Dror Bar-Natan: Talks: Geneva-231201:

Thanks for allowing me in Geneva!

Shifted Partial Quadratics, their Pushforwards, and Signature Invariants for Tangles

http://drorbn.net/ge23

Abstract. Following a general discussion of the computation of zombians of unfinished columbaria (with examples), I will tell you about my recent joint work w/ Jessica Liu on what we feel is the “textbook” extension of knot signatures to tangles, which for unknown reasons, is not in any of the textbooks that we know.

Jessica Liu

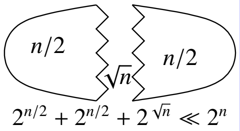
Columbaria in an East Sydney Cemetery

Image: Freepik.com

Jacobian, Hamiltonian, Zombian

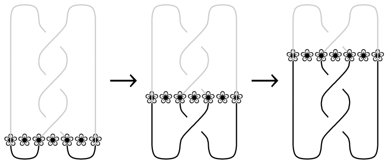
Prior Art on signatures for tangles / braids. Gambaudo and Ghys [GG], Cimasoni and Conway [CC], Conway [Co], Merz [Me]. All define signatures of tangles / braids by first closing them to links and then work hard to derive composition properties.

Why Tangles? • Faster!



- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:
  - ○ The Jones Polynomial ⇝ The Temperley-Lieb Algebra.
  - ○ Khovanov Homology ⇝ “Unfinished complexes”, complexes in a category.
  - ○ The Kontsevich Integral ⇝ Associators.
  - ○ HFK ⇝ OMG, type D, type A, $\mathcal{A}_\infty$, …

Knot-tease



Zombies: Freepik.com

Computing Zombians of Unfinished Columbaria.

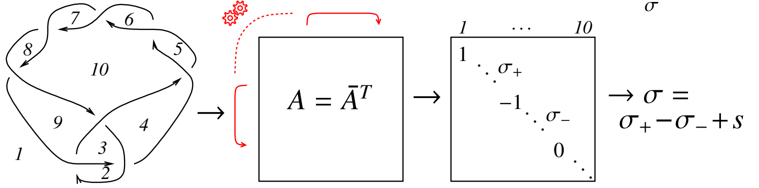
- Must be no slower than for finished ones.
- Future zombies must be able to complete the computation.
- Future zombies must not even know the size of the task that today’s zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!

Columbarium near Assen

Example / Exercise. Compute the determinant of a $1,000 \times 1,000$ matrix in which 50 entries are not yet given.

Homework / Research Projects. • What with ZPUCs? • Use this to get an Alexander tangle invariant.

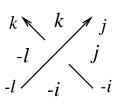
Reminders. {knots} $\rightrightarrows$ {matrices / quadratic forms} $\xrightarrow[\sigma]{\text{signature}} \mathbb{Z}$:



With $|\omega|=1$, $t=1-\omega$, $r=t+\bar{t}$, $v=\operatorname{Re}(\omega)$, and $u=\operatorname{Re}(\omega^{1/2})$:

Tristram-Levine *(TL)* Kashaev *(Kas)*

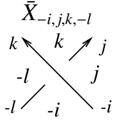
$X_{-i,j,k,-l}$



$$A \mathrel{+}= \left(\begin{array}{cccc|c} -r & -t & 2t & \bar{t} & i \\ -\bar{t} & 0 & \bar{t} & 0 & j \\ 2\bar{t} & t & -r & -\bar{t} & k \\ t & 0 & -t & 0 & l \end{array}\right) \quad s=0$$

$$A \mathrel{+}= \left(\begin{array}{cccc|c} v & u & 1 & u & i \\ u & 1 & u & 1 & j \\ 1 & u & v & u & k \\ u & 1 & u & 1 & l \end{array}\right) \quad s \mathrel{-}= 1$$

$\bar{X}_{-i,j,k,-l}$



$$A \mathrel{+}= \left(\begin{array}{cccc|c} r & -t & -2\bar{t} & \bar{t} & i \\ -\bar{t} & 0 & \bar{t} & 0 & j \\ -2t & t & r & -\bar{t} & k \\ t & 0 & -t & 0 & l \end{array}\right) \quad s=0$$

$$A \mathrel{-}= \left(\begin{array}{cccc|c} v & u & 1 & u & i \\ u & 1 & u & 1 & j \\ 1 & u & v & u & k \\ u & 1 & u & 1 & l \end{array}\right) \quad s \mathrel{+}= 1$$

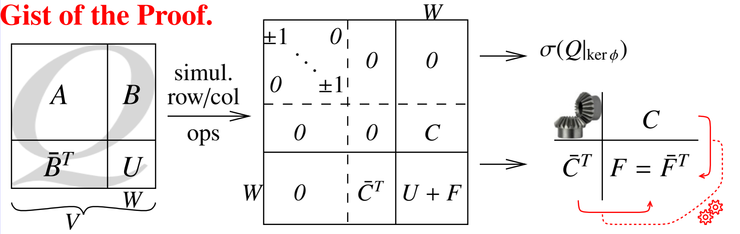
~~Kashaev’s Conjecture~~ [Ka] For knots, $\sigma_{Kas}=2\sigma_{TL}$.

Liu’s Theorem [Li].

A *Partial Quadratic (PQ)* on V is a quadratic Q defined only on a subspace $\mathcal{D}_Q \subset V$. We add PQs with $\mathcal{D}_{Q_1+Q_2} := \mathcal{D}_{Q_1} \cap \mathcal{D}_{Q_2}$. Given a linear $\psi\colon V \to W$ and a PQ Q on W, there is an obvious *pullback* ψ^*Q, a PQ on V.

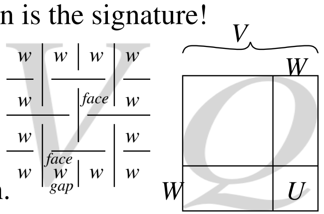
Theorem 1. Given a linear $\phi\colon V \to W$ and a PQ Q on V, there is a unique *pushforward* PQ ϕ_*Q on W such that for every *PQ U* on W, $\sigma_V(Q+\phi^*U)=\sigma_{\ker\phi}(Q|_{\ker\phi})+\sigma_W(U+\phi_*Q)$.
(If you must, $\mathcal{D}(\phi_*Q)=\phi(\operatorname{ann}_Q(\mathcal{D}(Q)\cap\ker\phi))$ and $(\phi_*Q)(w)=Q(v)$, where v is s.t. $\phi(v)=w$ and $Q(v,\operatorname{rad}Q|_{\ker\phi})=0$).

Gist of the Proof.



… and the quadratic $F =: \phi_*Q$ is well-defined only on $D := \ker C$.

Exactly what we want, if the Zombian is the signature!



V: The full space of *faces*.
W: The boundary, made of *gaps*.
Q: The known parts.
U: The part yet unknown.
$\sigma_V(Q+\phi^*(U))$: The overall Zombian.
$\sigma(Q|_{\ker\phi})$: An internal bit. $U+\phi_*Q$: A boundary bit.
And so our ZPUC is the pair $S=(\sigma(Q|_{\ker\phi}),\phi_*Q)$.

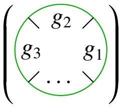
A *Shifted Partial Quadratic (SPQ)* on V is a pair $S=(s\in\mathbb{Z}, Q$ a PQ on $V)$. addition also adds the shifts, pullbacks keep the shifts, yet $\phi_*S := (s+\sigma_{\ker\phi}(Q|_{\ker\phi}),\phi_*Q)$ and $\sigma(S) := s+\sigma(Q)$.

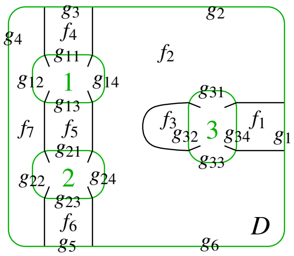
Theorem 1’ (*Reciprocity*). Given $\phi\colon V \to W$, for SPQs S on V and U on W we have $\sigma_V(S+\phi^*U)=\sigma_W(U+\phi_*S)$ (and this characterizes ϕ_*S). Note. ψ^* is additive but ϕ_* is not.

Theorem 2. ψ^* and ϕ_* are functorial.

$$\begin{tikzcd} V_0 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & V_1 \arrow[d, "\beta"] \\ V_2 \arrow[r, "\delta"'] \arrow[ur, dotted] & V_3 \end{tikzcd}$$

Theorem 3. If, as on the right, $\beta\alpha=\delta\gamma$ ~~and α and γ are surjective~~, then $\alpha_*\gamma^*=\beta^*\delta_*$. (Correction: `http://drorbn.net/ge23/PQ-UC.pdf`)

Definition. $\mathcal{S}$  $:= \left\{\begin{matrix}\text{SPQ } S \\ \text{on } \langle g_i\rangle\end{matrix}\right\}$.



Connection Diagram

Theorem 4 (?!). $\{\mathcal{S}(\text{cyclic sets})\}$ is a planar algebra, with compositions $\mathcal{S}(D)((S_i)) := \phi^D_*(\psi^*_D(\bigoplus_i S_i))$, where $\psi_D\colon\langle f_i\rangle \to \langle g_{\alpha i}\rangle$ maps every face of D to the sum of the input gaps adjacent to it and $\phi^D\colon\langle f_i\rangle \to \langle g_i\rangle$ maps every face to the sum of the output gaps adjacent to it. So for our D, ψ_D is $f_1 \mapsto g_{34}$, $f_2 \mapsto g_{31}+g_{14}+g_{24}+g_{33}$, $f_3 \mapsto g_{32}$, $f_4 \mapsto g_{11}$, $f_5 \mapsto g_{13}+g_{21}$, $f_6 \mapsto g_{23}$, $f_7 \mapsto g_{12}+g_{22}$ and ϕ^D is $f_1 \mapsto g_1$, $f_2 \mapsto g_2+g_6$, $f_3 \mapsto 0$, $f_4 \mapsto g_3$, $f_5 \mapsto 0$, $f_6 \mapsto g_5$, $f_7 \mapsto g_4$.

Theorem 5. *TL* and *Kas*, defined on X and $\bar{X}$ as before, extend to planar algebra morphisms {tangles} $\to \{\mathcal{S}\}$.

Levine

Tristram

Kashaev

Updated and corrected version: `http://drorbn.net/ge23/PQ-UC.pdf`

Implementation (sources: http://drorbn.net/icerm23/ap). I like it most when the implementation matches the math perfectly. We failed here.

```
Once[<< KnotTheory`];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

Utilities. The step function, algebraic numbers, canonical forms.

```
θ[x_] /; NumericQ[x] := UnitStep[x]
```

```
ω2[v_][p_] := Module[{q = Expand[p], n, c},
   If[q === 0, 0,
    c = Coefficient[q, ω, n = Exponent[q, ω]];
    c v^n + ω2[v][q - c (ω + ω^-1)^n]]];
```

```
sign[ℰ_] := Module[{n, d, v, p, rs, e, k},
  {n, d} = NumeratorDenominator[ℰ];
  {n, d} /= ω^(Exponent[n,ω]/2+Exponent[n,ω,Min]/2);
  p = Factor[ω2[v]@n * ω2[v]@d /. v → 4 u^2 - 2];
  rs = Solve[p == 0, u, Reals];
  If[rs === {}, Sign[p /. u → 0],
   rs = Union@(u /. rs);
   Sign[(-1)^(e=Exponent[p,u]) Coefficient[p, u, e]] + Sum[
     k = 0;
     While[(d = RootReduce[∂_{u,++k} p /. u → r]) == 0];
     If[EvenQ[k], 0, 2 Sign[d]] * θ[u - r],
     {r, rs}]
  ]
 ]
```

```
SetAttributes[B, Orderless];
CF[b_B] := RotateLeft[#, First@Ordering[#] - 1] & /@
  DeleteCases[b, {}]
```

```
CF[ℰ_] := Module[{γs = Union@Cases[ℰ, γ_ | γ̄_, ∞]},
  Total[CoefficientRules[ℰ, γs] /.
     (ps_ → c_) :→ Factor[c] × Times @@ γs^ps] ]
```

```
CF[{}] = {};
CF[𝒞_List] :=
 Module[{γs = Union@Cases[𝒞, γ_, ∞], γ},
  CF /@ DeleteCases[0][
    RowReduce[Table[∂_γ r, {r, 𝒞}, {γ, γs}]].γs] ]
```

```
(ℰ_)* := ℰ /. {γ̄ → γ, γ → γ̄, ω → ω^-1, c_Complex :→ c*};
r_Rule⁺ := {r, r*}
```

```
RulesOf[γ_i_ + rest_.] := (γ_i → -rest)⁺;
CF[PQ[𝒞_, q_]] := Module[{n𝒞 = CF[𝒞]},
  PQ[n𝒞, CF[q /. Union @@ RulesOf /@ n𝒞]] ]
```

```
CF[Σ_b_[σ_, pq_]] := Σ_CF[b][σ, CF[pq]]
```

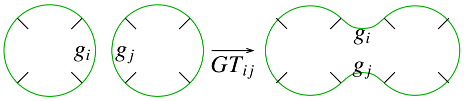
Pretty-Printing.

```
Format[Σ_b_B[σ_, PQ[𝒞_, q_]]] := Module[{γs},
   γs = γ_# & /@ Join @@ b;
   Column[{TraditionalForm@σ,
     TableForm[Join[
       Prepend[""] /@ Table[TraditionalForm[∂_c r],
         {r, 𝒞}, {c, γs}],
       {Prepend[""][
         Join @@
           (b /. {l_, m___, r_} :→
               {DisplayForm@RowBox[{"(", l}],
                m, DisplayForm@RowBox[{r, ")"}]}) /.
            i_Integer :→ γ_i ]},
       MapThread[Prepend,
        {Table[TraditionalForm[∂_{r,c} q], {r, γs*},
          {c, γs}], γs*}]
      ], TableAlignments → Center]
    }, Center] ];
```

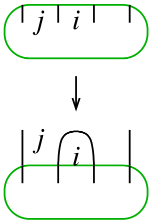
The Face-Centric Core.

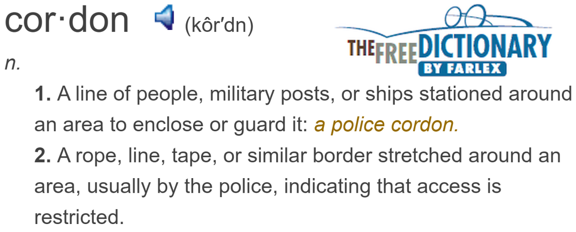
```
Σ_b1_[σ1_, PQ[𝒞1_, q1_]] ⊕ Σ_b2_[σ2_, PQ[𝒞2_, q2_]] ^:=
  CF@Σ_Join[b1,b2][σ1 + σ2, PQ[𝒞1 ⋃ 𝒞2, q1 + q2]];
```

GT for Gap Touch:



```
GT_i_,j_@Σ_B[{li___,i_,ri___},{lj___,j_,rj___},bs___][σ_,
   PQ[𝒞_, q_]] :=
 CF@Σ_B[{ri,li,j,rj,lj,i},bs][σ, PQ[𝒞 ⋃ {γ_i - γ_j}, q]]
```





cor·don (kôr′dn)

n.

1. A line of people, military posts, or ships stationed around an area to enclose or guard it: *a police cordon.*

2. A rope, line, tape, or similar border stretched around an area, usually by the police, indicating that access is restricted.

$$\begin{array}{c} s \\ i \\ \end{array}\left(\begin{array}{c|cc} 0 & \phi & C_{\text{rest}} \\ \hline \bar{\phi}^T & \lambda & \theta \\ \bar{C}^T_{\text{rest}} & \bar{\theta}^T & A_{\text{rest}} \end{array}\right) \rightarrow \begin{cases} \exists p\, \phi_p \neq 0 & \text{use } \phi_p \text{ to kill its row and column, drop a } \begin{pmatrix} 01 \\ 10 \end{pmatrix} \text{ summand} \\ \phi = 0, \lambda \neq 0 & \text{use } \lambda \text{ to kill } \theta\text{, let } s \mathrel{+}= \text{sign}(\lambda) \\ \phi = 0, \lambda = 0 & \text{append } \theta \text{ to } C_{\text{rest}}. \end{cases}$$

```
Cordon_i_@Σ_B[{li___,i_,ri___},bs___][σ_, PQ[𝒞_, q_]] :=
 Module[{ϕ = ∂_γi 𝒞, λ = ∂_{γ̄i,γi} q, nσ = σ, n𝒞, nq, p},
  {p} = FirstPosition[(# =!= 0) & /@ ϕ, True, {0}];
  {n𝒞, nq} = Which[
    p > 0,  {𝒞, q} /. (γ_i → -𝒞⟦p⟧ / ϕ⟦p⟧)⁺ /. (γ_i → 0)⁺,
    λ =!= 0, (nσ += sign[λ];
     {𝒞, q /. (γ_i → -(∂_γ̄i q) / λ)⁺ /. (γ_i → 0)⁺}),
    λ === 0, {𝒞 ⋃ {∂_γ̄i q}, q /. (γ_i → 0)⁺}];
  CF@Σ_B[Most@{ri,li},bs][nσ,
    PQ[n𝒞, nq] /. (γ_Last@{ri,li} → γ_First@{ri,li})⁺] ]
```

Strand Operations. c for contract, mc for magnetic contract:

```
c_{i_,j_}@t : Σ_{B[{li___,i_,ri___},{___,j_,___},___]}[__] :=
  t // GT_{j,First@{ri,li}} // Cordon_j
```

```
c_{i_,j_}@t : Σ_{B[{___,i_,j_,___},___]}[__] := Cordon_j@t
c_{i_,j_}@t : Σ_{B[{j_,___,i_},___]}[__] := Cordon_j@t
c_{i_,j_}@t : Σ_{B[{___,j_,i_,___},___]}[__] := Cordon_i@t
c_{i_,j_}@t : Σ_{B[{i_,___,j_},___]}[__] := Cordon_i@t
```

```
mc[ℰ_] := ℰ //.
   t : Σ_{B[{___,i_,___},{___,j_,___},___]}[__] |
       Σ_{B[{___,i_,j_,___},___]}[__] | Σ_{B[{j_,___,i_},___]}[__] /;
     i + j == 0 :→ c_{i,j}@t
```

The Crossings (and empty strands).

```
Kas@P_{i_,j_} := CF@Σ_{B[{i,j}]}[0, PQ[{}, 0]];
TL@P_{i_,j_} := CF@Σ_{B[{i,j}]}[0, PQ[{}, 0]]
```

```
Kas[x : X[i_, j_, k_, l_]] :=
  Kas@If[PositiveQ[x], X_{-i,j,k,-l}, X̄_{-j,k,l,-i}];
Kas[(x : X | X̄)_{fs__}] := Module[{v = 2 u^2 - 1, p, γs, m},
  γs = γ_# & /@ {fs}; p = (x === X);
                ( v  u  1  u )     ( v  u  1  u )
  m = If[p,     ( u  1  u  1 ) , - ( u  1  u  1 ) ];
                ( 1  u  v  u )     ( 1  u  v  u )
                ( u  1  u  1 )     ( u  1  u  1 )

  CF@Σ_{B[{fs}]}[If[p, -1, 1], PQ[{}, γs^*.m.γs]]]
```

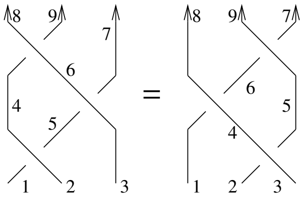
```
TL[x : X[i_, j_, k_, l_]] :=
  TL@If[PositiveQ[x], X_{-i,j,k,-l}, X̄_{-j,k,l,-i}];
TL[(x : X | X̄)_{fs__}] := Module[{t = 1 - ω, r, γs, m},
  r = t + t^*; γs = γ_# & /@ {fs};
  m = If[x === X,
     ( -r    -t   2 t   t^*  )   (  r     -t   -2 t^*   t^*  )
     ( -t^*   0   t^*   0    ) , ( -t^*    0    t^*     0    ) ];
     ( 2 t^*  t   -r    -t^* )   ( -2 t    t    r      -t^*  )
     (  t     0   -t    0    )   (  t      0   -t       0    )

  CF@Σ_{B[{fs}]}[0, PQ[{}, γs^*.m.γs]]]
```

Evaluation on Tangles and Knots.

```
Kas[K_] := Fold[mc[#1 ⊕ #2] &, Σ_{B[]}[0, PQ[{}, 0]],
    List @@ (Kas /@ PD@K)];
KasSig[K_] := Expand[Kas[K]〚1〛 / 2]
```

```
TL[K_] :=
  Fold[mc[#1 ⊕ #2] &, Σ_{B[]}[0, PQ[{}, 0]],
    List @@ (TL /@ PD@K)] /.
   θ[c_ + u] /; Abs[c] ≥ 1 :→ θ[c];
TLSig[K_] := TL[K]〚1〛
```

Reidemeister 3.



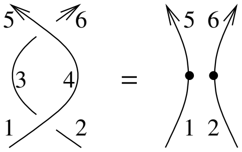
```
R3L = PD[X_{-2,5,4,-1}, X_{-3,7,6,-5},
    X_{-6,9,8,-4}];
R3R = PD[X_{-3,5,4,-2}, X_{-4,6,8,-1},
    X_{-5,7,9,-6}];
{TL@R3L == TL@R3R, Kas@R3L == Kas@R3R}
```

```
{True, True}
```

```
Kas@R3L
```

$$
\begin{array}{c}
2\,\theta\left(u-\frac{1}{2}\right)-2\,\theta\left(u+\frac{1}{2}\right)-2\\
\begin{array}{ccccccc}
 & (\gamma_{-3} & \gamma_{7} & \gamma_{9} & \gamma_{8} & \gamma_{-1} & \gamma_{-2}) \\
\bar{\gamma}_{-3} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} \\
\bar{\gamma}_{7} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2(2u^2-1)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} \\
\bar{\gamma}_{9} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} \\
\bar{\gamma}_{8} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} \\
\bar{\gamma}_{-1} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2(2u^2-1)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} \\
\bar{\gamma}_{-2} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}
\end{array}
\end{array}
$$

Reidemeister 2.



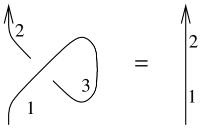
```
TL@PD[X_{-2,4,3,-1}, X̄_{-4,6,5,-3}]
```

$$
\begin{array}{c}
0\\
\begin{array}{ccccc}
 & 1 & 0 & -1 & 0 \\
 & (\gamma_{-2} & \gamma_{6} & \gamma_{5} & \gamma_{-1}) \\
\bar{\gamma}_{-2} & 0 & 0 & 0 & 0 \\
\bar{\gamma}_{6} & 0 & 0 & 0 & 0 \\
\bar{\gamma}_{5} & 0 & 0 & 0 & 0 \\
\bar{\gamma}_{-1} & 0 & 0 & 0 & 0
\end{array}
\end{array}
$$

```
{TL@PD[X_{-2,4,3,-1}, X̄_{-4,6,5,-3}] == GT_{5,-2}@TL@PD[P_{-1,5}, P_{-2,6}],
 Kas@PD[X_{-2,4,3,-1}, X̄_{-4,6,5,-3}] == GT_{5,-2}@Kas@PD[P_{-1,5}, P_{-2,6}]}
```

```
{True, True}
```

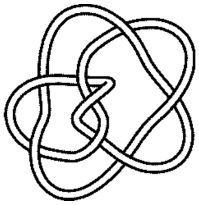
Reidemeister 1.



```
{TL@PD[X_{-3,3,2,-1}] == TL@P_{-1,2},
 Kas@PD[X_{-3,3,2,-1}] == Kas@P_{-1,2}}
```

```
{True, True}
```

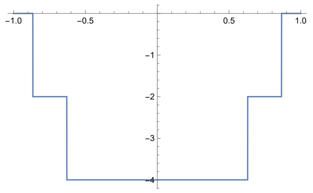
A Knot.



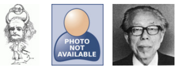
```
f = TLSig[Knot[8, 5]]
```

$$
2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]-2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]-2\,\theta\left[u-\boxed{-0.630\ldots}\right]+2\,\theta\left[u-\boxed{0.630\ldots}\right]
$$

```
Plot[f, {u, -1, 1}]
```

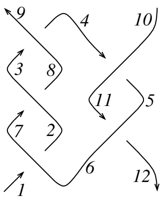


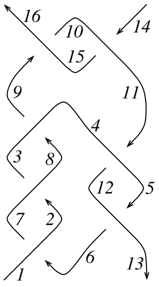
The Conway-Kinoshita-Terasaka Tangles.


Conway Kinoshita Terasaka

$$\mathtt{T1} = \mathtt{PD}\left[\bar{X}_{-6,2,7,-1},\ \bar{X}_{-2,8,3,-7},\ \bar{X}_{-8,4,9,-3},\ X_{-11,6,12,-5},\ X_{-4,11,5,-10}\right];$$

$$\mathtt{T2} = \mathtt{PD}\left[X_{-6,2,7,-1},\ X_{-2,8,3,-7},\ X_{-8,4,9,-3},\ \bar{X}_{-12,6,13,-5},\ \bar{X}_{-4,12,5,-11},\ \bar{X}_{-10,15,11,-14},\ \bar{X}_{-15,10,16,-9}\right];$$





Column@{TL[T1], Kas[T1]}

$$-2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-1$$

$$\begin{array}{ccccc}
 & (\gamma_{-10} & \gamma_{9} & \gamma_{-1} & \gamma_{12}) \\
\bar\gamma_{-10} & 0 & 1-\omega & 0 & \omega-1 \\
\bar\gamma_{9} & \frac{\omega-1}{\omega} & \frac{2\,\omega}{\omega^2-\omega+1} & -\frac{\omega-1}{\omega} & -\frac{2\,\omega}{\omega^2-\omega+1} \\
\bar\gamma_{-1} & 0 & \omega-1 & 0 & 1-\omega \\
\bar\gamma_{12} & -\frac{\omega-1}{\omega} & -\frac{2\,\omega}{\omega^2-\omega+1} & \frac{\omega-1}{\omega} & \frac{2\,\omega}{\omega^2-\omega+1}
\end{array}$$

$$-2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-1$$

$$\begin{array}{ccccc}
 & (\gamma_{-10} & \gamma_{9} & \gamma_{-1} & \gamma_{12}) \\
\bar\gamma_{-10} & 2\,(u-1)\,(u+1)\,(4\,u^2-3) & 0 & -2\,(u-1)\,(u+1)\,(4\,u^2-3) & 0 \\
\bar\gamma_{9} & 0 & \frac{1}{2\,(4\,u^2-3)} & 0 & -\frac{1}{2\,(4\,u^2-3)} \\
\bar\gamma_{-1} & -2\,(u-1)\,(u+1)\,(4\,u^2-3) & 0 & 2\,(u-1)\,(u+1)\,(4\,u^2-3) & 0 \\
\bar\gamma_{12} & 0 & -\frac{1}{2\,(4\,u^2-3)} & 0 & \frac{1}{2\,(4\,u^2-3)}
\end{array}$$

Column@{TL[T2], Kas[T2]}

$$0$$

$$\begin{array}{ccccc}
 & (\gamma_{-14} & \gamma_{16} & \gamma_{-1} & \gamma_{13}) \\
\bar\gamma_{-14} & 0 & 1-\omega & 0 & \omega-1 \\
\bar\gamma_{16} & \frac{\omega-1}{\omega} & -\frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1} & -\frac{\omega-1}{\omega} & \frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1} \\
\bar\gamma_{-1} & 0 & \omega-1 & 0 & 1-\omega \\
\bar\gamma_{13} & -\frac{\omega-1}{\omega} & \frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1} & \frac{\omega-1}{\omega} & -\frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1}
\end{array}$$

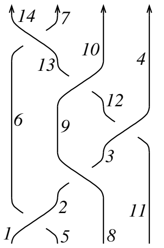
$$1$$

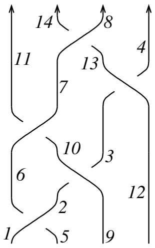
$$\begin{array}{ccccc}
 & (\gamma_{-14} & \gamma_{16} & \gamma_{-1} & \gamma_{13}) \\
\bar\gamma_{-14} & \frac{1}{2}\,(-16\,u^4+28\,u^2-13) & 0 & \frac{1}{2}\,(16\,u^4-28\,u^2+13) & 0 \\
\bar\gamma_{16} & 0 & -\frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13} & 0 & \frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13} \\
\bar\gamma_{-1} & \frac{1}{2}\,(16\,u^4-28\,u^2+13) & 0 & \frac{1}{2}\,(-16\,u^4+28\,u^2-13) & 0 \\
\bar\gamma_{13} & 0 & \frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13} & 0 & -\frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13}
\end{array}$$

Examples with non-trivial co-dimension.

$$\mathtt{B1} = \mathtt{PD}\left[X_{-5,2,6,-1},\ \bar{X}_{-8,3,9,-2},\ X_{-11,4,12,-3},\ X_{-12,10,13,-9},\ \bar{X}_{-13,7,14,-6}\right];$$

$$\mathtt{B2} = \mathtt{PD}\left[X_{-5,2,6,-1},\ \bar{X}_{-9,3,10,-2},\ X_{-10,7,11,-6},\ \bar{X}_{-12,4,13,-3},\ X_{-13,8,14,-7}\right];$$





Column@{TL[B1], Kas[B1]}

$$0$$

$$\begin{array}{ccccccccc}
 & 1 & 0 & -1 & 0 & \frac{1}{\omega} & 0 & -\frac{1}{\omega} & 0 \\
 & 0 & 0 & 0 & -1 & \frac{1}{\omega} & 0 & -\frac{1}{\omega} & 1 \\
 & (\gamma_{-11} & \gamma_{4} & \gamma_{10} & \gamma_{7} & \gamma_{14} & \gamma_{-1} & \gamma_{-5} & \gamma_{-8}) \\
\bar\gamma_{-11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{4} & 0 & 0 & 0 & 0 & \frac{\omega-1}{\omega^2} & 0 & -\frac{\omega-1}{\omega^2} & 0 \\
\bar\gamma_{10} & 0 & 0 & 0 & 0 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 \\
\bar\gamma_{7} & 0 & 0 & 0 & 0 & \frac{(\omega-1)^2}{\omega^2} & 0 & -\frac{(\omega-1)^2}{\omega^2} & 0 \\
\bar\gamma_{14} & 0 & -((\omega-1)\,\omega) & \omega-1 & (\omega-1)^2 & 0 & -\frac{\omega-1}{\omega} & \frac{\omega-1}{\omega} & 0 \\
\bar\gamma_{-1} & 0 & 0 & 0 & 0 & \omega-1 & 0 & 1-\omega & 0 \\
\bar\gamma_{-5} & 0 & (\omega-1)\,\omega & 1-\omega & -(\omega-1)^2 & 1-\omega & \frac{\omega-1}{\omega} & \frac{(\omega-1)^2}{\omega} & 0 \\
\bar\gamma_{-8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$$0$$

$$\begin{array}{ccccccccc}
 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\
 & (\gamma_{-11} & \gamma_{4} & \gamma_{10} & \gamma_{7} & \gamma_{14} & \gamma_{-1} & \gamma_{-5} & \gamma_{-8}) \\
\bar\gamma_{-11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{4} & 0 & 0 & 0 & -1 & -u & 0 & u & 1 \\
\bar\gamma_{10} & 0 & 0 & 0 & -u & 1-2\,u^2 & 0 & 2\,u^2-1 & u \\
\bar\gamma_{7} & 0 & -1 & -u & 2\,u^2-3 & -u & -1 & 0 & 1 \\
\bar\gamma_{14} & 0 & -u & 1-2\,u^2 & -u & -1 & -u & -2\,(u-1)\,(u+1) & u \\
\bar\gamma_{-1} & 0 & 0 & 0 & -1 & -u & 0 & u & 1 \\
\bar\gamma_{-5} & 0 & u & 2\,u^2-1 & 0 & -2\,(u-1)\,(u+1) & u & 4\,u^2-3 & 0 \\
\bar\gamma_{-8} & 0 & 1 & u & 1 & u & 1 & 0 & 1-2\,u^2
\end{array}$$

Column@{TL[B2], Kas[B2]}

$$0$$

$$\begin{array}{ccccccccc}
 & (\gamma_{-12} & \gamma_{4} & \gamma_{8} & \gamma_{14} & \gamma_{11} & \gamma_{-1} & \gamma_{-5} & \gamma_{-9}) \\
\bar\gamma_{-12} & \frac{(\omega-1)^2}{\omega} & \omega-1 & -2\,(\omega-1) & \frac{2\,(\omega-1)^2}{\omega} & \frac{2\,(\omega-1)}{\omega^2} & 0 & -\frac{2\,(\omega-1)}{\omega^2} & -\frac{(\omega-1)\,(2\,\omega-3)}{\omega} \\
\bar\gamma_{4} & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{8} & \frac{2\,(\omega-1)}{\omega} & 1-\omega & \frac{(\omega-1)^2}{\omega} & -\frac{(\omega-1)\,(2\,\omega-3)}{\omega} & -\frac{2\,(\omega-1)}{\omega^2} & 0 & \frac{2\,(\omega-1)}{\omega^2} & \frac{2\,(\omega-2)\,(\omega-1)}{\omega} \\
\bar\gamma_{14} & \frac{2\,(\omega-1)^2}{\omega} & 0 & -\frac{(\omega-1)\,(3\,\omega-2)}{\omega} & \frac{3\,(\omega-1)^2}{\omega} & -\frac{(\omega-2)\,(\omega-1)}{\omega^2} & 0 & -\frac{2\,(\omega-1)}{\omega^2} & -\frac{2\,(\omega-2)\,(\omega-1)}{\omega} \\
\bar\gamma_{11} & -2\,(\omega-1)\,\omega & 0 & 2\,(\omega-1)\,\omega & -((\omega-1)\,(2\,\omega-1)) & \frac{(\omega-1)^2}{\omega} & -\frac{\omega-1}{\omega} & \frac{2\,(\omega-1)}{\omega} & 2\,(\omega-1)^2 \\
\bar\gamma_{-1} & 0 & 0 & 0 & 0 & \omega-1 & 0 & 1-\omega & 0 \\
\bar\gamma_{-5} & 2\,(\omega-1)\,\omega & 0 & -2\,(\omega-1)\,\omega & 2\,(\omega-1)\,\omega & -2\,(\omega-1) & \frac{\omega-1}{\omega} & \frac{(\omega-1)^2}{\omega} & -((\omega-1)\,(2\,\omega-1)) \\
\bar\gamma_{-9} & -\frac{(\omega-1)\,(3\,\omega-2)}{\omega} & 0 & \frac{2\,(\omega-1)\,(2\,\omega-1)}{\omega} & -\frac{2\,(\omega-1)\,(2\,\omega-1)}{\omega} & \frac{2\,(\omega-1)^2}{\omega^2} & 0 & -\frac{(\omega-2)\,(\omega-1)}{\omega^2} & \frac{3\,(\omega-1)^2}{\omega}
\end{array}$$

$$2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)-2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)$$

$$\begin{array}{ccccccccc}
 & 1 & \frac{1}{2\,u} & 0 & -\frac{1}{2\,u} & -1 & -\frac{1}{2\,u} & 0 & \frac{1}{2\,u} \\
 & (\gamma_{-12} & \gamma_{4} & \gamma_{8} & \gamma_{14} & \gamma_{11} & \gamma_{-1} & \gamma_{-5} & \gamma_{-9}) \\
\bar\gamma_{-12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{4} & 0 & -\frac{(2\,u-1)\,(2\,u+1)\,(2\,u^2-1)}{4\,u^2\,(4\,u^2-3)} & -\frac{2\,u^2-1}{2\,u} & \frac{1}{4\,u^2\,(4\,u^2-3)} & 0 & -\frac{(2\,u-1)\,(2\,u+1)}{4\,u^2\,(4\,u^2-3)} & -\frac{1}{2\,u\,(4\,u^2-3)} & \frac{8\,u^4-6\,u^2-1}{4\,u^2\,(4\,u^2-3)} \\
\bar\gamma_{8} & 0 & -\frac{2\,u^2-1}{2\,u} & -2\,(u-1)\,(u+1) & \frac{2\,u^2-1}{2\,u} & 0 & -\frac{1}{2\,u} & 0 & \frac{1}{2\,u} \\
\bar\gamma_{14} & 0 & \frac{1}{4\,u^2\,(4\,u^2-3)} & \frac{2\,u^2-1}{2\,u} & \frac{(2\,u^2-1)\,(16\,u^4-16\,u^2+1)}{4\,u^2\,(4\,u^2-3)} & 0 & -\frac{8\,u^4-10\,u^2+1}{4\,u^2\,(4\,u^2-3)} & \frac{1}{2\,u\,(4\,u^2-3)} & \frac{1}{4\,u^2\,(4\,u^2-3)} \\
\bar\gamma_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{-1} & 0 & -\frac{(2\,u-1)\,(2\,u+1)}{4\,u^2\,(4\,u^2-3)} & -\frac{1}{2\,u} & -\frac{8\,u^4-10\,u^2+1}{4\,u^2\,(4\,u^2-3)} & 0 & \frac{8\,u^4-10\,u^2-1}{4\,u^2\,(4\,u^2-3)} & \frac{8\,u^4-10\,u^2+1}{2\,u\,(4\,u^2-3)} & \frac{16\,u^4-16\,u^2+1}{4\,u^2\,(4\,u^2-3)} \\
\bar\gamma_{-5} & 0 & -\frac{1}{2\,u\,(4\,u^2-3)} & 0 & \frac{1}{2\,u\,(4\,u^2-3)} & 0 & \frac{8\,u^4-10\,u^2+1}{2\,u\,(4\,u^2-3)} & \frac{2\,(u-1)\,(u+1)\,(2\,u-1)\,(2\,u+1)}{4\,u^2-3} & \frac{8\,u^4-6\,u^2-1}{2\,u\,(4\,u^2-3)} \\
\bar\gamma_{-9} & 0 & \frac{8\,u^4-6\,u^2-1}{4\,u^2\,(4\,u^2-3)} & \frac{1}{2\,u} & \frac{1}{4\,u^2\,(4\,u^2-3)} & 0 & \frac{16\,u^4-16\,u^2+1}{4\,u^2\,(4\,u^2-3)} & \frac{8\,u^4-6\,u^2-1}{2\,u\,(4\,u^2-3)} & -\frac{32\,u^6-64\,u^4+30\,u^2+1}{4\,u^2\,(4\,u^2-3)}
\end{array}$$

$$\begin{pmatrix}A & B\\ C & U\end{pmatrix}\xrightarrow{\det(A)}\begin{pmatrix}I & A^{-1}B\\ C & U\end{pmatrix}\xrightarrow{1}\begin{pmatrix}I & A^{-1}B\\ 0 & U-CA^{-1}B\end{pmatrix},$$

so $\det\begin{pmatrix}A & B\\ C & U\end{pmatrix}=\det(A)\det(U-CA^{-1}B)$.

Roughly, $\det(A)$ is "det on ker", $-CA^{-1}B$ is "a pushforward of $\begin{pmatrix}A & B\\ C & U\end{pmatrix}$". (what if $\nexists A^{-1}$?)

Questions. 1. Does this have a topological meaning? 2. Is there a version of the Kashaev Conjecture for tangles? 3. Find all solutions of R123 in our "algebra". 4. Braids and the Burau representation. 5. Recover the work in "Prior Art". 6. Are there any concordance properties? 7. What is the "SPQ group"? 8. The jumping points of signatures are the roots of the Alexander polynomial. Does this generalize to tangles? 9. Which of the three Cordon cases is the most common? 10. Are there interesting examples of tangles for which rels is non-trivial? 11. Is the *pq* part determined by Γ-calculus? 12. Is the *pq* part determined by finite type invariants? 13. Does it work with closed components / links? 14. Strand-doubling formulas? 15. A multivariable version? 16. Mutation invariance? 17. Ribbon knots? 18. Are there "face-virtual knots"? 19. Does the pushforward story extend to ranks? To formal Gaussian measures? To super Gaussian measures?

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Proof of Theorem 1'. Fix W and consider triples $(V, S, \phi\colon V \to W)$ where $S = (s, D, Q)$ is an SPQ on V. Declare $(V_1, S_1, \phi_1) \sim (V_2, S_2, \phi_2)$ if for every quadratic U on W,

$$\sigma_{V_1}(S_1 + \phi_1^* U) = \sigma_{V_2}(S_2 + \phi_2^* U).$$

Given our (V, S, ϕ), we need to show:
1. There is an SPQ S' on W such that $(V, S, \phi) \sim (W, S', I)$.
2. If $(W, S', I) \sim (W, S'', I)$ then $S' = S''$.

Property 2 is easy. Property 1 follows from the following four claims, each of which is easy.

Claim 1. $(V, S, \phi) \sim (D(S), S, \phi|_{D(S)})$, so wlog, S is "full", meaning $D(S) = V$. $\square$

Claim 2. If S is full, $v \in \ker\phi$, and $\lambda := Q(v) \neq 0$, then $(V, S, \phi) \sim$

$$\left(V/\langle v\rangle, \left(s + \operatorname{sign}(\lambda), V/\langle v\rangle, Q - \frac{Q(-,v)\otimes Q(v,-)}{|\lambda|^2}\right), \phi/\langle v\rangle\right).$$

So wlog $Q|_{\ker\phi} = 0$ (meaning, $Q|_{\ker\phi\otimes\ker\phi} = 0$). $\square$

Claim 3. If $Q|_{\ker\phi} = 0$ and $v \in \ker\phi$, let $V' = \ker Q(v, -)$ and then $(V, S, \phi) \sim (V', S|_{V'}, \phi|_{V'})$ so wlog $Q|_{V\otimes\ker\phi + \ker\phi\otimes V} = 0$. $\square$

Claim 4. If $Q|_{V\otimes\ker\phi+\ker\phi\otimes V} = 0$ then $S = \phi^* S'$ for some SPQ S' on $\operatorname{im}\phi$ and then $(V, S, \phi) \sim (W, S', I)$. $\square$ $\square$

Proof of Theorem 2. The functoriality of pullbacks needs no proof. Now assume $V_0 \xrightarrow{\alpha} V_1 \xrightarrow{\beta} V_2$ and that S is an SPQ on V_0. Then for every SPQ U on V_2 we have, using reciprocity three times, that $\sigma(\beta_*\alpha_* S + U) = \sigma(\alpha_* S + \beta^* U) = \sigma(S + \alpha^*\beta^* U) = \sigma(S + (\beta\alpha)^* U) = \sigma((\beta\alpha)_* S + U)$. Hence $\beta_*\alpha_* S = (\beta\alpha)_* S$. $\square$

Lemma 1. $\phi_*\phi^* S = S|_{\operatorname{im}\phi}$.

Proof. For every PQ U with $D(U) = \operatorname{im}\phi$ we have $\sigma(S|_{\operatorname{im}\phi} + U) = \sigma(S + U) = \sigma(\phi^*(S + U)) = \sigma(\phi^* S + \phi^* U) = \sigma(\phi_*\phi^* S + U)$ where for the second equality we use the fact that a pullback by a surjective map does not change the signature, and the last equality is the reciprocity property. $\square$

Lemma 2. Under the conditions of Theorem 3, if S_i is an SPQ on V_i for $i = 1, 2$, then $\sigma(\alpha^* S_1 + \gamma^* S_2) = \sigma(\beta_* S_1 + \delta_* S_2)$.

Froof. Let $\pi := \beta\alpha = \delta\gamma$. Then (in order) by reciprocity with $U = 0$, by the functoriality of pushforwards, by Lemma 1, and using the surjectivity of α and of γ, $\sigma(\alpha^* S_1 + \gamma^* S_2) = \sigma(\pi_*(\alpha^* S_1 + \gamma^* S_2)) \neq \sigma(\beta_*\alpha_*\alpha^* S_1 + \delta_*\gamma_*\gamma^* S_2) = \sigma(\beta_*(S_1|_{\operatorname{im}\alpha}) + \delta_*(S_2|_{\operatorname{im}\gamma})) = \sigma(\beta_* S_1 + \delta_* S_2)$ $\square$

Froof of Theorem 3. Given S on V_2, for every U on V_1 we have using reciprocity, Lemma 2, and reciprocity again, that $\sigma(\alpha_*\gamma^* S + U) = \sigma(\gamma^* S + \alpha^* U) = \sigma(\delta_* S + \beta_* U) = \sigma(\beta^*\delta_* S + U)$. Hence $\alpha_*\gamma^* S = \beta^*\delta_* S$. $\square$