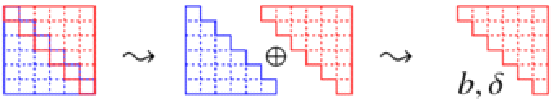
Possible Clips

May 20, 2019 10:41 AM

From Ohio-1901:

Solvable Approximation. In gl_n, half is enough! Indeed $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b\colon ◹ \otimes ◹ \to ◹$
$b(◺) \rightsquigarrow \delta\colon ◹ \to ◹ \otimes ◹$

Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon ◺$, and $[◹, ◺] = ◺ + \epsilon ◹$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1} = 0$ always yields a solvable Lie algebra.

***CU* and *QU*.** Starting from sl_2, get $CU_\epsilon = \langle y, a, x, t\rangle / ([t, -] = 0, [a, y] = -y, [a, x] = x, [x, y] = 2\epsilon a - t)$. Quantize using standard tools (I'm sorry) and get $QU_\epsilon = \langle y, a, x, t\rangle / ([t, -] = 0, [a, y] = -y, [a, x] = x, xy - \mathbb{e}^{\hbar\epsilon} yx = (1 - T\mathbb{e}^{-2\hbar\epsilon a})/\hbar)$.

5. The "Weyl form of the canonical commutation relations" states that if $[y, x] = tI$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. So with

$SW_{xy} \circlearrowright \mathcal{S}(y, x) \overset{\mathbb{O}_{xy}}{\underset{\mathbb{O}_{yx}}{\rightrightarrows}} \mathcal{U}(y, x)$ we have $\widetilde{SW}_{xy} = \mathbb{e}^{\eta y + \xi x - \eta\xi t}$.

The Real Thing. In the algebra QU_ϵ, over $\mathbb{Q}[\![\hbar]\!]$ using the $yaxt$ order, $T = \mathbb{e}^{\hbar t}$, $\bar{T} = T^{-1}$, $\mathcal{A} = \mathbb{e}^{\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

$$\tilde{R}_{ij} = \mathbb{e}^{\hbar(y_i x_j - t_i a_j)} \left(1 + \epsilon\hbar\left(a_i a_j - \hbar^2 y_i^2 x_j^2/4\right) + O(\epsilon^2)\right)$$

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$$\tilde{m} = \mathbb{e}^{(\alpha_1+\alpha_2)a + \eta_2\xi_1(1-T)/\hbar + (\xi_1\bar{\mathcal{A}}_2 + \xi_2)x + (\eta_1 + \eta_2\bar{\mathcal{A}}_1)y} \left(1 + \epsilon\lambda + O(\epsilon^2)\right),$$

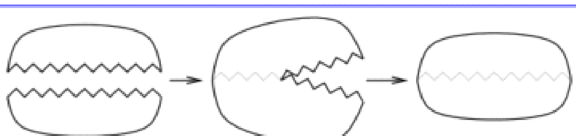
where $\lambda = 2a\eta_2\xi_1 T + \eta_2^2\xi_1^2\left(3T^2 - 4T + 1\right)/4\hbar - \eta_2\xi_1^2(3T - 1)x\bar{\mathcal{A}}_2/2$
$-\eta_2^2\xi_1(3T - 1)y\bar{\mathcal{A}}_1/2 + \eta_2\xi_1 xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$.

Finally,

$$\tilde{\Delta} = \mathbb{e}^{\tau(t_1+t_1) + \eta(y_1 + T_1 y_2) + \alpha(a_1 + a_2) + \xi(x_1 + x_2)} \left(1 + O(\epsilon)\right) \in \mathcal{S}(B^*, B_1, B_2),$$

and $\tilde{S} = \mathbb{e}^{-\tau t - \alpha a - \eta\xi(1-\bar{T})\mathcal{A}/\hbar - \bar{T}\eta y\mathcal{A} - \xi x\mathcal{A}} \left(1 + O(\epsilon)\right) \in \mathcal{S}(B^*, B)$.

The Zipping Issue. (between unbound and bound lies half-zipped).



Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set $\left\langle P(\zeta^j, z_i)\right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i=0}$. (E.g., if $P = \sum a_{nm}\zeta^n z^m$ then $\langle P\rangle_\zeta = \sum a_{nm}\, \partial_z^n z^m\Big|_{z=0} = \sum n! a_{nn}$).

The Zipping / Contraction Theorem. If $P = P(\zeta^j, z_i)$ has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P \mathbb{e}^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde{q})\mathbb{e}^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P \,\middle|\, \begin{matrix}\zeta^j\to\zeta^j+\eta^i\tilde{q}^j_i \\ z_i\to\tilde{q}^k_i(z_k+y_k)\end{matrix}\right\rangle_{(\zeta^j)}$$

where $\tilde{q}$ is the inverse matrix of $1-q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$.

The 2D Lie Algebra. One may show* that if $[a,x]=\gamma x$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$. Ergo with

$$SW_{ax} \circlearrowright \mathcal{S}(a,x) \underset{\mathbb{O}_{xa}}{\overset{\mathbb{O}_{ax}}{\rightrightarrows}} \mathcal{U}(a,x)$$

we have $\widetilde{SW}_{ax} = \mathbb{e}^{\alpha a+\mathbb{e}^{-\gamma\alpha}\xi x}$.

$*$ Indeed $xa = (a-\gamma)x$ thus $xa^n = (a-\gamma)^n x$ thus $x\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha(a-\gamma)}x = \mathbb{e}^{-\gamma\alpha}\mathbb{e}^{\alpha a}x$ thus $x^n\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}(\mathbb{e}^{-\gamma\alpha})^n x^n$ thus $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$.

Faddeev's Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n-1}{q-1}$, with $[n]_q! := [1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}^x_q := \sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have

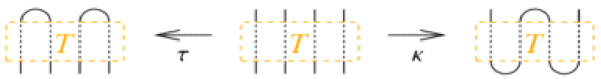
$$\log \mathbb{e}^x_q = \sum_{k\geq 1}\frac{(1-q)^k x^k}{k(1-q^k)} = x + \frac{(1-q)^2x^2}{2(1-q^2)} + \dots.$$

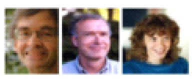
Proof. We have that $\mathbb{e}^x_q = \frac{\mathbb{e}^{qx}_q - \mathbb{e}^x_q}{qx-x}$ ("the q-derivative of $\mathbb{e}^x_q$ is itself"), and hence $\mathbb{e}^{qx}_q = (1+(1-q)x)\mathbb{e}^x_q$, and

$$\log \mathbb{e}^{qx}_q = \log(1+(1-q)x) + \log \mathbb{e}^x_q.$$

Writing $\log \mathbb{e}^x_q = \sum_{k\geq 1} a_k x^k$ and comparing powers of x, we get $q^k a_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

From Matemale-1804:



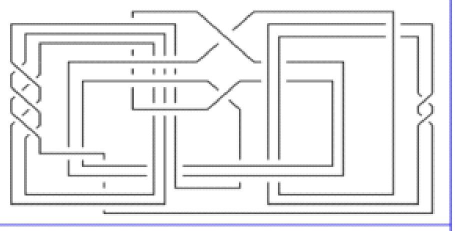


Gompf, Scharlemann, Thompson [GST]

$U \in \mathcal{T}_n$ $\quad 1 \in \mathcal{A}_n$

$\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$ with $\mathcal{R} := \kappa(\tau^{-1}(1))$

ribbon $K \in \mathcal{T}_1$ $\quad z(K) \in \mathcal{R} \subseteq \mathcal{A}_1$



Faster is better, leaner is meaner!

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Faddeev Quesne Zagier

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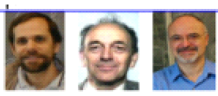
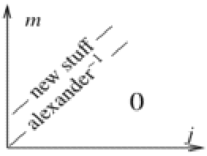
Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing


Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar) = 1$.



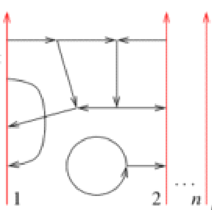
"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

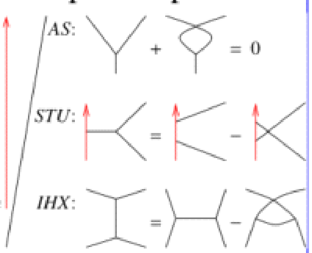
$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty \frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

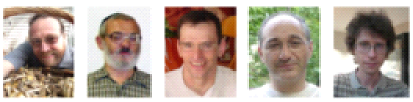
From MIT-1602:

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z\colon \begin{Bmatrix} S\text{-component} \\ (v/b\text{-})\text{tangles}\end{Bmatrix} \to \mathcal{A}_S^v :=$$






Etingof Kazhdan Haviv Enriquez Ševera