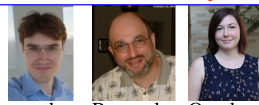
Cars, Interchanges, Traffic Counters, and a Pretty Darned Good Knot Invariant

Accompanies ωεβ/APAI

Abstract. Reporting on joint work with Roland van der Veen, I'll tell you some stories about ρ_1, an easy to define, strong, fast to compute, homomorphic, and well-connected knot invariant. ρ_1 was first studied by Rozansky and Overbay [Ro1, Ro2, Ro3, Ov], it has far-reaching generalizations, it is dominated by the coloured Jones polynomial, and I wish I understood it. **Common misconception.** "Dominated" ⇏ "lesser".



van der Veen Rozansky Overbay

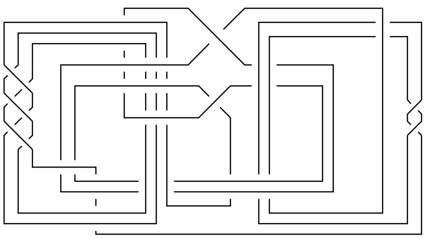
ωεβ/J

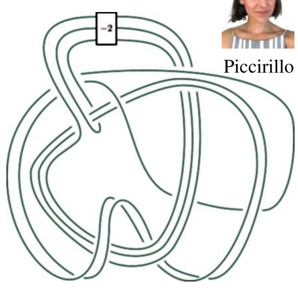
We seek strong, fast, homomorphic knot and tangle invariants.

Strong. Having a small "kernel".

Fast. Computable even for large knots (best: poly time).

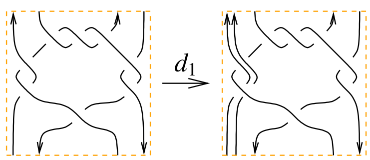
Piccirillo



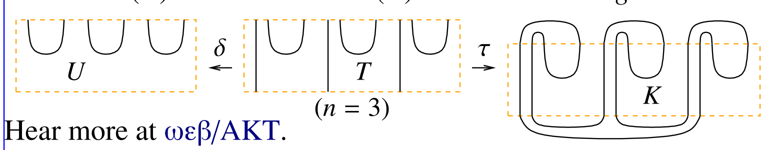


Gompf–Scharlemann–Thompson

Homomorphic. Extends to tangles and behaves under tangle operations; especially gluings and doublings:



Why care for "Homomorphic"? Theorem. A knot K is *ribbon* iff there exists a $2n$-component tangle T with skeleton as below such that $\tau(T) = K$ and where $\delta(T) = U$ is the *untangle*:



Hear more at ωεβ/AKT.

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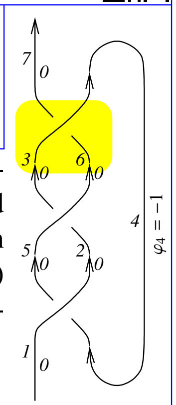
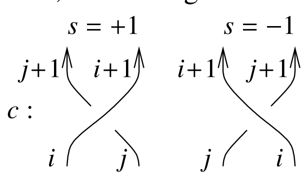
Jones:

Formulas stay;
interpretations change with time.



Formulas. Draw an n-crossing knot K as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \ldots, 2n+1\}$ and with rotation numbers φ_k. Let A be the $(2n+1)\times(2n+1)$ matrix constructed by starting with the identity matrix I, and adding a 2×2 block for each crossing:

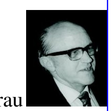


A	col $i+1$	col $j+1$
row i	$-T^s$	$T^s - 1$
row j	0	-1

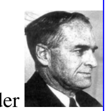
Let $G = (g_{\alpha\beta}) = A^{-1}$. For the trefoil example, it is:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

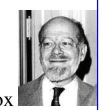
$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$



Burau

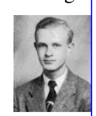


Alexander



Fox

Wirtinger



Blanchfield

"The Green Function"

Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A), \qquad \text{with } \varphi = \sum_k \varphi_k,\ w = \sum_c s.$$

Classical Topologists: This is boring. Yawn.

Formulas, continued. Finally, set

$$R_1(c) := s\left(g_{ji}\left(g_{j+1,j} + g_{j,j+1} - g_{ij}\right) - g_{ii}\left(g_{j,j+1} - 1\right) - 1/2\right)$$

$$\rho_1 := \Delta^2\left(\sum_c R_1(c) - \sum_k \varphi_k\left(g_{kk} - 1/2\right)\right).$$

In our example $\rho_1 = -T^2 + 2T - 2 + 2T^{-1} - T^{-2}$.

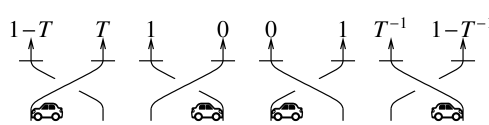
Theorem. ρ_1 is a knot invariant. Proof: later.

Classical Topologists: Whiskey Tango Foxtrot?

Cars, Interchanges, and Traffic Counters. Cars always drive forward. When a car crosses over a bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0^*$. See also [Jo, LTW].

Jones Lin Tian Wang

$p = 1 - T^s$



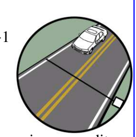


image credits: diamondtraffic.com

* In algebra $x \sim 0$ if for every y in the ideal generated by x, $1 - y$ is invertible.

Preliminaries

This is Rho1.nb of http://drorbn.net/j22/ap.

```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/j22/ap
  to compute rotation numbers.
```

The Program

```
R1[s_, i_, j_] :=
  s (g[j,i] (g[j+1,j] + g[j,j+1] - g[i,j]) - g[i,i] (g[j,j+1] - 1) - 1/2);
ρ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :>
     (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                        0      -1   ))];
   Δ = T^((-Total[φ]-Total[Cs[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ρ1 = Sum[R1 @@ Cs[[k]], {k,1,n}] - Sum[φ[[k]] (g[k,k] - 1/2), {k,1,2 n}];
   Factor@{Δ, Δ^2 ρ1 /. g[α_,β_] :> G[[α, β]]}];
```

The First Few Knots

```
Table[K → ρ[K], {K, AllKnots[{3, 6}]}]
```

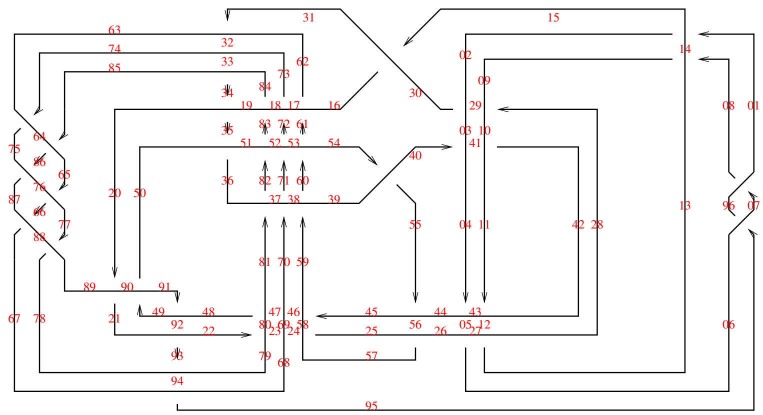
$$\left\{\text{Knot}[3,1]\to\left\{\frac{1-T+T^2}{T},\ \frac{(-1+T)^2\,(1+T^2)}{T^2}\right\},\ \text{Knot}[4,1]\to\left\{-\frac{1-3T+T^2}{T},\ 0\right\},\ \text{Knot}[5,1]\to\left\{\frac{1-T+T^2-T^3+T^4}{T^2},\ \frac{(-1+T)^2\,(1+T^2)\,(2+T^2+2T^4)}{T^4}\right\},\right.$$

$$\text{Knot}[5,2]\to\left\{\frac{2-3T+2T^2}{T},\ \frac{(-1+T)^2\,(5-4T+5T^2)}{T^2}\right\},\ \text{Knot}[6,1]\to\left\{-\frac{(-2+T)\,(-1+2T)}{T},\ \frac{(-1+T)^2\,(1-4T+T^2)}{T^2}\right\},$$

$$\text{Knot}[6,2]\to\left\{-\frac{1-3T+3T^2-3T^3+T^4}{T^2},\ \frac{(-1+T)^2\,(1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}\right\},\ \left.\text{Knot}[6,3]\to\left\{\frac{1-3T+5T^2-3T^3+T^4}{T^2},\ 0\right\}\right\}$$

$p = 1 - T^s$

Fast!



```
Timing@
```

$$\rho[\text{EPD}[X_{14,1}, \overline{X}_{2,29}, X_{3,40}, X_{43,4}, \overline{X}_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, \overline{X}_{9,28}, X_{10,41}, X_{42,11}, \overline{X}_{27,12}, X_{30,15}, \overline{X}_{16,61}, \overline{X}_{17,72}, \overline{X}_{18,83}, X_{19,34}, \overline{X}_{89,20}, \overline{X}_{21,92}, \overline{X}_{79,22}, \overline{X}_{68,23}, \overline{X}_{57,24}, \overline{X}_{25,56}, X_{62,31}, X_{73,32}, X_{84,33}, \overline{X}_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, \overline{X}_{39,54}, X_{44,55}, X_{58,45}, \overline{X}_{69,46}, \overline{X}_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, \overline{X}_{63,74}, \overline{X}_{64,85}, \overline{X}_{76,65}, \overline{X}_{87,66}, \overline{X}_{67,94}, \overline{X}_{75,86}, \overline{X}_{88,77}, \overline{X}_{78,93}]]$$

$$\left\{86.2031,\ \left\{-\frac{1}{T^8}\left(-1+2T-T^2-T^3+2T^4-T^5+T^8\right)\left(-1+T^3-2T^4+T^5+T^6-2T^7+T^8\right),\ \frac{1}{T^{16}}(-1+T)^2\left(5-18T+33T^2-32T^3+2T^4+42T^5-62T^6-8T^7+166T^8-242T^9+108T^{10}+132T^{11}-226T^{12}+148T^{13}-11T^{14}-36T^{15}-11T^{16}+148T^{17}-226T^{18}+132T^{19}+108T^{20}-242T^{21}+166T^{22}-8T^{23}-62T^{24}+42T^{25}+2T^{26}-32T^{27}+33T^{28}-18T^{29}+5T^{30}\right)\right\}\right\}$$

Strong!

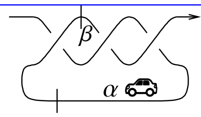
```
{NumberOfKnots[{3, 12}],
 Length@
  Union@Table[ρ[K], {K, AllKnots[{3, 12}]}],
 Length@
  Union@Table[{HOMFLYPT[K], Kh[K]},
    {K, AllKnots[{3, 12}]}]}
```

```
{2977, 2882, 2785}
```

So the pair (Δ, ρ_1) attains 2,882 distinct values on the 2,977 prime knots with up to 12 crossings (a deficit of 95), whereas the pair (HOMFLYPT, Khovanov Homology) attains only 2,785 distinct values on the same knots (a deficit of 192).

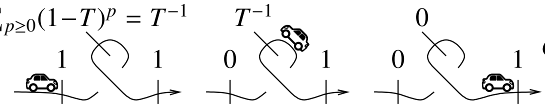
Hoste Ocneanu Millett Freyd Lickorish Yetter Przytycki Traczyk Khovanov

Theorem. The Green function $g_{\alpha\beta}$ is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point).



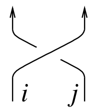
Example.

$\sum_{p\geq 0}(1-T)^p = T^{-1}$



$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof. Near a crossing c with sign s, incoming upper edge i and incoming lower edge j, both sides satisfy the *g-rules*:



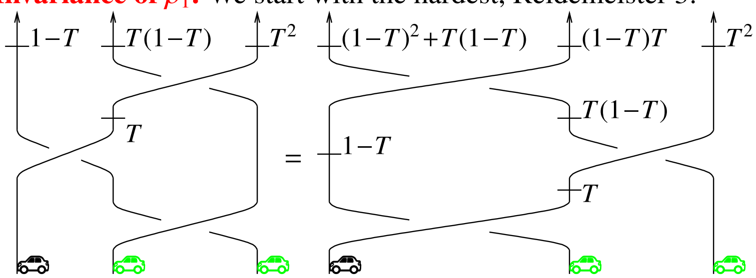
$$g_{i\beta} = \delta_{i\beta} + T^s g_{i+1,\beta} + (1-T^s)g_{j+1,\beta}, \qquad g_{j\beta} = \delta_{j\beta} + g_{j+1,\beta},$$

and always, $g_{\alpha,2n+1} = 1$: use common sense and $AG = I$ $(= GA)$.

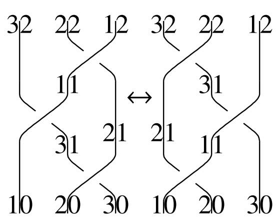
Bonus. Near c, both sides satisfy the further *g-rules*:

$$g_{\alpha i} = T^{-s}(g_{\alpha,i+1} - \delta_{\alpha,i+1}), \quad g_{\alpha j} = g_{\alpha,j+1} - (1-T^s)g_{\alpha i} - \delta_{\alpha,j+1}.$$

Invariance of ρ_1. We start with the hardest, Reidemeister 3:



⇒ Overall traffic patterns are unaffected by Reid3!
⇒ Green's $g_{\alpha\beta}$ is unchanged by Reid3, provided the cars injection site α and the traffic counters β are away.
⇒ Only the contribution from the R_1 terms within the Reid3 move matters, and using g-rules the relevant $g_{\alpha\beta}$'s can be pushed outside of the Reid3 area:

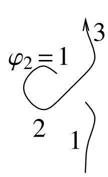



```
δ_{i_,j_} := If[i == j, 1, 0];
gRules_{s_,i_,j_} :=
  {g_{iβ_} :> δ_{iβ} + T^s g_{i+1,β} + (1 - T^s) g_{j+1,β},
   g_{jβ_} :> δ_{jβ} + g_{j+1,β}, g_{α_,i} :> T^-s (g_{α,i+1} - δ_{α,i+1}),
   g_{α_j} :> g_{α,j+1} - (1 - T^s) g_{αi} - δ_{α,j+1}}
lhs = R1[1, 20, 30] + R1[1, 10, 31] + R1[1, 11, 21] //.
    gRules_{1,20,30} ∪ gRules_{1,10,31} ∪ gRules_{1,11,21};
rhs = R1[1, 10, 20] + R1[1, 11, 30] + R1[1, 21, 31] //.
    gRules_{1,10,20} ∪ gRules_{1,11,30} ∪ gRules_{1,21,31};
Simplify[lhs == rhs]
True
```

Next comes Reid1, where we use results from an earlier example:

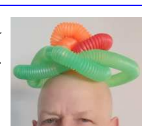
```
R1[1, 2, 1] - 1 (g22 - 1/2) /. g_{α_,β_} :> {{1, T^-1, 1}, {0, T^-1, 1}, {0, 0, 1}}[[α, β]]
```

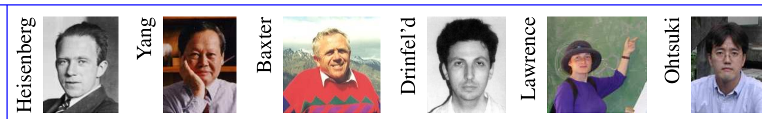
$\frac{1}{T^2} - \frac{1}{T} - \frac{-1+\frac{1}{T}}{T}$ = 0



Invariance under the other moves is proven similarly.

Wearing my Topology hat the formula for R_1, and even the idea to look for R_1, remain a complete mystery to me.





Heisenberg Yang Baxter Drinfel'd Lawrence Ohtsuki

Wearing my Quantum Algebra hat, I spy a Heisenberg algebra $\mathbb{H} = A\langle p, x\rangle/([p,x] = 1)$:

$$\text{cars} \leftrightarrow p \qquad \text{traffic counters} \leftrightarrow x$$

Where did it come from? Consider $\mathfrak{g}_\epsilon := sl_{2+}^\epsilon := L\langle y, b, a, x\rangle$ with relations

$$[b,x] = \epsilon x, \quad [b,y] = -\epsilon y, \quad [b,a] = 0,$$
$$[a,x] = x, \quad [a,y] = -y, \quad [x,y] = b + \epsilon a.$$

At invertible ϵ, it is isomorphic to sl_2 plus a central factor, and it can be quantized à la Drinfel'd [Dr] much like sl_2 to get an algebra $QU = A\langle y, b, a, x\rangle$ subject to (with $q = e^{\hbar\epsilon}$):

$$[b,a] = 0, \quad [b,x] = \epsilon x, \quad [b,y] = -\epsilon y,$$
$$[a,x] = x, \quad [a,y] = -y, \quad xy - qyx = \frac{1 - e^{-\hbar(b+\epsilon a)}}{\hbar}.$$

Now QU has an R-matrix solving Yang-Baxter (meaning Reid3),

$$R = \sum_{m,n\geq 0} \frac{y^n b^m \otimes (\hbar a)^m (\hbar x)^n}{m![n]_q!}, \qquad ([n]_q! \text{ is a "quantum factorial"})$$

and so it has an associated "universal quantum invariant" à la Lawrence and Ohtsuki [La, Oh], $Z_\epsilon(K) \in QU$.
Now $QU \cong \mathcal{U}(\mathfrak{g}_\epsilon)$ (only as algebras!) and $\mathcal{U}(\mathfrak{g}_\epsilon)$ represents into $\mathbb{H}$ via

$$y \to -tp - \epsilon \cdot xp^2, \quad b \to t + \epsilon \cdot xp, \quad a \to xp, \quad x \to x,$$

(abstractly, $\mathfrak{g}_\epsilon$ acts on its Verma module

$$\mathcal{U}(\mathfrak{g}_\epsilon)/(\mathcal{U}(\mathfrak{g}_\epsilon)\langle y, a, b - \epsilon a - t\rangle) \cong \mathbb{Q}[x]$$

by differential operators, namely via $\mathbb{H}$), so R can be pushed to $\mathcal{R} \in \mathbb{H} \otimes \mathbb{H}$.
Everything still makes sense at $\epsilon = 0$ and can be expanded near $\epsilon = 0$ resulting with $\mathcal{R} = \mathcal{R}_0(1 + \epsilon\mathcal{R}_1 + \cdots)$, with $\mathcal{R}_0 = e^{t(xp\otimes 1 - x\otimes p)}$ and $\mathcal{R}_1$ a quartic polynomial in p and x. So p's and x's get created along K and need to be pushed around to a standard location ("normal ordering"). This is done using

$$(p \otimes 1)\mathcal{R}_0 = \mathcal{R}_0(T(p\otimes 1) + (1-T)(1 \otimes p)),$$
$$(1 \otimes p)\mathcal{R}_0 = \mathcal{R}_0(1 \otimes p),$$

and when the dust settles, we get our formulas for ρ_1. But QU is a quasi-triangular Hopf algebra, and hence **ρ_1 is homomorphic**. Read more at [BV1, BV2] and hear more at ωεβ/SolvApp, ωεβ/Dogma, ωεβ/DoPeGDO, ωεβ/FDA, ωεβ/AQDW.
Also, we can (and know how to) look at higher powers of ϵ and we can (and more or less know how to) replace sl_2 by arbitrary semi-simple Lie algebra (e.g., [Sch]). So **ρ_1 is not alone!**

Schaveling

These constructions are very similar to Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] and hence to the "loop expansion" of the Kontsevich integral and the coloured Jones polynomial.

If this all reads like **insanity** to you, it should (and you haven't seen half of it). Simple things should have simple explanations. Hence,

Homework. Explain ρ_1 with no reference to quantum voodoo and find it a topology home (large enough to house generalizations!). Make explicit the homomorphic properties of ρ_1. Use them to do topology!

The Most Important Missing Infrastructure Project in Knot Theory

January-23-12
10:12 AM

An "infrastructure project" is hard (and sometimes non-glorious) work that's done now and pays off later.

An example, and the most important one within knot theory, is the tabulation of knots up to 10 crossings. I think it precedes Rolfsen, yet the result is often called "the Rolfsen Table of Knots", as it is famously printed as an appendix to the famous book by Rolfsen. There is no doubt the production of the Rolfsen table was hard and non-glorious. Yet its impact was and is tremendous. Every new thought in knot theory is tested against the Rolfsen table, and it is hard to find a paper in knot theory that doesn't refer to the Rolfsen table in one way or another.

A second example is the Hoste-Thistlethwaite tabulation of knots with up to 17 crossings. Perhaps more fun to do as the real hard work was delegated to a machine, yet hard it certainly was: a proof is in the fact that nobody so far had tried to replicate their work, not even to a smaller crossing number. Yet again, it is hard to overestimate the value of that project: in many ways the Rolfsen table is "not yet generic", and many phenomena that appear to be rare when looking at the Rolfsen table become the rule when the view is expanded. Likewise, other phenomena only appear for the first time when looking at higher crossing numbers.

But as I like to say, knots are the wrong object to study in knot theory. Let me quote (with some variation) my own (with Dancso) "[WKO]" paper:

> Studying knots on their own is the parallel of studying cakes and pastries as they come out of the bakery - we sure want to make them our own, but the theory of desserts is more about the ingredients and how they are put together than about the end products. In algebraic knot theory this reflects through the fact that knots are not finitely generated in any sense (hence they must be made of some more basic ingredients), and through the fact that there are very few operations defined on knots (connected sums and satellite operations being the main exceptions), and thus most interesting properties of knots are transcendental, or non-algebraic, when viewed from within the algebra of knots and operations on knots (see [AKT-CFA]).
>
> The right objects for study in knot theory are thus the ingredients that make up knots and that permit a richer algebraic structure. These are braids (which are already well-studied and tabulated) and even more so tangles and tangled graphs.

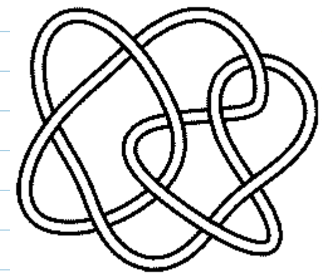
Thus in my mind the most important missing infrastructure project in knot theory is the tabulation of tangles to as high a crossing number as practical. This will enable a great amount of testing and experimentation for which the grounds are now still missing. The existence of such a tabulation will greatly impact the direction of knot theory, as many tangle theories and issues that are now ignored for the lack of scope, will suddenly become alive and relevant. The overall influence of such a tabulation, if done right, will be comparable to the influence of the Rolfsen table.

Aside. What are tangles? Are they embedded in a disk? A ball? Do they have an "up side" and a "down side"? Are the strands oriented? Do we mod out by some symmetries or figure out the action of some symmetries? Shouldn't we also calculate the affect of various tangle operations (strand doubling and deletion, juxtapositions, etc.)? Shouldn't we also enumerate virtual tangles? w-tangles? Tangled graphs?

In my mind it would be better to leave these questions to the tabulator. Anything is better than nothing, yet good tabulators would try to tabulate the more general things from which the more special ones can be sieved relatively easily, and would see that their programs already contain all that would be easy to implement within their frameworks. Counting legs is easy and can be left to the end user. Determining symmetries is better done along with the enumeration itself, and so it should.

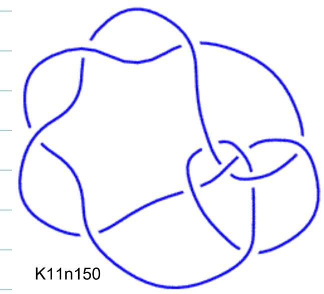
An even better tabulation should come with a modern front-end - a set of programs for basic manipulations of tangles, and a web-based "tangle atlas" for an even easier access.

Overall this would be a major project, well worthy of your time.



(KnotPlot image)

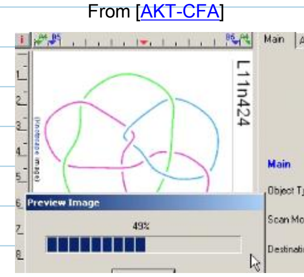
9_42 is Alexander Stoimenow's favourite



(Knotscape image)

The interchange of I-95 and I-695, northeast of Baltimore. (more)

From [AKT-CFA]



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