In[∘]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< IType.m];
```

$T_3 = T_1 T_2;$

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.

Syntax::sntx: Invalid syntax in or before "\*SubscriptBox[\(∂\), \(vs[[\(a\)\(]\)],
                                            vs[[\(b\)\(]\)]\)]L0\))/.Thread[vs->0]/.Subscript[(p|x),
                                            __]->0,{a,n},{b,n}];

                                            ^
    ". (line 34 of "IType.m")
```

exec

In[∘]:=

```
nb2tex$PDFWidth *= 1.25;
```

The Programs

tex

{\red\bf A faster program,} in which the Feynman diagrams are ``pre-computed'' (see theta.nb at \web{ap}):

pdf

In[∘]:=

$$F_1[\{s\_, i\_, j\_\}] := CF\Big[s \Big(1/2 - g_{3ii} + T_2^s g_{1ii} g_{2ji} - g_{1ii} g_{2jj} - (T_2^s - 1) g_{2ji} g_{3ii} + 2 g_{2jj} g_{3ii} - (1 - T_3^s) g_{2ji} g_{3ji} - g_{2ii} g_{3jj} - T_2^s g_{2ji} g_{3jj} + g_{1ii} g_{3jj} + \Big((T_1^s - 1) g_{1ji} (T_2^{2s} g_{2ji} - T_2^s g_{2jj} + T_2^s g_{3jj}) + (T_3^s - 1) g_{3ji} \big(1 - T_2^s g_{1ii} - (T_1^s - 1)(T_2^s + 1) g_{1ji} + (T_2^s - 2) g_{2jj} + g_{2ij}\big)\Big) \Big/ (T_2^s - 1)\Big)\Big]$$

pdf

In[∘]:=

$$F_2[\{s0\_, i0\_, j0\_\}, \{s1\_, i1\_, j1\_\}] := CF\Big[s1 (T_1^{s0} - 1)(T_2^{s1} - 1)^{-1} (T_3^{s1} - 1) g_{1,j1,i0} g_{3,j0,i1} \big((T_2^{s0} g_{2,i1,i0} - g_{2,i1,j0}) - (T_2^{s0} g_{2,j1,i0} - g_{2,j1,j0})\big)\Big]$$

pdf

In[∘]:=

$$F_3[\varphi\_, k\_] = \varphi\, g_{3kk} - \varphi / 2;$$

tex

We call the invariant computed θ:

pdf

In[∘]:=
```
Θ[K_] := Θ[K] = Module[{X, φ, n, A, Δ, G, ev, θ},
   {X, φ} = Rot[K]; n = Length[X];
   A = IdentityMatrix[2 n + 1];
   Cases[X, {s_, i_, j_} ⧴ (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                              0     -1    ))];
   Δ = T^((-Total[φ]-Total[X[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} ⧴ (G[[α, β]] /. T → T_ν)];
   θ = ev[Sum[F_1[X[[k]]], {k, 1, n}]];
   θ += ev[Sum[Sum[F_2[X[[k1]], X[[k2]]], {k2, 1, n}], {k1, 1, n}]];
   θ += ev[Sum[F_3[φ[[k]], k], {k, 1, 2 n}]];
   Factor@{Δ, (Δ /. T → T_1) (Δ /. T → T_2) (Δ /. T → T_3) θ}
  ];
```

exec

```
nb2tex$PDFWidth /= 1.25;
```

Some Knots

tex

\needspace{15mm}
{\bf\red Some Knots.}

pdf

In[∘]:=
```
Expand[Θ[Knot[3, 1]]]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$

pdf

In[∘]:=
```
(* PolyPlot suppressed *)
```

In[∘]:=
```
Options[PolyPlot1] = {Labeled → False};
PolyPlot1[Δ_, OptionsPattern[]] := Module[{crs, m, maxc, minc, s, rect},
  rect = {{0, 0}, {1, 0}, {1, 1}, {0, 1}};
  If[Expand[Δ] === 0, Graphics[],
   m = Max[-Exponent[Δ, T, Min], Exponent[Δ, T, Max]];
   crs = CoefficientRules[T^m Δ, {T}];
   maxc = N@Log@Max@Abs[Last /@ crs];
   minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
   If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
```

```
    Graphics[crs /. ({x_} → c_) :→ {
        Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
        Tooltip[Polygon[({x + m - 1/2, 0} + #) & /@ rect], c T^(x-m)],
        If[Not@OptionValue[Labeled], {},
          {Black, FontSize → Scaled[1/((m + 1) (6 + maxc))], Text[c T^(x-m), {x + m, 0.5}]}]
      }, AspectRatio → Min[1/5, 1/Sqrt[m + 1]],
     ImagePadding → None, PlotRangePadding → None]
   ]];
Options[PolyPlot2] = {Labeled → False};
PolyPlot2[θ_, OptionsPattern[]] := Module[{crs, m1, m2, maxc, minc, s, hex, p},
    If[Expand[θ] === 0, Graphics[{White, Disk[]}],
     hex = Table[{Cos[α], Sin[α]} / Cos[2 π / 12] / 2, {α, 2 π / 12, 2 π, 2 π / 6}];
     m1 = Max[-Exponent[θ, T1, Min], Exponent[θ, T1, Max]];
     m2 = Max[-Exponent[θ, T2, Min], Exponent[θ, T2, Max]];
     crs = CoefficientRules[T1^m1 T2^m2 θ, {T1, T2}];
     maxc = N@Log@Max@Abs[Last /@ crs];
     minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
     If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
     Graphics[{(*{Yellow,Disk[{0,0},1+Cos[2π/12]Norm[{m1,m2}]/Sqrt[2]]},*)
       crs /. ({x1_, x2_} → c_) :→ {
           Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
           p = {{1, -1/2}, {0, Sqrt[3]/2}}.{x1 - m1, x2 - m2};
           Tooltip[Polygon[(p + #) & /@ hex], c T1^(x1-m1) T2^(x2-m2)],
           If[Not@OptionValue[Labeled], {},
            {Black, FontSize → Scaled[1/(Max[m1, m2] (10 + maxc))], Text[c T1^(x1-m1) T2^(x2-m2), p]}]
          }
      }, ImagePadding → None, PlotRangePadding → None]
    ]];
PolyPlot[{Δ_, θ_}, opts___Rule] := GraphicsColumn[
    {PolyPlot1[Δ, FilterRules[{opts}, Options[PolyPlot1]]],
     PolyPlot2[θ, FilterRules[{opts}, Options[PolyPlot2]]]},
    Spacings → Scaled@0.08, ImagePadding → None, PlotRangePadding → None,
    FilterRules[{opts}, Options[GraphicsColumn]]
   ];
```

tex

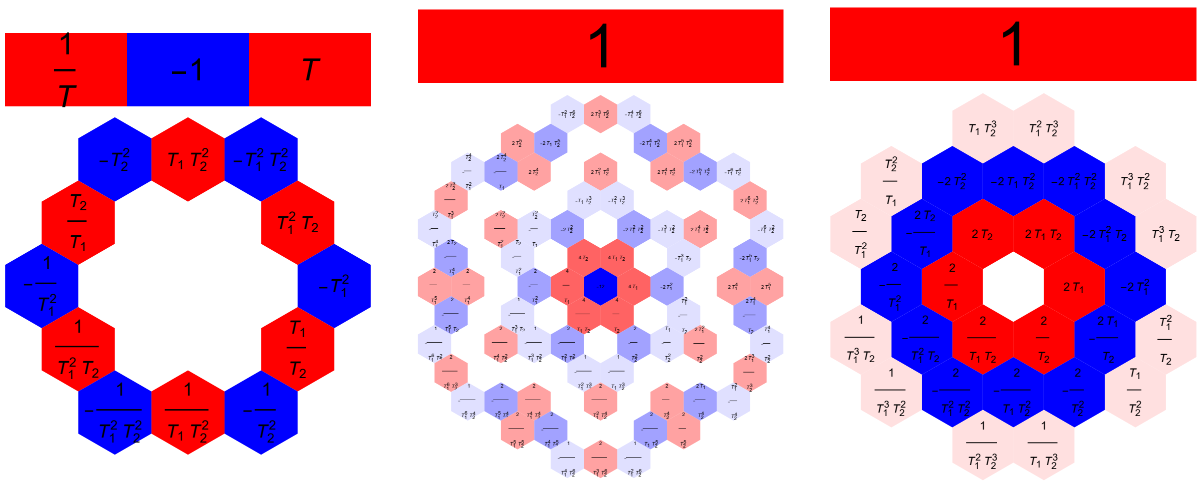
\needspace{25mm}
\parpic[r]{$

{\includegraphics[height=0.5in]{../Beijing-2407/K11n34.png}
\atop\text{\tiny K11n34}}
{\includegraphics[height=0.5in]{../Beijing-2407/K11n42.png}
\atop\text{\tiny K11n42}}
$}

pdf

In[ ]:=
```
GraphicsRow[PolyPlot[Θ[Knot[#]],
    Labeled → True] &
  /@ {"3_1", "K11n34", "K11n42"}]
```

Out[ ]=
pdf



tex

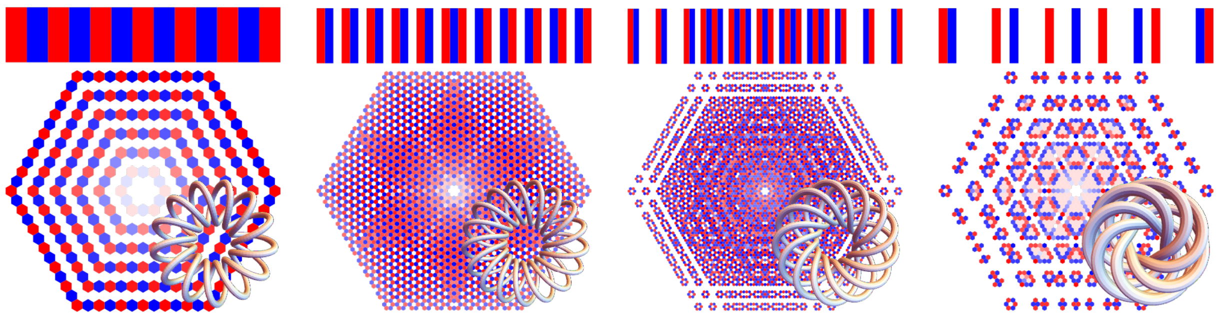
\parpic[r]{$
{\includegraphics[height=0.6in]{../../Projects/Gallery/Conway.png}
\atop\text{\scriptsize Conway}}
{\includegraphics[height=0.6in]{../../Projects/Gallery/PhotoNotAvailable.png}
\atop\text{\scriptsize Kinoshita}}
{\includegraphics[height=0.6in]{../../Projects/Gallery/Terasaka.jpg}
\atop\text{\scriptsize Terasaka}}
$}

So θ detects knot mutation and separates the Conway knot K11n34 from the Kinoshita-Terasaka knot K11n42!

pdf

In[◦]:=
```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
  ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```

Out[◦]=
pdf



tex

The torus knot $T_{22/7}$:
\newline\includegraphics[width=\linewidth]{T227.pdf}

\needspace{5cm}
Next, a random 300 crossing knot from~\cite{DHOEBL:Random} (more\newline
at \web{DK}):
\[\includegraphics[width=0.8\linewidth]{../UBC-241004/DK300.pdf} \]

tex

\newcolumn
{\red\bf Unproven Fact.} For any knot K, twice its genus $g(K)$ bounds the T_1 degree of θ:
$\deg_{T_1}\theta(K)\leq 2g(K)$.
\parpic[r]{$
 {\includegraphics[height=0.6in]{../../Projects/Gallery/Gompf.jpg}
  \atop\text{\scriptsize Gompf}}
 {\includegraphics[height=0.6in]{../../Projects/Gallery/Scharlemann.jpg}
  \atop\text{\scriptsize Scharlemann}}
 {\includegraphics[height=0.6in]{../../Projects/Gallery/Thompson.jpg}
  \atop\text{\scriptsize Thompson}}
$}
The 48-crossing Gompf-Scharlemann-Thompson GST_{48} knot \cite{GompfScharlemannThompson:Counterexample} is significant because it may be a counterexample to the slice-ribbon conjecture:

\[\resizebox{\linewidth}{!}{\import{../Waco-2203/}{GST48-Marked.pdf_t}} \]

exec

```
nb2tex$PDFWidth *= 1.25;
```

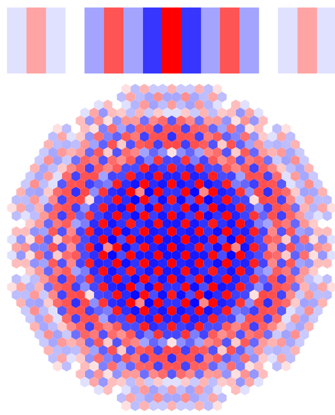
pdf

In[∘]:=

$$\mathrm{GST}_{48} = \mathrm{EPD}\big[X_{14,1}, \overline{X}_{2,29}, X_{3,40}, X_{43,4}, \overline{X}_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, \overline{X}_{9,28}, X_{10,41}, X_{42,11}, \overline{X}_{27,12}, X_{30,15}, \overline{X}_{16,61}, \overline{X}_{17,72}, \overline{X}_{18,83}, X_{19,34}, \overline{X}_{89,20}, \overline{X}_{21,92}, \overline{X}_{79,22}, \overline{X}_{68,23}, \overline{X}_{57,24}, \overline{X}_{25,56}, X_{62,31}, X_{73,32}, X_{84,33}, \overline{X}_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, \overline{X}_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, \overline{X}_{63,74}, \overline{X}_{64,85}, \overline{X}_{76,65}, \overline{X}_{87,66}, \overline{X}_{67,94}, \overline{X}_{75,86}, \overline{X}_{88,77}, \overline{X}_{78,93}\big];$$

$$\mathrm{AbsoluteTiming}[\mathrm{PolyPlot}[\Theta_{48} = \Theta @ \mathrm{GST}_{48},\ \mathrm{ImageSize} \to \mathrm{Small}]]$$

Out[∘]=
pdf

{17.1103,  }

pdf

In[∘]:=

$$\{\mathrm{Exponent}[\Theta_{48}[\![1]\!], T], \mathrm{Floor}[\mathrm{Exponent}[\Theta_{48}[\![2]\!], T_2] / 2]\}$$

Out[∘]=
pdf

{8, 10}

tex

\vfill\hfill So Θ knows things about GST_{48} that Δ doesn’t!

exec

In[∘]:=

```
nb2tex$PDFWidth /= 1.25;
```