PQ on  $V$ .

**Theorem 1.** Given a linear  $\phi: V \rightarrow W$  and a PQ  $Q$  on  $V$ , there is a unique **pushforward** PQ  $\phi_*Q$  on  $W$  such that for every PQ  $U$  on  $W$ ,  $\sigma_V(Q + \phi^*U) = \sigma_{\ker \phi}(Q|_{\ker \phi}) + \sigma_W(U + \phi_*Q)$ .

(If you must,  $\mathcal{D}(\phi_*Q) = \phi(\text{ann}_Q(\mathcal{D}(Q) \cap \ker \phi))$  and  $(\phi_*Q)(w) = Q(v)$ , where  $v$  is s.t.  $\phi(v) = w$  and  $Q(v, \text{rad } Q|_{\ker \phi}) = 0$ ).

**Prior Art** on signatures for tangles / braids. Gambaudo and Ghys [GG], Cimasoni and Conway [CC], Conway [Co], Merz [Me]. All define signatures of tangles / braids by first closing them to links and then work hard to derive composition properties.

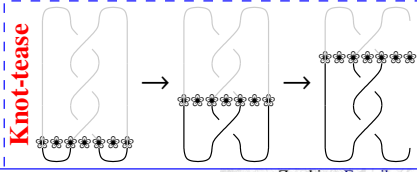
**Gist of the Proof.**



**Why Tangles?** • Faster!

- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:
  - The Jones Polynomial  $\rightsquigarrow$  The Temperley-Lieb Algebra.
  - Khovanov Homology  $\rightsquigarrow$  “Unfinished complexes”, complexes in a category.
  - The Kontsevich Integral  $\rightsquigarrow$  Associators.
  - HFK  $\rightsquigarrow$  OMG, type  $D$ , type  $A$ ,  $\mathcal{A}_\infty, \dots$

$$2^{n/2} + 2^{n/2} + 2\sqrt{n} \ll 2^n$$



**Computing Zombians of Unfinished Columbaria.**

- Must be no slower than for finished ones.
- Future zombies must be able to complete the computation.
- Future zombies must not even know the size of the task that today's zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!

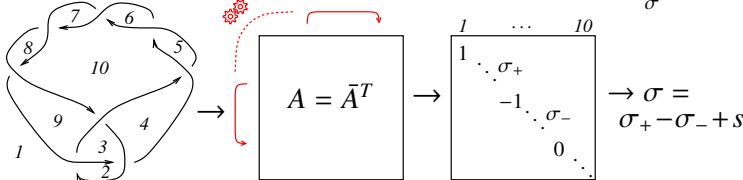


Columbarium near Assen

**Example / Exercise.** Compute the determinant of a  $1,000 \times 1,000$  matrix in which 50 entries are not yet given.

**Homework / Research Projects.** • What with ZPUCs? • Use this to get an Alexander tangle invariant.

**Reminders.** {knots}  $\rightleftharpoons$  {matrices / quadratic forms}  $\xrightarrow{\text{signature } \sigma} \mathbb{Z}$ :



With  $|\omega| = 1$ ,  $t = 1 - \omega$ ,  $r = t + \bar{t}$ ,  $v = \text{Re}(\omega)$ , and  $u = \text{Re}(\omega^{1/2})$ :

$X_{-i,j,k,-l}$	Tristram-Levine (TL)	Kashaev (Kas)
	$A += \begin{pmatrix} -r & -t & 2t & \bar{t} \\ -\bar{t} & 0 & \bar{t} & 0 \\ 2\bar{t} & t & -r & -\bar{t} \\ t & 0 & -t & 0 \end{pmatrix}$	$A += \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}$
$\bar{X}_{-i,j,k,-l}$	$A += \begin{pmatrix} r & -t & -2\bar{t} & \bar{t} \\ -\bar{t} & 0 & \bar{t} & 0 \\ -2t & t & r & -\bar{t} \\ t & 0 & -t & 0 \end{pmatrix}$	$A -= \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}$

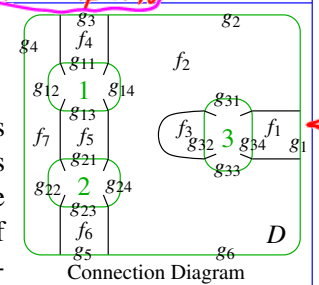
A **Shifted Partial Quadratic (SPQ)** on  $V$  is a pair  $S = (s \in \mathbb{Z}, Q \text{ a PQ on } V)$ . addition also adds the shifts, pullbacks keep the shifts, yet  $\phi_*S := (s + \sigma_{\ker \phi}(Q|_{\ker \phi}), \phi_*Q)$  and  $\sigma(S) := s + \sigma(Q)$ .

**Theorem 1' (Reciprocity).** Given  $\phi: V \rightarrow W$ , for SPQs  $S$  on  $V$  and  $U$  on  $W$  we have  $\sigma_V(S + \phi^*U) = \sigma_W(U + \phi_*S)$  (and this characterizes  $\phi_*S$ ).

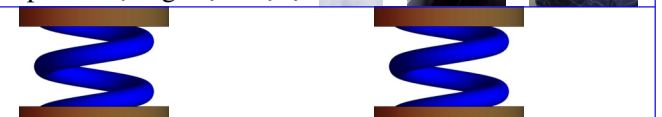
**Theorem 2.**  $\psi^*$  and  $\phi_*$  are functorial. Also, if  $V_0 \xrightarrow{\alpha} V_1$ ,  $V_1 \xrightarrow{\gamma} V_2$ ,  $V_2 \xrightarrow{\delta} V_3$  and  $\beta\alpha = \delta\gamma$  and  $\alpha$  and  $\gamma$  are surjective, then  $\alpha_*\gamma_* = \beta_*\delta_*$ . Finally,  $\psi^*$  is additive but  $\phi_*$  is not.

**Definition.**  $S \left( \begin{matrix} g_2 \\ g_3 \\ \dots \end{matrix} \right) := \{ \text{SPQ } S \text{ on } \langle g_i \rangle \}$ .

**Theorem 3** ~~X~~  $\{S(\text{cyclic sets})\}$  is a planar algebra, with compositions  $S(D)((S_i)) := \phi_*^D(\psi_D^*(\bigoplus_i S_i))$ , where  $\psi_D: \langle f_i \rangle \rightarrow \langle g_{\alpha i} \rangle$  maps every face of  $D$  to the sum of the input gaps adjacent to it and  $\phi^D: \langle f_i \rangle \rightarrow \langle g_i \rangle$  maps every face to the sum of the output gaps adjacent to it. So for our  $D$ ,  $\psi_D$  is  $f_1 \mapsto g_{34}$ ,  $f_2 \mapsto g_{31} + g_{14} + g_{24} + g_{33}$ ,  $f_3 \mapsto g_{32}$ ,  $f_4 \mapsto g_{11}$ ,  $f_5 \mapsto g_{13} + g_{21}$ ,  $f_6 \mapsto g_{23}$ ,  $f_7 \mapsto g_{12} + g_{22}$  and  $\phi^D$  is  $f_1 \mapsto g_1$ ,  $f_2 \mapsto g_2 + g_6$ ,  $f_3 \mapsto 0$ ,  $f_4 \mapsto g_3$ ,  $f_5 \mapsto 0$ ,  $f_6 \mapsto g_5$ ,  $f_7 \mapsto g_4$ .



**Theorem 4.** TL and Kas, defined on  $X$  and  $\bar{X}$  as before, extend to planar algebra morphisms  $\{ \text{tangles} \} \rightarrow \{ S \}$ .



**Proof of Theorem 1'.** Fix  $W$  and consider triples  $(V, S, \phi: V \rightarrow W)$  where  $S = (s, D, Q)$  is an SPQ on  $V$ . Declare  $(V_1, S_1, \phi_1) \sim (V_2, S_2, \phi_2)$  if for every quadratic  $U$  on  $W$ ,

$$\sigma_{V_1}(S_1 + \phi_1^* U) = \sigma_{V_2}(S_2 + \phi_2^* U).$$

Given our  $(V, S, \phi)$ , we need to show:

1. There is an SPQ  $S'$  on  $W$  such that  $(V, S, \phi) \sim (W, S', I)$ .
2. If  $(W, S', I) \sim (W, S'', I)$  then  $S' = S''$ .

Property 2 is easy. Property 1 follows from the following four claims, each of which is easy.

**Claim 1.**  $(V, S, \phi) \sim (D(S), S, \phi|_{D(S)})$ , so wlog,  $S$  is "full", meaning  $D(S) = V$ .  $\square$

**Claim 2.** If  $S$  is full,  $v \in \ker \phi$ , and  $\lambda := Q(v) \neq 0$ , then  $(V, S, \phi) \sim (V/\langle v \rangle, (s + \text{sign}(\lambda), V/\langle v \rangle, Q - \frac{Q(-, v) \otimes Q(v, -)}{|\lambda|^2}), \phi/\langle v \rangle)$ .  $\square$

So wlog  $Q|_{\ker \phi} = 0$  (meaning,  $Q|_{\ker \phi \otimes \ker \phi} = 0$ ).  $\square$

**Claim 3.** If  $Q|_{\ker \phi} = 0$  and  $v \in \ker \phi$ , let  $V' = \ker Q(v, -)$  and then  $(V, S, \phi) \sim (V', S|_{V'}, \phi|_{V'})$  so wlog  $Q|_{V \otimes \ker \phi + \ker \phi \otimes V} = 0$ .  $\square$

**Claim 4.** If  $Q|_{V \otimes \ker \phi + \ker \phi \otimes V} = 0$  then  $S = \phi^* S'$  for some SPQ  $S'$  on  $\text{im } \phi$  and then  $(V, S, \phi) \sim (W, S', I)$ .  $\square \square$

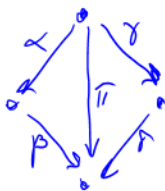
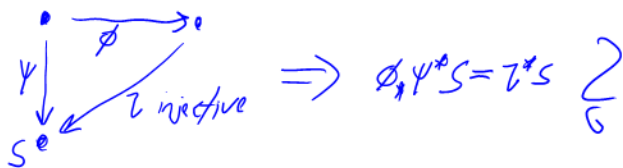
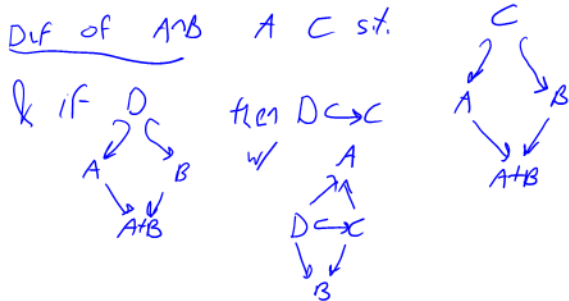
**Lemma 1.**  $\phi_* \phi^* S = S|_{\text{im } \phi}$ .

**Proof.** For every PQ  $U$  with  $D(U) = \text{im } \phi$  we have  $\sigma(Q|_{\text{im } \phi} + U) = \sigma(Q + U) = \sigma(\phi^*(Q + U)) = \sigma(\phi^* Q + \phi^* U) = \sigma(\phi_* \phi^* S + U)$  where for the second equality we use the fact that a pullback by a surjective map does not change the signature, and the last equality is the reciprocity property.  $\square$

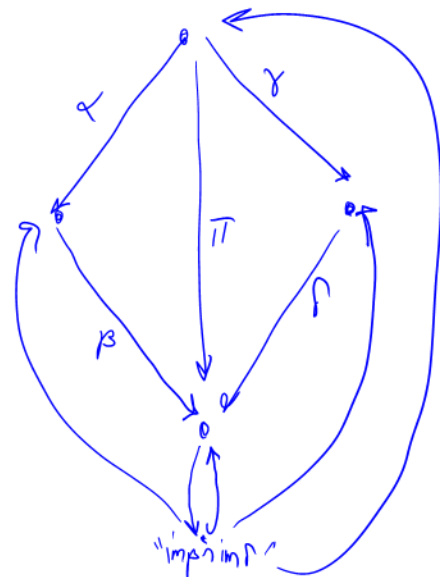
**Lemma 2.** Under the conditions of Theorem 2, if  $S_i$  is an SPQ on  $V_i$  for  $i = 1, 2$ , then  $\sigma(\alpha_* S_1 + \gamma_* S_2) = \sigma(\beta_* S_1 + \delta_* S_2)$ .

**Proof.** Let  $\pi := \beta\alpha = \delta\gamma$ . Then (in order) by reciprocity with  $U = 0$ , by the functoriality of pushforwards, by Lemma 1, and using the surjectivity of  $\alpha$  and of  $\gamma$ ,  $\sigma(\alpha_* S_1 + \gamma_* S_2) = \sigma(\pi_*(\alpha_* S_1 + \gamma_* S_2)) = \sigma(\beta_* \alpha_* \alpha^* S_1 + \delta_* \gamma_* \gamma^* S_2) = \sigma(\beta_* (S_1|_{\text{im } \alpha}) + \delta_* (S_2|_{\text{im } \gamma})) = \sigma(\beta_* S_1 + \delta_* S_2)$ .  $\square$

**Proof of Theorem 2.** Given  $S$  on  $V_2$ , for every  $U$  on  $V_1$  we have using reciprocity, Lemma 2, and reciprocity again, that  $\sigma(\alpha_* \gamma^* S + U) = \sigma(\gamma^* S + \alpha^* U) = \sigma(\delta_* S + \beta_* U) = \sigma(\beta^* \delta_* S + U)$ . Hence  $\alpha_* \gamma^* S = \beta^* \delta_* S$ .  $\square$



$\text{im } \pi = \text{im } \beta \text{ im } \delta$



pf of functoriality  $S \xrightarrow{\alpha} S' \xrightarrow{\beta} S''$

$$\begin{aligned} \sigma(\beta_* \alpha^* S + U) &= \sigma(\alpha_* S + \beta^* U) = \sigma(S + \alpha^* \beta^* U) \\ &= \sigma(S + (\beta\alpha)^* U) = \sigma((\beta\alpha)_* S + U) \end{aligned}$$

Include the functoriality proof!

**Implementation** (sources: <http://drorbn.net/icerm23/ap>). I like it most when the implementation matches the math perfectly. We failed here.

Once[<< KnotTheory`];

Loading KnotTheory` version

of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

**Utilities.** The step function, algebraic numbers, canonical forms.

$\theta[x\_]$  /; NumericQ[x] := UnitStep[x]

```
 $\omega 2[v\_][p\_]$  := Module[{q = Expand[p], n, c},
  If[q === 0, 0,
    c = Coefficient[q,  $\omega$ , n = Exponent[q,  $\omega$ ]];
     $c v^n + \omega 2[v][q - c(\omega + \omega^{-1})^n]$ ];
```

```
sign[ $\mathcal{E}$ _] := Module[{n, d, v, p, rs, e, k},
  {n, d} = NumeratorDenominator[ $\mathcal{E}$ ];
  {n, d} /=  $\omega^{\text{Exponent}[n, \omega]/2 + \text{Exponent}[n, \omega, \text{Min}]/2}$ ;
  p = Factor[ $\omega 2[v]@n * \omega 2[v]@d /. v \rightarrow 4 u^2 - 2$ ];
  rs = Solve[p == 0, u, Reals];
  If[rs === {}, Sign[p /. u -> 0],
    rs = Union@{u /. rs};
    Sign[(-1)e=Exponent[p, u] Coefficient[p, u, e]] + Sum[
      k = 0;
      While[{d = RootReduce[ $\partial_{\{u, ++k\}} p /. u \rightarrow r$ ]} == 0];
      If[EvenQ[k], 0, 2 Sign[d]] *  $\theta[u - r]$ ,
      {r, rs}]]
]
```

SetAttributes[B, Orderless];

```
CF[b_B] := RotateLeft[#, First@Ordering[#] - 1] & /@
DeleteCases[b, {}]
```

```
CF[ $\mathcal{E}$ _] := Module[{ $\gamma$ s = Union@Cases[ $\mathcal{E}$ ,  $\gamma$ _ |  $\bar{\gamma}$ _,  $\infty$ ]},
  Total[CoefficientRules[ $\mathcal{E}$ ,  $\gamma$ s] /.
    (ps_ -> c_) := Factor[c]  $\times$  Times@@ $\gamma$ sps]]
```

CF[{}] = {};

CF[C\_List] :=

```
Module[{ $\gamma$ s = Union@Cases[C,  $\gamma$ _,  $\infty$ ],  $\gamma$ },
```

```
CF /@ DeleteCases[0] [
  RowReduce[Table[ $\partial_{\gamma} r$ , {r, C}, { $\gamma$ ,  $\gamma$ s}]]  $\cdot \gamma$ s ]
```

( $\mathcal{E}$ \_)\* :=  $\mathcal{E} /. \{\bar{\gamma} \rightarrow \gamma, \gamma \rightarrow \bar{\gamma}, \omega \rightarrow \omega^{-1}, c\_Complex \rightarrow c^*\}$ ;

r\_Rule\* := {r, r\*}

RulesOf[ $\gamma_i$  + rest\_] := ( $\gamma_i \rightarrow -rest$ )\*;

```
CF[PQ[C_, q_]] := Module[{nC = CF[C]},
  PQ[nC, CF[q /. Union@@ RulesOf /@ nC]] ]
```

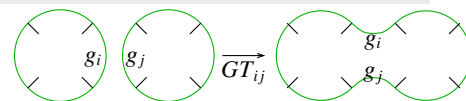
CF[ $\Sigma_b$ [ $\sigma$ \_, pq\_]] :=  $\Sigma_{CF[b]}$ [ $\sigma$ , CF[pq]]

## Pretty-Printing.

```
Format[ $\Sigma_{b,B}[\sigma_, PQ[C_, q_]]$ ] := Module[{ $\gamma$ s},
   $\gamma$ s =  $\gamma$ # & /@ Join@@ b;
  Column[{TraditionalForm@ $\sigma$ ,
    TableForm[Join[
      Prepend[""] /@ Table[TraditionalForm[ $\partial_c r$ ],
        {r, C}, {c,  $\gamma$ s}],
      {Prepend[""] [
        Join@@
          (b /. {L_, m___, r_} =>
            {DisplayForm@RowBox[{"(", L}],
              m, DisplayForm@RowBox[{r, ")"}]}) /-
            i_Integer =>  $\gamma_i$  ]}],
      MapThread[Prepend,
        {Table[TraditionalForm[ $\partial_{r,c} q$ ], {r,  $\gamma$ s*}],
          {c,  $\gamma$ s}],  $\gamma$ s*}]]
    ], TableAlignments -> Center]
  ], Center] ];
```

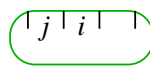
## The Face-Centric Core.

```
 $\Sigma_{b1}[\sigma_1, PQ[C1_, q1_]] \oplus \Sigma_{b2}[\sigma_2, PQ[C2_, q2_]] \wedge :=$ 
  CF@ $\Sigma_{\text{Join}[b1, b2]}$ [ $\sigma_1 + \sigma_2, PQ[C1 \cup C2, q1 + q2]$ ];
```



GT for Gap Touch:

```
GTi,j@ $\Sigma_B[\{li___, i, ri___\}, \{lj___, j, rj___\}, bs___]$ [ $\sigma$ _,
  PQ[C_, q_]] :=
  CF@ $\Sigma_B[\{ri, li, j, rj, lj, i\}, bs]$ [ $\sigma$ _, PQ[C  $\cup$  { $\gamma_i - \gamma_j$ }, q]]
```



cordon (kôr'dn)  
n.



1. A line of people, military posts, or ships stationed around an area to enclose or guard it: a *police cordon*.
2. A rope, line, tape, or similar border stretched around an area, usually by the police, indicating that access is restricted.

$$s \begin{pmatrix} 0 & \phi C_{rest} \\ \bar{\phi}^T & \lambda \theta \\ \bar{C}_{rest}^T & \bar{\theta}^T A_{rest} \end{pmatrix} \rightarrow \begin{cases} \exists p \phi_p \neq 0 & \text{use } \phi_p \text{ to kill its row and} \\ \phi = 0, \lambda \neq 0 & \text{column, drop a } \begin{pmatrix} 01 \\ 10 \end{pmatrix} \text{ summand} \\ \phi = 0, \lambda = 0 & \text{use } \lambda \text{ to kill } \theta, \text{ let } s += \text{sign}(\lambda) \\ & \text{append } \theta \text{ to } C_{rest}. \end{cases}$$

```
Cordoni@ $\Sigma_B[\{li___, i, ri___\}, bs___]$ [ $\sigma$ _, PQ[C_, q_]] :=
```

```
Module[{ $\phi = \partial_{\gamma_i} C$ ,  $\lambda = \partial_{\bar{\gamma}_i, \gamma_i} q$ , n $\sigma = \sigma$ , nC, nq, p},
  {p} = FirstPosition[ (# != 0) & /@  $\phi$ , True, {0}];
  {nC, nq} = Which[
    p > 0, {C, q} /. ( $\gamma_i \rightarrow -C[[p]] / \phi[[p]]$ )* /. ( $\gamma_i \rightarrow \theta$ )*,
     $\lambda \neq 0$ , (n $\sigma += \text{sign}[\lambda]$ ;
      {C, q /. ( $\gamma_i \rightarrow -(\partial_{\bar{\gamma}_i} q) / \lambda$ )* /. ( $\gamma_i \rightarrow \theta$ )*}),
     $\lambda === 0$ , {C  $\cup$  { $\partial_{\bar{\gamma}_i} q$ }, q /. ( $\gamma_i \rightarrow \theta$ )*}];
  CF@ $\Sigma_B[\text{Most}@\{ri, li\}, bs]$ [n $\sigma$ ,
    PQ[nC, nq] /. ( $\gamma_{\text{Last}@\{ri, li\}} \rightarrow \gamma_{\text{First}@\{ri, li\}}$ )* ] ]
```

**Strand Operations.** c for contract, mc for magnetic contract:

$$C_{i,j}@t : \Sigma_B[\{li\_ , i, ri\_ \}, \{ \_ , j, \_ \}, \_ ] [ \_ ] := t // GT_{j, First\{ri, li\}} // Cordon_j$$

$$C_{i,j}@t : \Sigma_B[\{ \_ , i, j, \_ \}, \_ ] [ \_ ] := Cordon_j @ t$$

$$C_{i,j}@t : \Sigma_B[\{j, \_ , i, \_ \}, \_ ] [ \_ ] := Cordon_j @ t$$

$$C_{i,j}@t : \Sigma_B[\{ \_ , j, i, \_ \}, \_ ] [ \_ ] := Cordon_i @ t$$

$$C_{i,j}@t : \Sigma_B[\{i, \_ , j, \_ \}, \_ ] [ \_ ] := Cordon_i @ t$$

$$mc[\mathcal{E}_] := \mathcal{E} //$$

$$t : \Sigma_B[\{ \_ , i, \_ \}, \{ \_ , j, \_ \}, \_ ] [ \_ ] | \Sigma_B[\{ \_ , i, j, \_ \}, \_ ] [ \_ ] | \Sigma_B[\{j, \_ , i, \_ \}, \_ ] [ \_ ] / ; i + j == \theta \Rightarrow C_{i,j}@t$$

**The Crossings** (and empty strands).

$$Kas@P_{i,j} := CF@ \Sigma_B[\{i,j\}] [\theta, PQ[\{\}, \theta]] ;$$

$$TL@P_{i,j} := CF@ \Sigma_B[\{i,j\}] [\theta, PQ[\{\}, \theta]]$$

$$Kas[x : X[i, j, k, l]] :=$$

$$Kas@If[PositiveQ[x], X_{-i,j,k,-l}, \bar{X}_{-j,k,l,-i}] ;$$

$$Kas[(x : X | \bar{X})_{fs\_}] := Module[\{v = 2u^2 - 1, p, \gamma s, m\},$$

$$\gamma s = \gamma_{\#} \& /@ \{fs\}; p = (x == X) ;$$

$$m = If[p, \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}, -\begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}] ;$$

$$CF@ \Sigma_B[\{fs\}] [If[p, -1, 1], PQ[\{\}, \gamma s^* . m . \gamma s]]$$

$$TL[x : X[i, j, k, l]] :=$$

$$TL@If[PositiveQ[x], X_{-i,j,k,-l}, \bar{X}_{-j,k,l,-i}] ;$$

$$TL[(x : X | \bar{X})_{fs\_}] := Module[\{t = 1 - \omega, r, \gamma s, m\},$$

$$r = t + t^*; \gamma s = \gamma_{\#} \& /@ \{fs\};$$

$$m = If[x == X,$$

$$\begin{pmatrix} -r & -t & 2t & t^* \\ -t^* & \theta & t^* & \theta \\ 2t^* & t & -r & -t^* \\ t & \theta & -t & \theta \end{pmatrix}, \begin{pmatrix} r & -t & -2t^* & t^* \\ -t^* & \theta & t^* & \theta \\ -2t & t & r & -t^* \\ t & \theta & -t & \theta \end{pmatrix}] ;$$

$$CF@ \Sigma_B[\{fs\}] [\theta, PQ[\{\}, \gamma s^* . m . \gamma s]]$$

**Evaluation on Tangles and Knots.**

$$Kas[K_] := Fold[mc[\#1 \oplus \#2] \&, \Sigma_B[] [\theta, PQ[\{\}, \theta]], List@@ (Kas /@ PD@K)] ;$$

$$KasSig[K_] := Expand[Kas[K][[1]] / 2]$$

$$TL[K_] :=$$

$$Fold[mc[\#1 \oplus \#2] \&, \Sigma_B[] [\theta, PQ[\{\}, \theta]], List@@ (TL /@ PD@K)] / .$$

$$\theta[c\_ + u] / ; Abs[c] \ge 1 \Rightarrow \theta[c] ;$$

$$TLSig[K_] := TL[K][[1]]$$

**Reidemeister 3.**

$$R3L = PD[X_{-2,5,4,-1}, X_{-3,7,6,-5}]$$

$$X_{-6,9,8,-4} ;$$

$$R3R = PD[X_{-3,5,4,-2}, X_{-4,6,8,-1}]$$

$$X_{-5,7,9,-6} ;$$

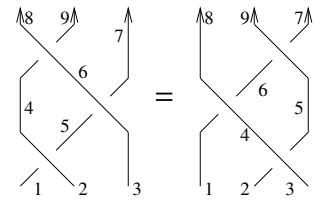
$$\{TL@R3L == TL@R3R, Kas@R3L == Kas@R3R\}$$

$$\{True, True\}$$

**Kas@R3L**

$$2\theta(u - \frac{1}{2}) - 2\theta(u + \frac{1}{2}) - 2$$

	$\gamma_3$	$\gamma_7$	$\gamma_9$	$\gamma_8$	$\gamma_{-1}$	$\gamma_{-2}$
$\bar{\gamma}_3$	$\frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$
$\bar{\gamma}_7$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2(2u^2-1)}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$
$\bar{\gamma}_9$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2u}{(2u-1)(2u+1)}$
$\bar{\gamma}_8$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$
$\bar{\gamma}_{-1}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2(2u^2-1)}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$
$\bar{\gamma}_{-2}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}$



**Reidemeister 2.**

$$TL@PD[X_{-2,4,3,-1}, \bar{X}_{-4,6,5,-3}]$$

$$\begin{matrix} & \theta & & & \\ & 1 & \theta & -1 & \theta \\ (\gamma_{-2} & & \gamma_6 & \gamma_5 & \gamma_{-1}) \\ \bar{\gamma}_{-2} & \theta & \theta & \theta & \theta \\ \bar{\gamma}_6 & \theta & \theta & \theta & \theta \\ \bar{\gamma}_5 & \theta & \theta & \theta & \theta \\ \bar{\gamma}_{-1} & \theta & \theta & \theta & \theta \end{matrix}$$

$$\{TL@PD[X_{-2,4,3,-1}, \bar{X}_{-4,6,5,-3}] == GT_{5,-2}@TL@PD[P_{-1,5}, P_{-2,6}], Kas@PD[X_{-2,4,3,-1}, \bar{X}_{-4,6,5,-3}] == GT_{5,-2}@Kas@PD[P_{-1,5}, P_{-2,6}]\}$$

$$\{True, True\}$$

**Reidemeister 1.**

$$\{TL@PD[X_{-3,3,2,-1}] == TL@P_{-1,2},$$

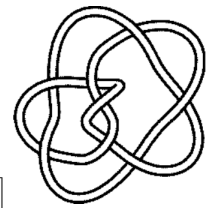
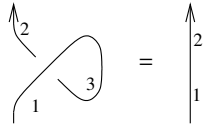
$$Kas@PD[X_{-3,3,2,-1}] == Kas@P_{-1,2}\}$$

$$\{True, True\}$$

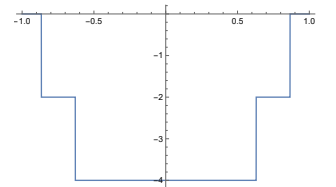
**A Knot.**

$$f = TLSig[Knot[8, 5]]$$

$$2\theta\left[-\frac{\sqrt{3}}{2} + u\right] - 2\theta\left[\frac{\sqrt{3}}{2} + u\right] - 2\theta\left[u - \left(\text{clockwise} - 0.630\dots\right)\right] + 2\theta\left[u - \left(\text{counterclockwise} 0.630\dots\right)\right]$$



$$Plot[f, \{u, -1, 1\}]$$

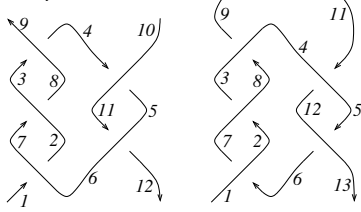


# The Conway-Kinoshita-Terasaka Tangles.



$$T1 = PD [\bar{X}_{-6,2,7,-1}, \bar{X}_{-2,8,3,-7}, \bar{X}_{-8,4,9,-3}, X_{-11,6,12,-5}, X_{-4,11,5,-10}];$$

$$T2 = PD [X_{-6,2,7,-1}, X_{-2,8,3,-7}, X_{-8,4,9,-3}, \bar{X}_{-12,6,13,-5}, \bar{X}_{-4,12,5,-11}, \bar{X}_{-10,15,11,-14}, \bar{X}_{-15,10,16,-9}];$$



## Column@{TL [T1], Kas [T1]}

$$-2\theta\left(u - \frac{\sqrt{3}}{2}\right) + 2\theta\left(u + \frac{\sqrt{3}}{2}\right) - 1$$

$(Y_{-10})$	$Y_9$	$Y_{-1}$	$Y_{12}$
$\bar{Y}_{-10}$	$\frac{\omega-1}{\omega}$	$\frac{1-\omega}{\omega^2-\omega+1}$	$\frac{\omega-1}{\omega}$
$\bar{Y}_9$	$\frac{2\omega}{\omega^2-\omega+1}$	$\frac{\omega-1}{\omega}$	$-\frac{2\omega}{\omega^2-\omega+1}$
$\bar{Y}_{-1}$	$\frac{\omega-1}{\omega}$	$\frac{1-\omega}{\omega^2-\omega+1}$	$\frac{2\omega}{\omega^2-\omega+1}$
$\bar{Y}_{12}$	$\frac{\omega-1}{\omega}$	$\frac{1-\omega}{\omega^2-\omega+1}$	$\frac{2\omega}{\omega^2-\omega+1}$

$$-2\theta\left(u - \frac{\sqrt{3}}{2}\right) + 2\theta\left(u + \frac{\sqrt{3}}{2}\right) - 1$$

$(Y_{-10})$	$Y_9$	$Y_{-1}$	$Y_{12}$
$\bar{Y}_{-10}$	$2(u-1)(u+1)(4u^2-3)$	$\frac{1}{2(4u^2-3)}$	$0$
$\bar{Y}_9$	$0$	$0$	$-\frac{1}{2(4u^2-3)}$
$\bar{Y}_{-1}$	$-2(u-1)(u+1)(4u^2-3)$	$0$	$0$
$\bar{Y}_{12}$	$0$	$-\frac{1}{2(4u^2-3)}$	$\frac{1}{2(4u^2-3)}$

## Column@{TL [T2], Kas [T2]}

$$1$$

$(Y_{-14})$	$Y_{16}$	$Y_{-1}$	$Y_{13}$
$\bar{Y}_{-14}$	$\frac{\omega-1}{\omega}$	$\frac{1-\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}$	$\frac{\omega-1}{\omega}$
$\bar{Y}_{16}$	$-\frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}$	$\frac{\omega-1}{\omega}$	$\frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}$
$\bar{Y}_{-1}$	$\frac{\omega-1}{\omega}$	$\frac{1-\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}$	$\frac{\omega-1}{\omega}$
$\bar{Y}_{13}$	$-\frac{\omega-1}{\omega}$	$-\frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}$	$-\frac{\omega-1}{\omega}$

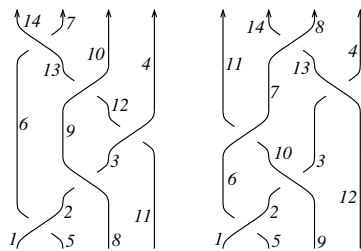
$$\frac{1}{2}(-16u^4 + 28u^2 - 13)$$

$(Y_{-14})$	$Y_{16}$	$Y_{-1}$	$Y_{13}$
$\bar{Y}_{-14}$	$\frac{1}{2}(-16u^4 + 28u^2 - 13)$	$\frac{1}{2}(16u^4 - 28u^2 + 13)$	$0$
$\bar{Y}_{16}$	$0$	$0$	$\frac{2(u-1)(u-1)}{16u^4-28u^2+13}$
$\bar{Y}_{-1}$	$\frac{1}{2}(16u^4 - 28u^2 + 13)$	$\frac{1}{2}(-16u^4 + 28u^2 - 13)$	$0$
$\bar{Y}_{13}$	$0$	$0$	$-\frac{2(u-1)(u-1)}{16u^4-28u^2+13}$

## Examples with non-trivial co-dimension.

$$B1 = PD [X_{-5,2,6,-1}, \bar{X}_{-8,3,9,-2}, X_{-11,4,12,-3}, X_{-12,10,13,-9}, \bar{X}_{-13,7,14,-6}];$$

$$B2 = PD [X_{-5,2,6,-1}, \bar{X}_{-9,3,10,-2}, X_{-10,7,11,-6}, \bar{X}_{-12,4,13,-3}, X_{-13,8,14,-7}];$$



## Column@{TL [B1], Kas [B1]}

$$0$$

$(Y_{-11})$	$Y_4$	$Y_{10}$	$Y_7$	$Y_{14}$	$Y_{-1}$	$Y_{-5}$	$Y_{-8}$
$\bar{Y}_{-11}$	$0$	$0$	$0$	$\frac{\omega-1}{\omega^2}$	$0$	$-\frac{\omega-1}{\omega^2}$	$0$
$\bar{Y}_4$	$0$	$0$	$0$	$-\frac{\omega-1}{\omega^2}$	$0$	$\frac{\omega-1}{\omega^2}$	$0$
$\bar{Y}_{10}$	$0$	$0$	$0$	$\frac{(\omega-1)^2}{\omega^2}$	$0$	$-\frac{(\omega-1)^2}{\omega^2}$	$0$
$\bar{Y}_7$	$0$	$0$	$0$	$-\frac{\omega-1}{\omega^2}$	$0$	$\frac{\omega-1}{\omega^2}$	$0$
$\bar{Y}_{14}$	$0$	$-(\omega-1)\omega$	$\omega-1$	$(\omega-1)^2$	$0$	$-\frac{\omega-1}{\omega^2}$	$0$
$\bar{Y}_{-1}$	$0$	$0$	$0$	$\omega-1$	$0$	$1-\omega$	$0$
$\bar{Y}_{-5}$	$0$	$(\omega-1)\omega$	$1-\omega$	$-(\omega-1)^2$	$1-\omega$	$\frac{\omega-1}{\omega^2}$	$\frac{(\omega-1)^2}{\omega}$
$\bar{Y}_{-8}$	$0$	$0$	$0$	$0$	$0$	$0$	$0$

$$0$$

$(Y_{-11})$	$Y_4$	$Y_{10}$	$Y_7$	$Y_{14}$	$Y_{-1}$	$Y_{-5}$	$Y_{-8}$
$\bar{Y}_{-11}$	$0$	$0$	$0$	$0$	$0$	$0$	$0$
$\bar{Y}_4$	$0$	$0$	$-1$	$-u$	$0$	$u$	$1$
$\bar{Y}_{10}$	$0$	$0$	$-u$	$1-2u^2$	$0$	$2u^2-1$	$0$
$\bar{Y}_7$	$0$	$-1$	$-u$	$-u$	$-1$	$0$	$1$
$\bar{Y}_{14}$	$0$	$-u$	$1-2u^2$	$-u$	$-1$	$-2(u-1)(u+1)$	$0$
$\bar{Y}_{-1}$	$0$	$0$	$-1$	$-u$	$0$	$u$	$1$
$\bar{Y}_{-5}$	$0$	$u$	$2u^2-1$	$0$	$-2(u-1)(u+1)$	$4u^2-3$	$0$
$\bar{Y}_{-8}$	$0$	$1$	$u$	$1$	$u$	$0$	$1-2u^2$

## Column@{TL [B2], Kas [B2]}

$$0$$

$(Y_{-12})$	$Y_4$	$Y_8$	$Y_{14}$	$Y_{11}$	$Y_{-1}$	$Y_{-5}$	$Y_{-9}$
$\bar{Y}_{-12}$	$\frac{\omega-1}{\omega}$	$\omega-1$	$-2(\omega-1)$	$\frac{2(\omega-1)^2}{\omega}$	$\frac{2(\omega-1)^2}{\omega}$	$0$	$-\frac{2(\omega-1)^2}{\omega}$
$\bar{Y}_4$	$0$	$0$	$0$	$0$	$0$	$0$	$0$
$\bar{Y}_8$	$\frac{\omega-1}{\omega}$	$1-\omega$	$0$	$-\frac{2(\omega-1)(2\omega-1)}{\omega}$	$-\frac{2(\omega-1)(2\omega-1)}{\omega}$	$0$	$\frac{2(\omega-1)(2\omega-1)}{\omega}$
$\bar{Y}_{14}$	$\frac{2(\omega-1)^2}{\omega}$	$0$	$0$	$\frac{2(\omega-1)^2}{\omega}$	$-\frac{2(\omega-1)^2}{\omega}$	$0$	$-\frac{2(\omega-1)^2}{\omega}$
$\bar{Y}_{11}$	$-2(\omega-1)\omega$	$0$	$2(\omega-1)\omega$	$-(\omega-1)(2\omega-1)$	$\frac{(\omega-1)^2}{\omega}$	$-\frac{(\omega-1)^2}{\omega}$	$\frac{2(\omega-1)^2}{\omega}$
$\bar{Y}_{-1}$	$0$	$0$	$0$	$0$	$\omega-1$	$0$	$1-\omega$
$\bar{Y}_{-5}$	$2(\omega-1)\omega$	$0$	$-2(\omega-1)\omega$	$2(\omega-1)\omega$	$-\frac{(\omega-1)^2}{\omega}$	$\frac{(\omega-1)^2}{\omega}$	$-(\omega-1)(2\omega-1)$
$\bar{Y}_{-9}$	$-\frac{2(\omega-1)(2\omega-1)}{\omega}$	$0$	$\frac{2(\omega-1)(2\omega-1)}{\omega}$	$-\frac{2(\omega-1)(2\omega-1)}{\omega}$	$\frac{2(\omega-1)^2}{\omega}$	$0$	$-\frac{2(\omega-1)(2\omega-1)}{\omega}$

$$2\theta\left(u - \frac{\sqrt{3}}{2}\right) - 2\theta\left(u + \frac{\sqrt{3}}{2}\right)$$

$(Y_{-12})$	$Y_4$	$Y_8$	$Y_{14}$	$Y_{11}$	$Y_{-1}$	$Y_{-5}$	$Y_{-9}$
$\bar{Y}_{-12}$	$\frac{1}{2u}$	$0$	$0$	$-\frac{1}{2u}$	$-1$	$-\frac{1}{2u}$	$0$
$\bar{Y}_4$	$0$	$0$	$0$	$0$	$0$	$0$	$0$
$\bar{Y}_8$	$0$	$-\frac{2(\omega-1)(2\omega-1)}{4u^2(4u^2-3)}$	$\frac{2(\omega-1)}{2u}$	$\frac{1}{4u^2(4u^2-3)}$	$0$	$-\frac{2(\omega-1)(2\omega-1)}{4u^2(4u^2-3)}$	$-\frac{1}{4u^2(4u^2-3)}$
$\bar{Y}_{14}$	$0$	$\frac{2(\omega-1)}{2u}$	$-2(u-1)(u+1)$	$\frac{1}{4u^2(4u^2-3)}$	$0$	$-\frac{1}{2u}$	$0$
$\bar{Y}_{11}$	$0$	$\frac{2(\omega-1)(2\omega-1)}{4u^2(4u^2-3)}$	$\frac{2(\omega-1)(2\omega-1)}{2u}$	$\frac{2(\omega-1)(2\omega-1)}{4u^2(4u^2-3)}$	$0$	$\frac{2(\omega-1)(2\omega-1)}{4u^2(4u^2-3)}$	$\frac{2(\omega-1)(2\omega-1)}{4u^2(4u^2-3)}$
$\bar{Y}_{-1}$	$0$	$0$	$0$	$0$	$0$	$0$	$0$
$\bar{Y}_{-5}$	$0$	$0$	$0$	$0$	$0$	$0$	$0$
$\bar{Y}_{-9}$	$0$	$\frac{2(\omega-1)(2\omega-1)}{4u^2(4u^2-3)}$	$\frac{1}{2u}$	$\frac{1}{4u^2(4u^2-3)}$	$0$	$\frac{2(\omega-1)(2\omega-1)}{4u^2(4u^2-3)}$	$\frac{2(\omega-1)(2\omega-1)}{4u^2(4u^2-3)}$

$\begin{pmatrix} A & B \\ C & U \end{pmatrix} \xrightarrow{\det(A)} \begin{pmatrix} I & A^{-1}B \\ C & U \end{pmatrix} \xrightarrow{1} \begin{pmatrix} I & A^{-1}B \\ 0 & U - CA^{-1}B \end{pmatrix}$ . Roughly,  $\det(A)$  is "det on ker",  $-CA^{-1}B$  is "a pushforward of  $\begin{pmatrix} A & B \\ C & U \end{pmatrix}$ ".  
so  $\det \begin{pmatrix} A & B \\ C & U \end{pmatrix} = \det(A) \det(U - CA^{-1}B)$ . (what if  $\mathbb{A}A^{-1}$ ?)

**Questions.** 1. Does this have a topological meaning? 2. Is there a version of the Kashaev Conjecture for tangles? 3. Find all solutions of R123 in our "algebra". 4. Braids and the Burau representation. 5. Recover the work in "Prior Art". 6. Are there any concordance properties? 7. What is the "SPQ group"? 8. The jumping points of signatures are the roots of the Alexander polynomial. Does this generalize to tangles? 9. Which of the three Cordon cases is the most common? 10. Are there interesting examples of tangles for which rels is non-trivial? 11. Is the  $pq$  part determined by  $\Gamma$ -calculus? 12. Is the  $pq$  part determined by finite type invariants? 13. Does it work with closed components / links? 14. Strand-doubling formulas? 15. A multivariable version? 16. Mutation invariance? 17. Ribbon knots? 18. Are there "face-virtual knots"? 19. Does the pushforward story extend to ranks? To formal Gaussian measures? To super Gaussian measures?

## References.

[CC] D. Cimasoni, A. Conway, *Colored Tangles and Signatures*, Math. Proc. Camb. Phil. Soc. **164** (2018) 493–530, arXiv:1507.07818.  
 [Co] A. Conway, *The Levine-Tristram Signature: A Survey*, arXiv:1903.04477.  
 [GG] J-M. Gambaudo, É. Ghys, *Braids and Signatures*, Bull. Soc. Math. France **133-4** (2005) 541–579.  
 [Ka] R. Kashaev, *On Symmetric Matrices Associated with Oriented Link Diagrams*, in *Topology and Geometry, A Collection of Essays Dedicated to Vladimir G. Turaev*, EMS Press 2021, arXiv:1801.04632.  
 [Li] J. Liu, *A Proof of the Kashaev Signature Conjecture*, arXiv:2311.01923.  
 [Me] A. Merz, *An Extension of a Theorem by Cimasoni and Conway*, arXiv:2104.02993.

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