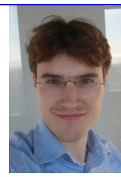
Car Traffic on Knot Diagrams and Some Cool Knot Invariants

Abstract. We will study some strange traffic rules for cars driving through an interchange: When they enter via an underpass, they just go through. But when they enter via an overpass, they fall down to the underpass with some small probability p, and then keep going unharmed, down under.
We will learn that to analyze this traffic game we need matrices and that to play it better we need probabilities that are not numbers between 0 and 1. We will also learn how this game can be used to define some knot invariants (a notion we will explain) which may be the best we presently have.
Joint with Roland van der Veen.


van der Veen

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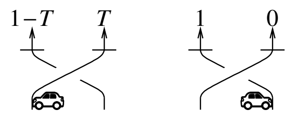
Model T Traffic Rules. Cars always drive forward. When a car crosses over a bridge it goes through with probability $T\sim 1$, but falls off with probability $1-T\sim 0$. On various edges *traffic counters* are placed.




$p=1-T^s$

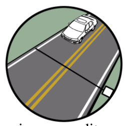
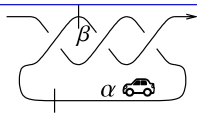
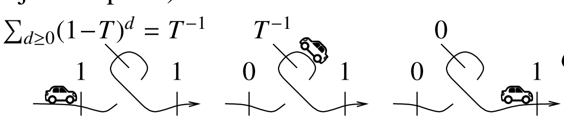


image credits: diamondtraffic.com

Definition. The *traffic function* $G=(g_{\alpha\beta})$ is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha=\beta$, the counter is *after* the injection point).



Example 1.

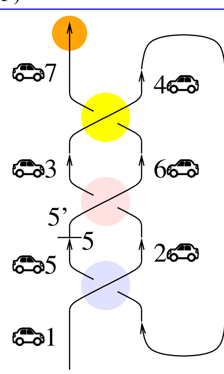


$$G=\begin{pmatrix}1 & T^{-1} & 1\\ 0 & T^{-1} & 1\\ 0 & 0 & 1\end{pmatrix}$$

Note. $\sum_{d\geq 0}p^d=\frac{1}{1-p}$ so $\sum_{d\geq 0}(1-T)^d=\frac{1}{1-(1-T)}=T^{-1}$.

What about the Trefoil? We get:

$$\begin{aligned}
g_{75} &= 0\\
g_{65}-g_{75} &= 0\\
g_{35}-Tg_{45}-(1-T)g_{75} &= 0\\
g_{25}-g_{35} &= 0\\
g_{55}-Tg_{65}-(1-T)g_{35} &= 1\\
g_{45}-g_{55} &= 0\\
g_{15}-T2_{65}-(1-T)g_{55} &= 0
\end{aligned}$$



In other words,

$$\begin{pmatrix}
1 & -T & 0 & 0 & T-1 & 0 & 0\\
0 & 1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -T & 0 & 0 & T-1\\
0 & 0 & 0 & 1 & -1 & 0 & 0\\
0 & 0 & T-1 & 0 & 1 & -T & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} g_{15}\\ g_{25}\\ g_{35}\\ g_{45}\\ g_{55}\\ g_{65}\\ g_{75}\end{pmatrix}
=\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix}.$$

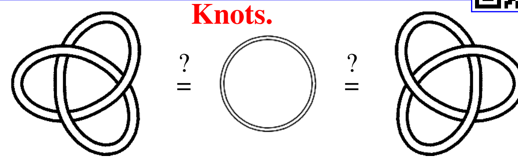
So with A the 7×7 matrix above, $g_{\alpha\beta}=(A^{-1})_{\alpha\beta}=:G_{\alpha\beta}$ with

$$G=\begin{pmatrix}
1 & T & 1 & T & 1 & T & 1\\
0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1\\
0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1\\
0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1\\
0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

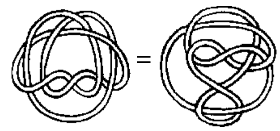
Computers don't care!

Knots.

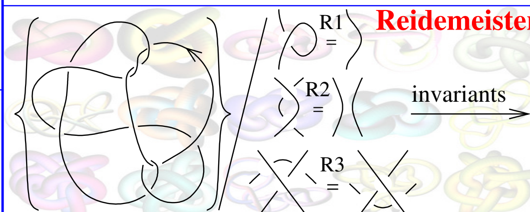

on my kitchen counter



For many years, the Perko pair (on the right) were thought to be different, until an undergrad, K. Perko, had shown them the same, in 1973. Seeing that eyeballing isn't good enough, how can we tell with certainty if two diagrams represent the same, or different, knots?



Reidemeister moves, invariants.



Tell knots apart? Alternating? Bound a genus 7 surface? Complement is fibered over S^1? Complement is hyperbolic? Bounds a disk with only ribbon singularities? Bounds a topological / smooth non-singular disk in B^4? ...

More? Start from http://drorbn.net/talks.

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