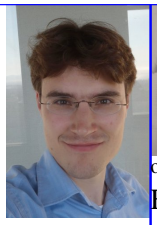
Car Traffic on Knot Diagrams and Some Cool Knot Invariants

Abstract. We will study some strange traffic rules for cars driving through an interchange: When they enter via an underpass, they just go through. But when they enter via an overpass, they fall down to the underpass with some small probability p, and then keep going unharmed, down under.

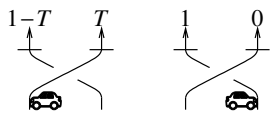
We will learn that to analyze this traffic game we need matrices and that to play it better we need probabilities that are not numbers between 0 and 1. We will also learn how this game can be used to define some knot invariants (a notion we will explain) which may be the best we presently have.



Joint with R. van der Veen

Acknowledgement. Work supported by NSERC grant RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

Model T Traffic Rules. Cars always drive forward. When a car crosses over a bridge it goes through with probability $T \sim 1$, but falls off with probability $p = 1 - T \sim 0$. On various edges *traffic counters* are placed.



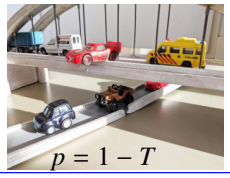
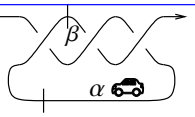


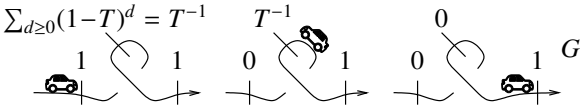
image credits: diamondtraffic.com

Definition. The *traffic function* $G = (g_{\alpha\beta})$ is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point).



(knots must be cut open!)

Example 1.

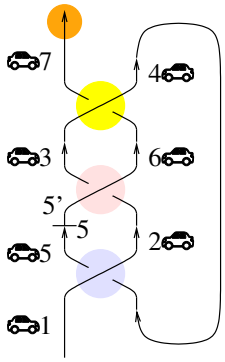


$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Note. $\sum_{d\ge0} p^d = \frac{1}{1-p}$ so $\sum_{d\ge0}(1-T)^d = \frac{1}{1-(1-T)} = T^{-1}$.

What about the Trefoil? We get:

$$\begin{aligned} g_{75} &= 0 \\ g_{65} - g_{75} &= 0 \\ g_{35} - Tg_{45} - (1-T)g_{75} &= 0 \\ g_{25} - g_{35} &= 0 \\ g_{55} - Tg_{65} - (1-T)g_{35} &= 1 \\ g_{45} - g_{55} &= 0 \\ g_{15} - T2_{65} - (1-T)g_{55} &= 0 \end{aligned}$$



In other words,

$$\begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g_{15} \\ g_{25} \\ g_{35} \\ g_{45} \\ g_{55} \\ g_{65} \\ g_{75} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

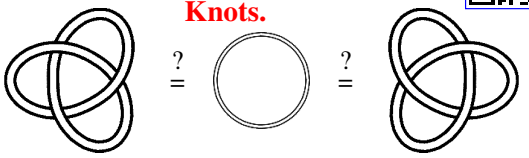
So with A the 7×7 matrix above, $g_{\alpha\beta} = (A^{-1}e_\alpha)_\beta = (A^{-1})_{\alpha\beta} =: G_{\alpha\beta}$ with

$$G = A^{-1} = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

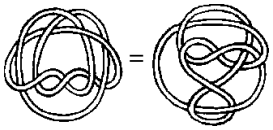
Computers don't care!

on my kitchen counter

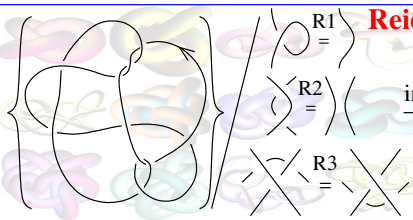
Knots.



For many years the Perko pair (on the right) were thought to be different, until in 1973 an undergrad, K. Perko, had shown them the same. Seeing that eyeballing isn't good enough, how can we tell with certainty if two diagrams represent the same, or different, knots?



Reidemeister moves, invariants.

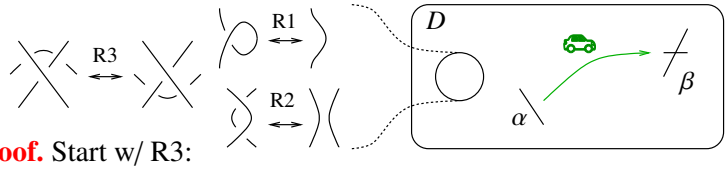


invariants →

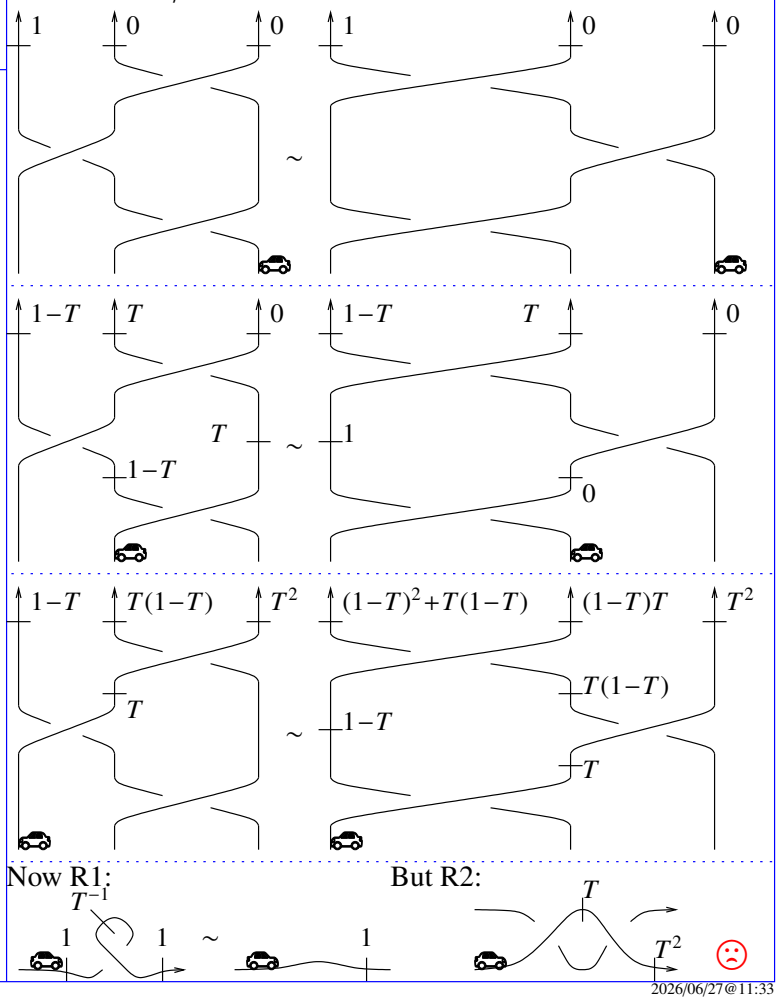
Something simple: numbers, polynomials, matrices, etc.

Tell knots apart? Alternating? Bound a genus 7 surface? Complement is fibered over S^1? Complement is hyperbolic? Bounds a disk with only ribbon singularities? Bounds a topological / smooth non-singular disk in B^4? ...

Fheorem. The traffic function $g_{\alpha\beta}$ is a "relative invariant":



Froof. Start w/ R3:

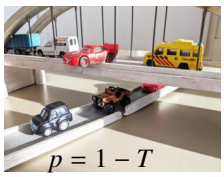


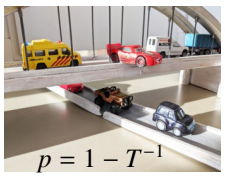
Now R1:

But R2:

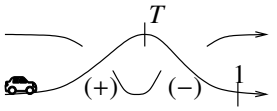
☹

Resolution. There are two types of crossings! In a "positive" crossing ⤫, cars on the underpass see the cars on the overpass going from left to right (Canada!). In a "negative" one ⤫, they go right to left (England!). In $(-)$ crossings, use T^{-1} instead of T:





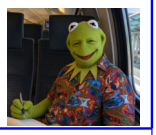
Problem Solved!



Small Print. Other lane orientations must also be checked. They work.

Waldorf. But T^{-1}? Probabilities bigger than 1?? But $1 - T^{-1}$? Negative probabilities??? That frog ain't a mathematician!

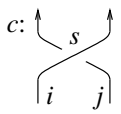
Statler. The show's so bad, let it go on. That frog must be an algebraist.



Me. Now the Fheorem is a Theorem!

Theorem 2. With $\Delta = T^{\text{easy}} \det(A)$ and

$$R_1(c) = s\Big[g_{ji}\left(g_{j+1,j} + g_{j,j+1} - g_{ij}\right) - g_{ii}\left(g_{j,j+1} - 1\right) - 1/2\Big],$$



Δ and $\rho_1 := \Delta^2\left(\sum_c R_1(c) + \begin{array}{c}\text{a minor}\\ \text{correction}\end{array}\right)$ are knot invariants.

Proof. An easy computation, but not today.

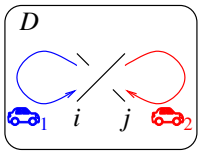
Theorem 3. Let T_1 and T_2 be variables and let $T_3 = T_1T_2$. Let $g_{\nu\alpha\beta} = g_{\alpha\beta}(T \to T_\nu)$ for $\nu = 1, 2, 3$ and let $\Delta_\nu = \Delta(T \to T_\nu)$. Let

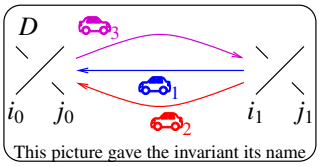
$$F_1(c) = s\Big[1/2 - g_{3ii} + T_2^s g_{1ii}g_{2ji} - T_2^s g_{3jj}g_{2ji} - (T_2^s - 1)g_{3ii}g_{2ji} + (T_3^s - 1)g_{2ji}g_{3ji} - g_{1ii}g_{2jj} + 2g_{3ii}g_{2jj} + g_{1ii}g_{3jj} - g_{2ii}g_{3jj}\Big] + \frac{s}{T_2^s - 1}\Big[(T_1^s - 1)T_2^s\left(g_{3jj}g_{1ji} - g_{2jj}g_{1ji} + T_2^s g_{1ji}g_{2ji}\right) + (T_3^s - 1)g_{3ji}\left(1 - T_2^s g_{1ii} + g_{2ij} + (T_2^s - 2)g_{2jj} - (T_1^s - 1)(T_2^s + 1)g_{1ji}\right)\Big],$$

$$F_2(c_0, c_1) = \frac{s_1(T_1^{s_0} - 1)(T_3^{s_1} - 1)g_{1j_1i_0}g_{3j_0i_1}}{T_2^{s_1} - 1} \cdot \left(T_2^{s_0}g_{2i_1i_0} + g_{2j_1j_0} - T_2^{s_0}g_{2j_1i_0} - g_{2i_1j_0}\right)$$

Computers don't care!

$$\theta(D) := \underbrace{\Delta_1\Delta_2\Delta_3}_{\begin{array}{c}\text{clears all}\\ \text{denominators}\end{array}}\left(\sum_c F_1(c) + \sum_{c_0,c_1} F_2(c_0,c_1) + \sum_k F_3(\varphi_k, k)\right),$$





This picture gave the invariant its name

a minor correction

Then $\Theta = (\Delta, \theta)$ is a knot invariant.

Proof. An easy computation, but not today.

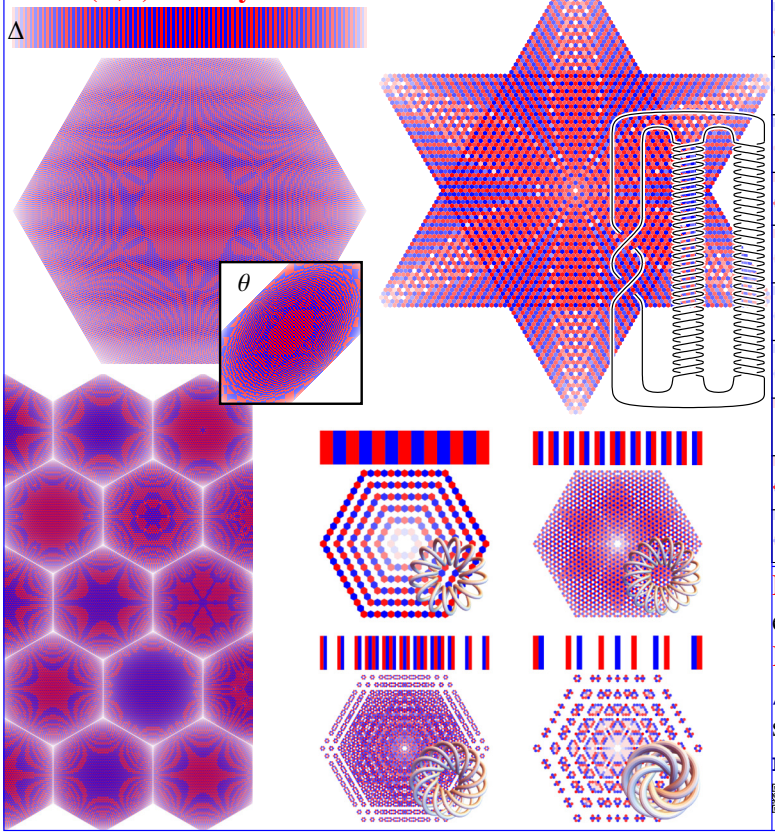
Fast! A key to the Jones polynomial is Kauffman's relation ⤫ → A)(+ A^{-1}⌒⌒. The Jones poly is much simpler but much slower!

Strong. Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis):

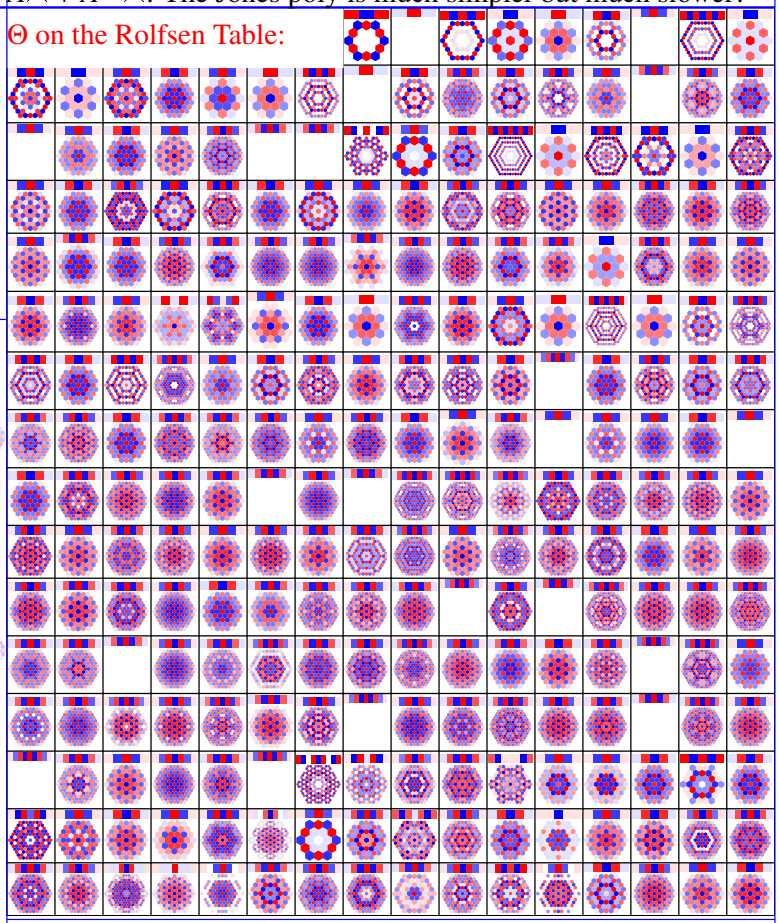
(Vol is approximate, $H \supset \Delta, J$)

	knots	$(H, Kh, Vol, \sigma_{LT})$	$\Theta = (\Delta, \theta)$	together
xing ≤ 10	249	249 (0)	249 (0)	249 (0)
xing ≤ 11	801	787 (14)	798 (3)	798 (3)
xing ≤ 12	2,977	(84)	(19)	(10)
xing ≤ 13	12,965	(911)	(194)	(169)
xing ≤ 14	59,937	(5,917)	(1,118)	(972)
xing ≤ 15	313,230	(41,434)	(6,758)	(6,304)

A $\Theta = (\Delta, \theta)$ Gallery.



Θ on the Rolfsen Table:



Morals. • Matrices are useful! • There's some probability theory even outside of $[0, 1]$! • There's a lot we still don't know!

Read More? • An article in Quanta Magazine: http://drorbn.net/cumc26/quanta. • Videos and handouts from several other talks I gave on and around θ: http://drorbn.net/talks. • Our original paper, with context, proofs and code: http://drorbn.net/cumc26/theta.