Pensieve header: Experiments in 2026: Alternative integrands for θ.

Initialization

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< "IType(SimplifiedFormatting).m"];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

Minor utilities

In[∘]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CCF[ℰ_] := Factor[ℰ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, ϵ | (p | x | π | ξ | g)__, F] /. F → CCF];
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

In[∘]:=
```
ℒ_i_[K_] := CF[ℒ_i /@ Features[K]〚2〛]
vs[K_] := Flatten@Table[{p_ν,i, x_ν,i}, {ν, 3}, {i, Features[K]〚1〛}]
```

In[∘]:=
```
{p*, x*, π*, ξ*} = {π, ξ, p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_νi__)* := (u*)_νi;
```

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

g2px and px2g

Modified from pensieve://Talks/Geneva-2408/DataConversions.nb

In[∘]:=
```
g2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {g_i_,j_ :→ λ p_i x_j}] /. {λ^k_. :→ 1 / k!}]
gν2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {g_ν_,i_,j_ :→ λ_ν p_ν,i x_ν,j}] /. {λ^k_._ :→ 1 / k!}]
```

In[∘]:=
```
Zip_{}[ℰ_] := ℰ;
Zip_{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip_{ξs}, ξ] /. f_. ξ^d_. :→ (D[f, {ξ*, d}])) /. ξ* → 0
```

In[∘]:=
```
px2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
  ps = Union[Cases[ℰ, p_, ∞]]; xs = Union[Cases[ℰ, x_, ∞]];
  Q = Sum[p0* x0* g_{p0[[2]],x0[[2]]}, {p0, ps}, {x0, xs}];
  Expand[Zip_{ps⋃xs}[ℰ e^Q]]
 ]
pxν2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
  ps = Union[Cases[ℰ, p__, ∞]]; xs = Union[Cases[ℰ, x__, ∞]];
  Q = Sum[p0* x0* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
  Expand[Zip_{ps⋃xs}[ℰ e^Q] /. g_{α_,β_,i_,j_} :→ If[α === β, g_{α,i,j}, 0]]
 ]
```

gRules

From pensieve://Talks/MonteVerita-2604/Theta.nb

In[∘]:=
$$gR_{s\_,i\_,j\_} := \{$$
$$g_{j\beta\_} :\to g_{j^+\beta} + \delta_{j\beta},\ g_{i\beta\_} :\to T^s g_{i^+\beta} + (1 - T^s)\, g_{j^+\beta} + \delta_{i\beta},$$
$$g_{\alpha\_i^+} :\to T^s g_{\alpha i} + \delta_{\alpha i^+},\ g_{\alpha\_j^+} :\to g_{\alpha j} + (1 - T^s)\, g_{\alpha i} + \delta_{\alpha j^+},$$
$$g_{\nu\_j\beta\_} :\to g_{\nu j^+\beta} + \delta_{j\beta},\ g_{\nu\_i\beta\_} :\to T_\nu^s g_{\nu i^+\beta} + (1 - T_\nu^s)\, g_{\nu j^+\beta} + \delta_{i\beta},$$
$$g_{\nu\_\alpha\_i^+} :\to T_\nu^s g_{\nu\alpha i} + \delta_{\alpha i^+},\ g_{\nu\_\alpha\_j^+} :\to g_{\nu\alpha j} + (1 - T_\nu^s)\, g_{\nu\alpha i} + \delta_{\alpha j^+}$$
$$\}$$
$$gR_{i\_} := \{g_{i,\beta\_} :\to g_{i^+,\beta} + \delta_{i,\beta},\ g_{\alpha\_,i^+} :\to g_{\alpha,i} + \delta_{\alpha,i^+}\}$$

Inverse gRules written locally:

In[∘]:=
$$igR_{s\_,i\_,j\_} := \{$$
$$g_{j^+,\beta\_} :\to g_{j,\beta} - \delta_{j,\beta},\ g_{i^+,\beta\_} :\to T^{-s} g_{i\beta} - (T^{-s} - 1)\, g_{j^+\beta} - T^{-s}\delta_{i\beta},$$
$$g_{\alpha\_,i} :\to T^{-s}\,(g_{\alpha,i^+} - \delta_{\alpha,i^+}),\ g_{\alpha\_,j} :\to g_{\alpha,j^+} - (1 - T^s)\, g_{\alpha,i} - \delta_{\alpha,j^+},$$
$$g_{\nu\_,j^+,\beta\_} :\to g_{\nu,j,\beta} - \delta_{j,\beta},\ g_{\nu\_,i^+,\beta\_} :\to T_\nu^{-s} g_{\nu i\beta} - (T_\nu^{-s} - 1)\, g_{\nu j^+\beta} - T_\nu^{-s}\delta_{i\beta},$$
$$g_{\nu\_,\alpha\_,i} :\to T_\nu^{-s}\,(g_{\nu,\alpha,i^+} - \delta_{\alpha,i^+}),\ g_{\nu\_,\alpha\_,j} :\to g_{\nu,\alpha,j^+} - (1 - T_\nu^s)\, g_{\nu,\alpha,i} - \delta_{\alpha,j^+}$$
$$\}$$

The Base Lagrangian

From pensieve://Talks/Rank2.nb.

pdf

In[∘]:=
```
T_3 = T_1 T_2;  i_^+ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /. {π_{is__} :→ B^-1 π_{is}, x_{is__} :→ B^-1 x_{is}, p_{is__} :→ B p_{is}} /. ϵ B^{b_} /; b < 0 → 0 /.
    B → 1) &;
```

In[∘]:=

$$\mathcal{L}_1[X_{i\_,j\_}[s\_]] := T_3^s\,\mathbb{E}\Big[\text{CF@Plus}\Big[$$
$$\sum_{\nu=1}^{3}\left(x_{\nu i}\,(p_{\nu i^+}-p_{\nu i}) + x_{\nu j}\,(p_{\nu j^+}-p_{\nu j}) + \left(T_\nu^s-1\right)x_{\nu i}\,(p_{\nu i^+}-p_{\nu j^+})\right),$$
$$\left(T_1^s-1\right)p_{3j}\,x_{1i}\left(T_2^s\,x_{2i}-x_{2j}\right),$$
$$\in s\,\left(T_3^s-1\right)p_{1j}\,(p_{2i}-p_{2j})\,x_{3i}\Big/\left(T_2^s-1\right),$$
$$\in s\,\Big(1/2 + T_2^s\,p_{1i}\,p_{2j}\,x_{1i}\,x_{2i} - p_{1i}\,p_{2j}\,x_{1i}\,x_{2j} - p_{3i}\,x_{3i} - \left(T_2^s-1\right)p_{2j}\,p_{3i}\,x_{2i}\,x_{3i} +$$
$$\left(T_3^s-1\right)p_{2j}\,p_{3j}\,x_{2i}\,x_{3i} + 2\,p_{2j}\,p_{3i}\,x_{2j}\,x_{3i} + p_{1i}\,p_{3j}\,x_{1i}\,x_{3j} - p_{2i}\,p_{3j}\,x_{2i}\,x_{3j} - T_2^s\,p_{2j}\,p_{3j}\,x_{2i}\,x_{3j} +$$
$$\Big(\left(T_1^s-1\right)p_{1j}\,x_{1i}\left(T_2^{2s}\,p_{2j}\,x_{2i} - T_2^s\,p_{2j}\,x_{2j} - \left(T_2^s+1\right)\left(T_3^s-1\right)p_{3j}\,x_{3i} + T_2^s\,p_{3j}\,x_{3j}\right) +$$
$$\left(T_3^s-1\right)p_{3j}\,x_{3i}\left(1 - T_2^s\,p_{1i}\,x_{1i} + p_{2i}\,x_{2j} + \left(T_2^s-2\right)p_{2j}\,x_{2j}\right)\Big)\Big/\left(T_2^s-1\right)\Big)\Big]\Big]$$
$$\mathcal{L}_1[C_{i\_}[\varphi\_]] := T_3^\varphi\,\mathbb{E}\Big[\sum_{\nu=1}^{3} x_{\nu i}\,(p_{\nu i^+}-p_{\nu i}) + \in\varphi\,(p_{3i}\,x_{3i}-1/2)\Big]$$

In[∘]:= $\$\mathcal{L} = \mathcal{L}_1;$

The Trefoil

In[∘]:= **K = Mirror@Knot[3, 1]; Features[K]**

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$\text{Features}[7, C_4[-1]\,X_{1,5}[1]\,X_{3,7}[1]\,X_{6,2}[1]]$

In[∘]:= $\{\text{vs}[K], \$\mathcal{L}[K]\}$

Out[∘]=

$$\{\{p_{1,1}, x_{1,1}, p_{1,2}, x_{1,2}, p_{1,3}, x_{1,3}, p_{1,4}, x_{1,4}, p_{1,5}, x_{1,5}, p_{1,6}, x_{1,6}, p_{1,7},$$
$$x_{1,7}, p_{2,1}, x_{2,1}, p_{2,2}, x_{2,2}, p_{2,3}, x_{2,3}, p_{2,4}, x_{2,4}, p_{2,5}, x_{2,5}, p_{2,6}, x_{2,6}, p_{2,7}, x_{2,7},$$
$$p_{3,1}, x_{3,1}, p_{3,2}, x_{3,2}, p_{3,3}, x_{3,3}, p_{3,4}, x_{3,4}, p_{3,5}, x_{3,5}, p_{3,6}, x_{3,6}, p_{3,7}, x_{3,7}\},$$
$$T_3^2\,\mathbb{E}\Big[2\in - p_{1,1}\,x_{1,1} + T_1\,p_{1,1^+}\,x_{1,1} + (1-T_1)\,p_{1,5^+}\,x_{1,1} - p_{1,2}\,x_{1,2} + p_{1,2^+}\,x_{1,2} - p_{1,3}\,x_{1,3} + T_1\,p_{1,3^+}\,x_{1,3} +$$
$$(1-T_1)\,p_{1,7^+}\,x_{1,3} - p_{1,4}\,x_{1,4} + p_{1,4^+}\,x_{1,4} - p_{1,5}\,x_{1,5} + p_{1,5^+}\,x_{1,5} - p_{1,6}\,x_{1,6} + (1-T_1)\,p_{1,2^+}\,x_{1,6} +$$
$$T_1\,p_{1,6^+}\,x_{1,6} - p_{1,7}\,x_{1,7} + p_{1,7^+}\,x_{1,7} - p_{2,1}\,x_{2,1} + T_2\,p_{2,1^+}\,x_{2,1} + (1-T_2)\,p_{2,5^+}\,x_{2,1} + \in T_2\,p_{1,1}\,p_{2,5}\,x_{1,1}\,x_{2,1} +$$
$$\frac{\in(-1+T_1)\,T_2^2\,p_{1,5}\,p_{2,5}\,x_{1,1}\,x_{2,1}}{-1+T_2} + (-1+T_1)\,T_2\,p_{3,5}\,x_{1,1}\,x_{2,1} - p_{2,2}\,x_{2,2} + p_{2,2^+}\,x_{2,2} -$$
$$\frac{\in(-1+T_1)\,T_2\,p_{1,2}\,p_{2,2}\,x_{1,6}\,x_{2,2}}{-1+T_2} - \in p_{1,6}\,p_{2,2}\,x_{1,6}\,x_{2,2} + (1-T_1)\,p_{3,2}\,x_{1,6}\,x_{2,2} - p_{2,3}\,x_{2,3} +$$
$$T_2\,p_{2,3^+}\,x_{2,3} + (1-T_2)\,p_{2,7^+}\,x_{2,3} + \in T_2\,p_{1,3}\,p_{2,7}\,x_{1,3}\,x_{2,3} + \frac{\in(-1+T_1)\,T_2^2\,p_{1,7}\,p_{2,7}\,x_{1,3}\,x_{2,3}}{-1+T_2} +$$
$$(-1+T_1)\,T_2\,p_{3,7}\,x_{1,3}\,x_{2,3} - p_{2,4}\,x_{2,4} + p_{2,4^+}\,x_{2,4} - p_{2,5}\,x_{2,5} + p_{2,5^+}\,x_{2,5} - \in p_{1,1}\,p_{2,5}\,x_{1,1}\,x_{2,5} -$$
$$\frac{\in(-1+T_1)\,T_2\,p_{1,5}\,p_{2,5}\,x_{1,1}\,x_{2,5}}{-1+T_2} + (1-T_1)\,p_{3,5}\,x_{1,1}\,x_{2,5} - p_{2,6}\,x_{2,6} + (1-T_2)\,p_{2,2^+}\,x_{2,6} +$$
$$T_2\,p_{2,6^+}\,x_{2,6} + \frac{\in(-1+T_1)\,T_2^2\,p_{1,2}\,p_{2,2}\,x_{1,6}\,x_{2,6}}{-1+T_2} + \in T_2\,p_{1,6}\,p_{2,2}\,x_{1,6}\,x_{2,6} + (-1+T_1)\,T_2\,p_{3,2}\,x_{1,6}\,x_{2,6} -$$
$$p_{2,7}\,x_{2,7} + p_{2,7^+}\,x_{2,7} - \in p_{1,3}\,p_{2,7}\,x_{1,3}\,x_{2,7} - \frac{\in(-1+T_1)\,T_2\,p_{1,7}\,p_{2,7}\,x_{1,3}\,x_{2,7}}{-1+T_2} +$$
$$(1-T_1)\,p_{3,7}\,x_{1,3}\,x_{2,7} + \frac{\in(-1+T_3)\,p_{1,5}\,p_{2,1}\,x_{3,1}}{-1+T_2} - \frac{\in(-1+T_3)\,p_{1,5}\,p_{2,5}\,x_{3,1}}{-1+T_2} -$$

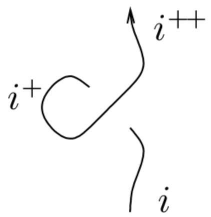
$$p_{3,1} x_{3,1} - \in p_{3,1} x_{3,1} + \frac{\in (-1+T_3)\, p_{3,5} x_{3,1}}{-1+T_2} + T_3 p_{3,1^+} x_{3,1} + (1-T_3)\, p_{3,5^+} x_{3,1} -$$
$$\frac{\in T_2 (-1+T_3)\, p_{1,1} p_{3,5} x_{1,1} x_{3,1}}{-1+T_2} - \frac{\in (-1+T_1)(1+T_2)(-1+T_3)\, p_{1,5} p_{3,5} x_{1,1} x_{3,1}}{-1+T_2} +$$
$$\in (1-T_2)\, p_{2,5} p_{3,1} x_{2,1} x_{3,1} + \in (-1+T_3)\, p_{2,5} p_{3,5} x_{2,1} x_{3,1} + 2 \in p_{2,5} p_{3,1} x_{2,5} x_{3,1} +$$
$$\frac{\in (-1+T_3)\, p_{2,1} p_{3,5} x_{2,5} x_{3,1}}{-1+T_2} + \frac{\in (-2+T_2)(-1+T_3)\, p_{2,5} p_{3,5} x_{2,5} x_{3,1}}{-1+T_2} - p_{3,2} x_{3,2} +$$
$$p_{3,2^+} x_{3,2} + \frac{\in (-1+T_1)\, T_2 p_{1,2} p_{3,2} x_{1,6} x_{3,2}}{-1+T_2} + \in p_{1,6} p_{3,2} x_{1,6} x_{3,2} - \in T_2 p_{2,2} p_{3,2} x_{2,6} x_{3,2} -$$
$$\in p_{2,6} p_{3,2} x_{2,6} x_{3,2} + \frac{\in (-1+T_3)\, p_{1,7} p_{2,3} x_{3,3}}{-1+T_2} - \frac{\in (-1+T_3)\, p_{1,7} p_{2,7} x_{3,3}}{-1+T_2} - p_{3,3} x_{3,3} - \in p_{3,3} x_{3,3} +$$
$$\frac{\in (-1+T_3)\, p_{3,7} x_{3,3}}{-1+T_2} + T_3 p_{3,3^+} x_{3,3} + (1-T_3)\, p_{3,7^+} x_{3,3} - \frac{\in T_2 (-1+T_3)\, p_{1,3} p_{3,7} x_{1,3} x_{3,3}}{-1+T_2} -$$
$$\frac{\in (-1+T_1)(1+T_2)(-1+T_3)\, p_{1,7} p_{3,7} x_{1,3} x_{3,3}}{-1+T_2} + \in (1-T_2)\, p_{2,7} p_{3,3} x_{2,3} x_{3,3} +$$
$$\in (-1+T_3)\, p_{2,7} p_{3,7} x_{2,3} x_{3,3} + 2 \in p_{2,7} p_{3,3} x_{2,7} x_{3,3} + \frac{\in (-1+T_3)\, p_{2,3} p_{3,7} x_{2,7} x_{3,3}}{-1+T_2} +$$
$$\frac{\in (-2+T_2)(-1+T_3)\, p_{2,7} p_{3,7} x_{2,7} x_{3,3}}{-1+T_2} - p_{3,4} x_{3,4} - \in p_{3,4} x_{3,4} + p_{3,4^+} x_{3,4} - p_{3,5} x_{3,5} +$$
$$p_{3,5^+} x_{3,5} + \in p_{1,1} p_{3,5} x_{1,1} x_{3,5} + \frac{\in (-1+T_1)\, T_2 p_{1,5} p_{3,5} x_{1,1} x_{3,5}}{-1+T_2} - \in p_{2,1} p_{3,5} x_{2,1} x_{3,5} -$$
$$\in T_2 p_{2,5} p_{3,5} x_{2,1} x_{3,5} - \frac{\in (-1+T_3)\, p_{1,2} p_{2,2} x_{3,6}}{-1+T_2} + \frac{\in (-1+T_3)\, p_{1,2} p_{2,6} x_{3,6}}{-1+T_2} +$$
$$\frac{\in (-1+T_3)\, p_{3,2} x_{3,6}}{-1+T_2} - p_{3,6} x_{3,6} - \in p_{3,6} x_{3,6} + (1-T_3)\, p_{3,2^+} x_{3,6} + T_3 p_{3,6^+} x_{3,6} -$$
$$\frac{\in (-1+T_1)(1+T_2)(-1+T_3)\, p_{1,2} p_{3,2} x_{1,6} x_{3,6}}{-1+T_2} - \frac{\in T_2 (-1+T_3)\, p_{1,6} p_{3,2} x_{1,6} x_{3,6}}{-1+T_2} +$$
$$\frac{\in (-2+T_2)(-1+T_3)\, p_{2,2} p_{3,2} x_{2,2} x_{3,6}}{-1+T_2} + \frac{\in (-1+T_3)\, p_{2,6} p_{3,2} x_{2,2} x_{3,6}}{-1+T_2} + 2 \in p_{2,2} p_{3,6} x_{2,2} x_{3,6} +$$
$$\in (-1+T_3)\, p_{2,2} p_{3,2} x_{2,6} x_{3,6} + \in (1-T_2)\, p_{2,2} p_{3,6} x_{2,6} x_{3,6} - p_{3,7} x_{3,7} + p_{3,7^+} x_{3,7} +$$
$$\in p_{1,3} p_{3,7} x_{1,3} x_{3,7} + \frac{\in (-1+T_1)\, T_2 p_{1,7} p_{3,7} x_{1,3} x_{3,7}}{-1+T_2} - \in p_{2,3} p_{3,7} x_{2,3} x_{3,7} - \in T_2 p_{2,7} p_{3,7} x_{2,3} x_{3,7} \Big] \Big\}$$

In[ ]:=
```
∫$ℒ[K] ⅆ vs[K]
```

Out[ ]=

$$-\frac{\mathbb{i}\, T_1^2 T_2^2\, \mathbb{E}\left[\frac{\in \left(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}$$

In[ ]:=
```
ℱ[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
vs_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
```



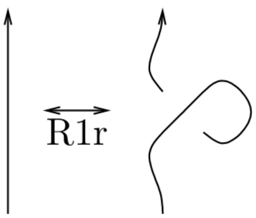
In[∘]:=

$$\text{lhs} = \int \mathcal{F}[\text{i}]\ \$\mathcal{L}\ /@\ (X_{i+2,i}[1]\ C_{i+1}[1])\ d\{vs_i,\ vs_{i^+},\ vs_{i+2}\}$$

$$\text{rhs} = \int \mathcal{F}[\text{i}]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{vs_i,\ vs_{i^+},\ vs_{i+2}\};$$

lhs == rhs // HL

» 1



In[∘]:=

$$\text{lhs} = \int \mathcal{F}[\text{i}]\ \$\mathcal{L}\ /@\ (X_{i,i+2}[1]\ C_{i+1}[-1])\ d\{vs_i,\ vs_{i^+},\ vs_{i+2}\}$$

$$\text{rhs} = \int \mathcal{F}[\text{i}]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{vs_i,\ vs_{i^+},\ vs_{i+2}\};$$

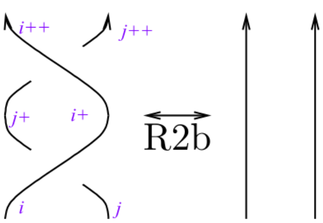
lhs == rhs // HL

Out[∘]=

$$-i\ \mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$$

Out[∘]=

True



Territory: $(x \mid p)_{(i|j)^+},\ x_{i|j}$.

In[∘]:=

$$\text{lhs} = \int \mathcal{F}[\text{i, j}]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ d\{vs_i,\ vs_j,\ vs_{i^+},\ vs_{j^+}\}$$

$$\text{rhs} = \int \mathcal{F}[\text{i, j}]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ d\{vs_i,\ vs_j,\ vs_{i^+},\ vs_{j^+}\};$$

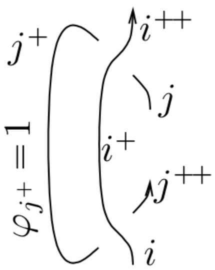
lhs == rhs // HL

Out[∘]=

$$\mathbb{E}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}]$$

Out[∘]=

True



In[]:=

$$\texttt{lhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L}\ /@\ (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\}$$

$$\texttt{rhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\};$$

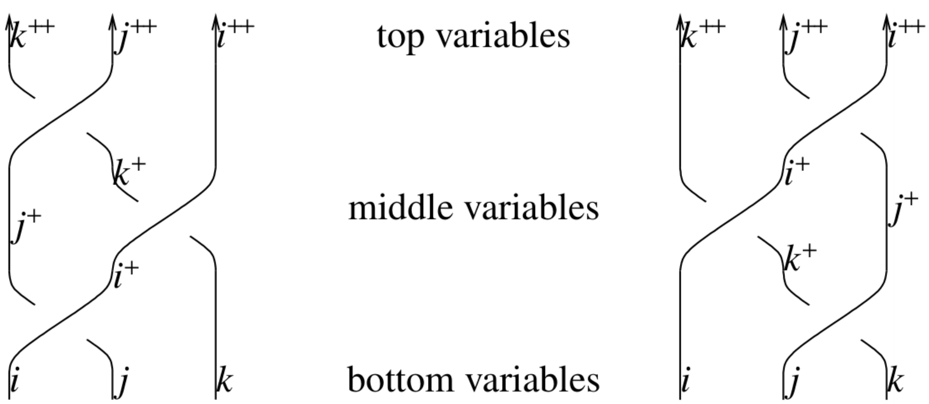
```
lhs == rhs // HL
```

Out[]=

$$-\mathbb{i}\ T_1\ T_2\ \mathbb{E}\left[\frac{\epsilon}{2} + p_{1,2+i}\ \pi_{1,i} + p_{1,3+j}\ \pi_{1,j} + p_{2,2+i}\ \pi_{2,i} + p_{2,3+j}\ \pi_{2,j} + p_{3,2+i}\ \pi_{3,i} + p_{3,3+j}\ \pi_{3,j} + \epsilon\ p_{3,3+j}\ \pi_{3,j}\right]$$

Out[]=

True



In[]:=

$$\texttt{Short}\left[\texttt{lhs} = \int (\mathcal{F}[i, j, k]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1]))\ d\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\}\right]$$

$$\texttt{rhs} = \int (\mathcal{F}[i, j, k]\ \$\mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1]))\ d\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\};$$

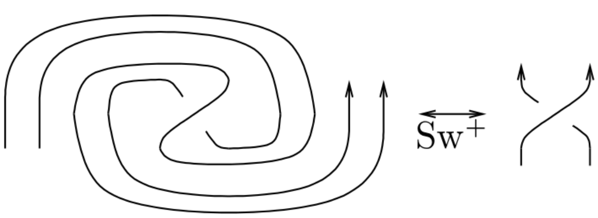
```
lhs == rhs // HL
```

Out[]//Short=

$$T_1^3\ T_2^3\ \mathbb{E}\left[\frac{3\,\epsilon}{2} + \ll 153 \gg\right]$$

Out[]=

True



In[ ]:=
```
Short[lhs =
   ∫𝓕[i, j] $ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1]) d{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{i+2}, vs_{j+2}}]
rhs =
   ∫𝓕[i, j] $ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0]) d{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{i+2}, vs_{j+2}};
lhs == rhs // HL
```

Out[ ]//Short=

$$T_1 \ll 1\gg \ll 1\gg \mathbb{E}\left[\frac{\epsilon}{2} + T_1\, p_{1,3+i}\, \pi_{1,i} + (1 - T_1)\, p_{1,3+j}\, \pi_{1,i} + p_{1,3+j}\, \pi_{1,j} + \ll 33\gg + \epsilon \ll 4\gg \pi_{3\,\ll 1\gg\,\ll 1\gg} + \frac{\epsilon\,(-1 + T_1)\ll 3\gg \pi_{3,j}}{-1 + T_2} - \epsilon\, T_2\, p_{2,3+i}\, p_{3,3+j}\, \pi_{2,i}\, \pi_{3,j} - \epsilon\, p_{2,3+j}\, p_{3,3+j}\, \pi_{2,i}\, \pi_{3,j}\right]$$

Out[ ]=

True

The pPush Lagrangian

In[ ]:=
```
pPush[ω_. 𝔼[L_]] := Module[{B, L1, Q, P0, P1a, P1b, θ0},
   L1 = CoefficientRules[B L /. {p_{αβ__} :→ B p_{αβ}, x_{αβ__} :→ B^-1 x_{αβ}}, {ϵ, B}];
   {ω, Q = {0, 1} /. L1, P0 = {0, 0} /. L1, P1a = {1, 2} /. L1, P1b = {1, 1} /. L1};
   θ0 = CF[
      px⌄2g[CF[(P0 /. Thread[{s, i, j} → {s0, i0, j0}])
            (P1a /. Thread[{s, i, j} → {s1, i1, j1}])
         ]] /. gR_{s0,i0,j0} /. gR_{s1,i1,j1} /. {δ_{j1,i0} | δ_{j0,i1} | δ_{i1,j0} → 0, δ_{i1,i0} | δ_{j1,j0} → B}
    ];
   θ1 = List @@ Factor[g⌄2px[θ0 /. B → 0]];
   {Y0, Y1} = {Times @@ Select[θ1, FreeQ[s1 | i1 | j1]],
      Times @@ Select[θ1, FreeQ[s0 | i0 | j0]]} /. {s0 | s1 → s, i0 | i1 → i, j0 | j1 → j};
   Times @@ θ1 == Y0 Y1;
   degθ = g⌄2px[Coefficient[θ0, B] /. {s0 | s1 → s, i0 | i1 → i, j0 | j1 → j}];
   ω 𝔼[CF@Plus[
       Q, Y0, ϵ Y1, ϵ degθ,
       ϵ g⌄2px[px⌄2g[P1b] /. gR_{s,i,j} /. {δ_{i,j} | δ_{j,i} → 0, δ_{i,i} | δ_{j,j} → 1}]
      ]]
  ];
```

(Alt) In[ ]:=
```
ℒ_1[C_i[φ]]
```

(Alt) Out[ ]=

$$(T_1\, T_2)^{\varphi}\, \mathbb{E}\left[(-p_{1,i} + p_{1,1+i})\, x_{1,i} + (-p_{2,i} + p_{2,1+i})\, x_{2,i} + (-p_{3,i} + p_{3,1+i})\, x_{3,i} + \epsilon\, \varphi \left(-\frac{1}{2} + p_{3,i}\, x_{3,i}\right)\right]$$

(Alt) In[∘]:=

$$\text{gv2px}\left[\text{pxv2g}\left[-\frac{1}{2}+p_{3,i}\,x_{3,i}\right]\;/.\;\text{gR}_i\;/.\;\{\delta_{i,i}\to 1\}\right]$$

(Alt) Out[∘]=

$$\frac{1}{2}+p_{3,1+i}\,x_{3,i}$$

In[∘]:=

$$\mathcal{L}_{\text{pPush}}[X_{i\_,j\_}[s\_]] = \text{pPush@}\mathcal{L}_1[X_{i,j}[s]];$$

$$\mathcal{L}_{\text{pPush}}[C_{i\_}[\varphi\_]] := T_3^{\varphi}\,\mathbb{E}\left[\text{Sum}[x_{\nu,i}\,(p_{\nu,i+1}-p_{\nu,i}),\,\{\nu,3\}]+\epsilon\,\varphi\left(\frac{1}{2}+p_{3,i+1}\,x_{3,i}\right)\right];$$

In[∘]:= $\$\mathcal{L} = \mathcal{L}_{\text{pPush}};$

In[∘]:= $\$\mathcal{L}[X_{i,j}[s]]$

$\$\mathcal{L}[C_i[\varphi]]$

Out[∘]=

$$\begin{aligned}
&(T_1 T_2)^s\,\mathbb{E}\Big[\frac{s\,\epsilon}{2}-p_{1,i}\,x_{1,i}+T_1^s\,p_{1,1+i}\,x_{1,i}+(1-T_1^s)\,p_{1,1+j}\,x_{1,i}-p_{1,j}\,x_{1,j}+\\
&\quad p_{1,1+j}\,x_{1,j}-p_{2,i}\,x_{2,i}+T_2^s\,p_{2,1+i}\,x_{2,i}-s\,\epsilon\,T_2^s\,p_{2,1+i}\,x_{2,i}+(1-T_2^s)\,p_{2,1+j}\,x_{2,i}+\\
&\quad s\,\epsilon\,T_1^s\,T_2^s\,p_{1,1+i}\,p_{2,1+j}\,x_{1,i}\,x_{2,i}+\frac{s\,\epsilon\,(-1+T_1^s)\,T_2^s\,p_{1,1+j}\,p_{2,1+j}\,x_{1,i}\,x_{2,i}}{-1+T_2^s}+\\
&\quad (-1+T_1^s)\,T_2^s\,p_{3,1+j}\,x_{1,i}\,x_{2,i}-p_{2,j}\,x_{2,j}+p_{2,1+j}\,x_{2,j}+s\,\epsilon\,p_{2,1+j}\,x_{2,j}-\\
&\quad s\,\epsilon\,T_1^s\,p_{1,1+i}\,p_{2,1+j}\,x_{1,i}\,x_{2,j}-\frac{s\,\epsilon\,(-1+T_1^s)\,p_{1,1+j}\,p_{2,1+j}\,x_{1,i}\,x_{2,j}}{-1+T_2^s}+(1-T_1^s)\,p_{3,1+j}\,x_{1,i}\,x_{2,j}+\\
&\quad \frac{s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{1,1+j}\,p_{2,1+i}\,x_{3,i}}{-1+T_2^s}-\frac{s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{1,1+j}\,p_{2,1+j}\,x_{3,i}}{-1+T_2^s}-\\
&\quad p_{3,i}\,x_{3,i}+(T_1 T_2)^s\,p_{3,1+i}\,x_{3,i}+s\,\epsilon\,(T_1 T_2)^s\,p_{3,1+i}\,x_{3,i}+(1-(T_1 T_2)^s)\,p_{3,1+j}\,x_{3,i}-\\
&\quad \frac{s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{3,1+j}\,x_{3,i}}{-1+T_2^s}-\frac{s\,\epsilon\,T_1^s\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{1,1+i}\,p_{3,1+j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s}+\\
&\quad \frac{s\,\epsilon\,(-1+T_1^s)\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{1,1+j}\,p_{3,1+j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s}-s\,\epsilon\,(T_1 T_2)^s\,(-1+T_2^s)\,p_{2,1+j}\,p_{3,1+i}\,x_{2,i}\,x_{3,i}+\\
&\quad s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{2,1+j}\,p_{3,1+j}\,x_{2,i}\,x_{3,i}+2\,s\,\epsilon\,(T_1 T_2)^s\,p_{2,1+j}\,p_{3,1+i}\,x_{2,j}\,x_{3,i}+\\
&\quad \frac{s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{2,1+i}\,p_{3,1+j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s}-\frac{s\,\epsilon\,(-1+2\,T_2^s)\,(-1+(T_1 T_2)^s)\,p_{2,1+j}\,p_{3,1+j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s}-\\
&\quad p_{3,j}\,x_{3,j}+p_{3,1+j}\,x_{3,j}+s\,\epsilon\,T_1^s\,p_{1,1+i}\,p_{3,1+j}\,x_{1,i}\,x_{3,j}+\frac{s\,\epsilon\,(-1+T_1^s)\,p_{1,1+j}\,p_{3,1+j}\,x_{1,i}\,x_{3,j}}{-1+T_2^s}-\\
&\quad s\,\epsilon\,T_2^s\,p_{2,1+i}\,p_{3,1+j}\,x_{2,i}\,x_{3,j}-s\,\epsilon\,p_{2,1+j}\,p_{3,1+j}\,x_{2,i}\,x_{3,j}\Big]
\end{aligned}$$

Out[∘]=

$$(T_1 T_2)^{\varphi}\,\mathbb{E}\left[(-p_{1,i}+p_{1,1+i})\,x_{1,i}+(-p_{2,i}+p_{2,1+i})\,x_{2,i}+(-p_{3,i}+p_{3,1+i})\,x_{3,i}+\epsilon\,\varphi\left(\frac{1}{2}+p_{3,1+i}\,x_{3,i}\right)\right]$$

In[]:=
```
Collect[$ℒ[X_{i,j}[s]][[2, 1]], ϵ]
Collect[$ℒ[C_i[φ]][[2, 1]], ϵ]
```

Out[]=

$$-p_{1,i}\,x_{1,i}+T_1^s\,p_{1,1+i}\,x_{1,i}+(1-T_1^s)\,p_{1,1+j}\,x_{1,i}-p_{1,j}\,x_{1,j}+p_{1,1+j}\,x_{1,j}-p_{2,i}\,x_{2,i}+T_2^s\,p_{2,1+i}\,x_{2,i}+(1-T_2^s)\,p_{2,1+j}\,x_{2,i}+(-1+T_1^s)\,T_2^s\,p_{3,1+j}\,x_{1,i}\,x_{2,i}-p_{2,j}\,x_{2,j}+p_{2,1+j}\,x_{2,j}+(1-T_1^s)\,p_{3,1+j}\,x_{1,i}\,x_{2,j}-p_{3,i}\,x_{3,i}+(T_1\,T_2)^s\,p_{3,1+i}\,x_{3,i}+(1-(T_1\,T_2)^s)\,p_{3,1+j}\,x_{3,i}-p_{3,j}\,x_{3,j}+p_{3,1+j}\,x_{3,j}+\epsilon\left(\frac{s}{2}-s\,T_2^s\,p_{2,1+i}\,x_{2,i}+s\,T_1^s\,T_2^s\,p_{1,1+i}\,p_{2,1+j}\,x_{1,i}\,x_{2,i}+\frac{s\,(-1+T_1^s)\,T_2^s\,p_{1,1+j}\,p_{2,1+j}\,x_{1,i}\,x_{2,i}}{-1+T_2^s}+s\,p_{2,1+j}\,x_{2,j}-s\,T_1^s\,p_{1,1+i}\,p_{2,1+j}\,x_{1,i}\,x_{2,j}-\frac{s\,(-1+T_1^s)\,p_{1,1+j}\,p_{2,1+j}\,x_{1,i}\,x_{2,j}}{-1+T_2^s}+\frac{s\,T_2^s\,(-1+(T_1\,T_2)^s)\,p_{1,1+j}\,p_{2,1+i}\,x_{3,i}}{-1+T_2^s}-\frac{s\,T_2^s\,(-1+(T_1\,T_2)^s)\,p_{1,1+j}\,p_{2,1+j}\,x_{3,i}}{-1+T_2^s}+s\,(T_1\,T_2)^s\,p_{3,1+i}\,x_{3,i}-\frac{s\,T_2^s\,(-1+(T_1\,T_2)^s)\,p_{3,1+j}\,x_{3,i}}{-1+T_2^s}-\frac{s\,T_1^s\,T_2^s\,(-1+(T_1\,T_2)^s)\,p_{1,1+i}\,p_{3,1+j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s}+\frac{s\,(-1+T_1^s)\,T_2^s\,(-1+(T_1\,T_2)^s)\,p_{1,1+j}\,p_{3,1+j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s}-s\,(T_1\,T_2)^s\,(-1+T_2^s)\,p_{2,1+j}\,p_{3,1+i}\,x_{2,i}\,x_{3,i}+s\,T_2^s\,(-1+(T_1\,T_2)^s)\,p_{2,1+j}\,p_{3,1+j}\,x_{2,i}\,x_{3,i}+2\,s\,(T_1\,T_2)^s\,p_{2,1+j}\,p_{3,1+i}\,x_{2,j}\,x_{3,i}+\frac{s\,T_2^s\,(-1+(T_1\,T_2)^s)\,p_{2,1+i}\,p_{3,1+j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s}-\frac{s\,(-1+2\,T_2^s)\,(-1+(T_1\,T_2)^s)\,p_{2,1+j}\,p_{3,1+j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s}+s\,T_1^s\,p_{1,1+i}\,p_{3,1+j}\,x_{1,i}\,x_{3,j}+\frac{s\,(-1+T_1^s)\,p_{1,1+j}\,p_{3,1+j}\,x_{1,i}\,x_{3,j}}{-1+T_2^s}-s\,T_2^s\,p_{2,1+i}\,p_{3,1+j}\,x_{2,i}\,x_{3,j}-s\,p_{2,1+j}\,p_{3,1+j}\,x_{2,i}\,x_{3,j}\right)$$

Out[]=

$$(-p_{1,i}+p_{1,1+i})\,x_{1,i}+(-p_{2,i}+p_{2,1+i})\,x_{2,i}+(-p_{3,i}+p_{3,1+i})\,x_{3,i}+\epsilon\,\varphi\left(\frac{1}{2}+p_{3,1+i}\,x_{3,i}\right)$$

The Trefoil

(Alt) In[]:=
```
K = Mirror@Knot[3, 1]; {vs[K], $ℒ[K]}
```

KnotTheory: Loading precomputed data in PD4Knots`.

(Alt) Out[]=

$$\{\{p_{1,1},x_{1,1},p_{1,2},x_{1,2},p_{1,3},x_{1,3},p_{1,4},x_{1,4},p_{1,5},x_{1,5},p_{1,6},x_{1,6},p_{1,7},x_{1,7},p_{2,1},x_{2,1},p_{2,2},x_{2,2},p_{2,3},x_{2,3},p_{2,4},x_{2,4},p_{2,5},x_{2,5},p_{2,6},x_{2,6},p_{2,7},x_{2,7},p_{3,1},x_{3,1},p_{3,2},x_{3,2},p_{3,3},x_{3,3},p_{3,4},x_{3,4},p_{3,5},x_{3,5},p_{3,6},x_{3,6},p_{3,7},x_{3,7}\},$$

$$T_1^2\,T_2^2\,\mathbb{E}\Big[\epsilon-p_{1,1}\,x_{1,1}+T_1\,p_{1,2}\,x_{1,1}+(1-T_1)\,p_{1,6}\,x_{1,1}-p_{1,2}\,x_{1,2}+p_{1,3}\,x_{1,2}-p_{1,3}\,x_{1,3}+T_1\,p_{1,4}\,x_{1,3}+(1-T_1)\,p_{1,8}\,x_{1,3}-p_{1,4}\,x_{1,4}+p_{1,5}\,x_{1,4}-p_{1,5}\,x_{1,5}+p_{1,6}\,x_{1,5}+(1-T_1)\,p_{1,3}\,x_{1,6}-p_{1,6}\,x_{1,6}+T_1\,p_{1,7}\,x_{1,6}-p_{1,7}\,x_{1,7}+p_{1,8}\,x_{1,7}-p_{2,1}\,x_{2,1}+T_2\,p_{2,2}\,x_{2,1}-\epsilon\,T_2\,p_{2,2}\,x_{2,1}+(1-T_2)\,p_{2,6}\,x_{2,1}+\epsilon\,T_1\,T_2\,p_{1,2}\,p_{2,6}\,x_{1,1}\,x_{2,1}+\frac{\epsilon\,(-1+T_1)\,T_2\,p_{1,6}\,p_{2,6}\,x_{1,1}\,x_{2,1}}{-1+T_2}+(-1+T_1)\,T_2\,p_{3,6}\,x_{1,1}\,x_{2,1}-p_{2,2}\,x_{2,2}+p_{2,3}\,x_{2,2}+\epsilon\,p_{2,3}\,x_{2,2}-\frac{\epsilon\,(-1+T_1)\,p_{1,3}\,p_{2,3}\,x_{1,6}\,x_{2,2}}{-1+T_2}-\epsilon\,T_1\,p_{1,7}\,p_{2,3}\,x_{1,6}\,x_{2,2}+(1-T_1)\,p_{3,3}\,x_{1,6}\,x_{2,2}-p_{2,3}\,x_{2,3}+T_2\,p_{2,4}\,x_{2,3}-\epsilon\,T_2\,p_{2,4}\,x_{2,3}+(1-T_2)\,p_{2,8}\,x_{2,3}+$$

$$\in T_1 T_2 p_{1,4} p_{2,8} x_{1,3} x_{2,3} + \frac{\in (-1+T_1) T_2 p_{1,8} p_{2,8} x_{1,3} x_{2,3}}{-1+T_2} + (-1+T_1) T_2 p_{3,8} x_{1,3} x_{2,3} - p_{2,4} x_{2,4} +$$
$$p_{2,5} x_{2,4} - p_{2,5} x_{2,5} + p_{2,6} x_{2,5} + \in p_{2,6} x_{2,5} - \in T_1 p_{1,2} p_{2,6} x_{1,1} x_{2,5} - \frac{\in (-1+T_1) p_{1,6} p_{2,6} x_{1,1} x_{2,5}}{-1+T_2} +$$
$$(1-T_1) p_{3,6} x_{1,1} x_{2,5} + (1-T_2) p_{2,3} x_{2,6} - p_{2,6} x_{2,6} + T_2 p_{2,7} x_{2,6} - \in T_2 p_{2,7} x_{2,6} +$$
$$\frac{\in (-1+T_1) T_2 p_{1,3} p_{2,3} x_{1,6} x_{2,6}}{-1+T_2} + \in T_1 T_2 p_{1,7} p_{2,3} x_{1,6} x_{2,6} + (-1+T_1) T_2 p_{3,3} x_{1,6} x_{2,6} -$$
$$p_{2,7} x_{2,7} + p_{2,8} x_{2,7} + \in p_{2,8} x_{2,7} - \in T_1 p_{1,4} p_{2,8} x_{1,3} x_{2,7} - \frac{\in (-1+T_1) p_{1,8} p_{2,8} x_{1,3} x_{2,7}}{-1+T_2} +$$
$$(1-T_1) p_{3,8} x_{1,3} x_{2,7} + \frac{\in T_2 (-1+T_1 T_2) p_{1,6} p_{2,2} x_{3,1}}{-1+T_2} - \frac{\in T_2 (-1+T_1 T_2) p_{1,6} p_{2,6} x_{3,1}}{-1+T_2} -$$
$$p_{3,1} x_{3,1} + T_1 T_2 p_{3,2} x_{3,1} + \in T_1 T_2 p_{3,2} x_{3,1} + (1-T_1 T_2) p_{3,6} x_{3,1} - \frac{\in T_2 (-1+T_1 T_2) p_{3,6} x_{3,1}}{-1+T_2} -$$
$$\frac{\in T_1 T_2 (-1+T_1 T_2) p_{1,2} p_{3,6} x_{1,1} x_{3,1}}{-1+T_2} + \frac{\in (-1+T_1) T_2 (-1+T_1 T_2) p_{1,6} p_{3,6} x_{1,1} x_{3,1}}{-1+T_2} -$$
$$\in T_1 (-1+T_2) T_2 p_{2,6} p_{3,2} x_{2,1} x_{3,1} + \in T_2 (-1+T_1 T_2) p_{2,6} p_{3,6} x_{2,1} x_{3,1} + 2 \in T_1 T_2 p_{2,6} p_{3,2} x_{2,5} x_{3,1} +$$
$$\frac{\in T_2 (-1+T_1 T_2) p_{2,2} p_{3,6} x_{2,5} x_{3,1}}{-1+T_2} - \frac{\in (-1+2 T_2) (-1+T_1 T_2) p_{2,6} p_{3,6} x_{2,5} x_{3,1}}{-1+T_2} - p_{3,2} x_{3,2} +$$
$$p_{3,3} x_{3,2} + \frac{\in (-1+T_1) p_{1,3} p_{3,3} x_{1,6} x_{3,2}}{-1+T_2} + \in T_1 p_{1,7} p_{3,3} x_{1,6} x_{3,2} - \in p_{2,3} p_{3,3} x_{2,6} x_{3,2} -$$
$$\in T_2 p_{2,7} p_{3,3} x_{2,6} x_{3,2} + \frac{\in T_2 (-1+T_1 T_2) p_{1,8} p_{2,4} x_{3,3}}{-1+T_2} - \frac{\in T_2 (-1+T_1 T_2) p_{1,8} p_{2,8} x_{3,3}}{-1+T_2} -$$
$$p_{3,3} x_{3,3} + T_1 T_2 p_{3,4} x_{3,3} + \in T_1 T_2 p_{3,4} x_{3,3} + (1-T_1 T_2) p_{3,8} x_{3,3} - \frac{\in T_2 (-1+T_1 T_2) p_{3,8} x_{3,3}}{-1+T_2} -$$
$$\frac{\in T_1 T_2 (-1+T_1 T_2) p_{1,4} p_{3,8} x_{1,3} x_{3,3}}{-1+T_2} + \frac{\in (-1+T_1) T_2 (-1+T_1 T_2) p_{1,8} p_{3,8} x_{1,3} x_{3,3}}{-1+T_2} -$$
$$\in T_1 (-1+T_2) T_2 p_{2,8} p_{3,4} x_{2,3} x_{3,3} + \in T_2 (-1+T_1 T_2) p_{2,8} p_{3,8} x_{2,3} x_{3,3} + 2 \in T_1 T_2 p_{2,8} p_{3,4} x_{2,7} x_{3,3} +$$
$$\frac{\in T_2 (-1+T_1 T_2) p_{2,4} p_{3,8} x_{2,7} x_{3,3}}{-1+T_2} - \frac{\in (-1+2 T_2) (-1+T_1 T_2) p_{2,8} p_{3,8} x_{2,7} x_{3,3}}{-1+T_2} - p_{3,4} x_{3,4} -$$
$$\in p_{3,4} x_{3,4} + p_{3,5} x_{3,4} - p_{3,5} x_{3,5} + p_{3,6} x_{3,5} + \in T_1 p_{1,2} p_{3,6} x_{1,1} x_{3,5} + \frac{\in (-1+T_1) p_{1,6} p_{3,6} x_{1,1} x_{3,5}}{-1+T_2} -$$
$$\in T_2 p_{2,2} p_{3,6} x_{2,1} x_{3,5} - \in p_{2,6} p_{3,6} x_{2,1} x_{3,5} - \frac{\in T_2 (-1+T_1 T_2) p_{1,3} p_{2,3} x_{3,6}}{-1+T_2} +$$
$$\frac{\in T_2 (-1+T_1 T_2) p_{1,3} p_{2,7} x_{3,6}}{-1+T_2} + (1-T_1 T_2) p_{3,3} x_{3,6} - \frac{\in T_2 (-1+T_1 T_2) p_{3,3} x_{3,6}}{-1+T_2} -$$
$$p_{3,6} x_{3,6} + T_1 T_2 p_{3,7} x_{3,6} + \in T_1 T_2 p_{3,7} x_{3,6} + \frac{\in (-1+T_1) T_2 (-1+T_1 T_2) p_{1,3} p_{3,3} x_{1,6} x_{3,6}}{-1+T_2} -$$
$$\frac{\in T_1 T_2 (-1+T_1 T_2) p_{1,7} p_{3,3} x_{1,6} x_{3,6}}{-1+T_2} - \frac{\in (-1+2 T_2) (-1+T_1 T_2) p_{2,3} p_{3,3} x_{2,2} x_{3,6}}{-1+T_2} +$$

$$\frac{\in T_2\ (-1+T_1 T_2)\ p_{2,7}\, p_{3,3}\, x_{2,2}\, x_{3,6}}{-1+T_2} + 2 \in T_1\, T_2\, p_{2,3}\, p_{3,7}\, x_{2,2}\, x_{3,6} + \in T_2\ (-1+T_1 T_2)\ p_{2,3}\, p_{3,3}\, x_{2,6}\, x_{3,6} -$$
$$\in T_1\ (-1+T_2)\ T_2\, p_{2,3}\, p_{3,7}\, x_{2,6}\, x_{3,6} - p_{3,7}\, x_{3,7} + p_{3,8}\, x_{3,7} + \in T_1\, p_{1,4}\, p_{3,8}\, x_{1,3}\, x_{3,7} +$$
$$\frac{\in\ (-1+T_1)\ p_{1,8}\, p_{3,8}\, x_{1,3}\, x_{3,7}}{-1+T_2} - \in T_2\, p_{2,4}\, p_{3,8}\, x_{2,3}\, x_{3,7} - \in p_{2,8}\, p_{3,8}\, x_{2,3}\, x_{3,7}\Big]\Big\}$$

(Alt) In[ ]:=

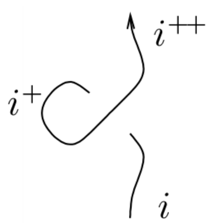
```
∫$ℒ[K] ⅆ vs[K]
```

(Alt) Out[ ]=

$$-\frac{\mathbb{i}\ T_1^2\, T_2^2\ \mathbb{E}\left[\frac{\in\ \left(1-T_1+T_1^2-T_2-T_1^3\, T_2+T_2^2+T_1^4\, T_2^2-T_1\, T_2^3-T_1^4\, T_2^3+T_1^2\, T_2^4-T_1^3\, T_2^4+T_1^4\, T_2^4\right)}{\left(1-T_1+T_1^2\right)\ \left(1-T_2+T_2^2\right)\ \left(1-T_1\, T_2+T_1^2\, T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\ \left(1-T_2+T_2^2\right)\ \left(1-T_1\, T_2+T_1^2\, T_2^2\right)}$$

In[ ]:=

```
𝓕[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
vs_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
```



In[ ]:=

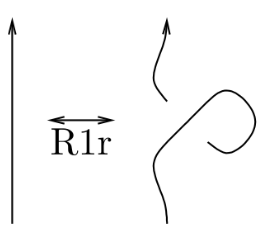
```
lhs = ∫𝓕[i] $ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{vs_i, vs_{i⁺}, vs_{i+2}}
rhs = ∫𝓕[i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{vs_i, vs_{i⁺}, vs_{i+2}};
lhs == rhs // HL
```

Out[ ]=

$$-\mathbb{i}\ \mathbb{E}\left[p_{1,3+i}\, \pi_{1,i} + p_{2,3+i}\, \pi_{2,i} + p_{3,3+i}\, \pi_{3,i}\right]$$

Out[ ]=

True



(Alt) In[ ]:=

$$\text{lhs} = \int \mathcal{F}[\text{i}]\ \$\mathcal{L}\ \text{/@}\ (X_{\text{i},\text{i+2}}[1]\ C_{\text{i+1}}[-1])\ d\{vs_{\text{i}}, vs_{\text{i}^+}, vs_{\text{i+2}}\}$$

$$\text{rhs} = \int \mathcal{F}[\text{i}]\ \$\mathcal{L}\ \text{/@}\ (C_{\text{i}}[0]\ C_{\text{i+1}}[0]\ C_{\text{i+2}}[0])\ d\{vs_{\text{i}}, vs_{\text{i}^+}, vs_{\text{i+2}}\};$$

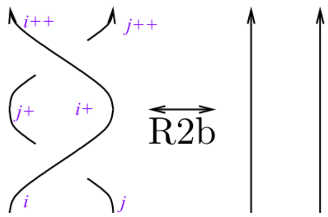
$$\text{lhs} == \text{rhs}\ \text{//}\ \text{HL}$$

(Alt) Out[ ]=

$$-\mathbb{i}\ \mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$$

(Alt) Out[ ]=

True



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

(Alt) In[ ]:=

$$\text{lhs} = \int \mathcal{F}[\text{i}, \text{j}]\ \$\mathcal{L}\ \text{/@}\ (X_{\text{i},\text{j}}[1]\ X_{\text{i+1},\text{j+1}}[-1])\ d\{vs_{\text{i}}, vs_{\text{j}}, vs_{\text{i}^+}, vs_{\text{j}^+}\}$$

$$\text{rhs} = \int \mathcal{F}[\text{i}, \text{j}]\ \$\mathcal{L}\ \text{/@}\ (C_{\text{i}}[0]\ C_{\text{i+1}}[0]\ C_{\text{j}}[0]\ C_{\text{j+1}}[0])\ d\{vs_{\text{i}}, vs_{\text{j}}, vs_{\text{i}^+}, vs_{\text{j}^+}\};$$

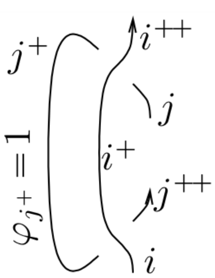
$$\text{lhs} == \text{rhs}\ \text{//}\ \text{HL}$$

(Alt) Out[ ]=

$$\mathbb{E}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}]$$

(Alt) Out[ ]=

True



(Alt) In[ ]:=

$$\text{lhs} = \int \mathcal{F}[\text{i}, \text{j}]\ \$\mathcal{L}\ \text{/@}\ (X_{\text{i+1},\text{j}}[1]\ X_{\text{i},\text{j+2}}[-1]\ C_{\text{j+1}}[1])\ d\{vs_{\text{i}}, vs_{\text{j}}, vs_{\text{i}^+}, vs_{\text{j}^+}, vs_{\text{j+2}}\}$$

$$\text{rhs} = \int \mathcal{F}[\text{i}, \text{j}]\ \$\mathcal{L}\ \text{/@}\ (C_{\text{i}}[0]\ C_{\text{i+1}}[0]\ C_{\text{j}}[0]\ C_{\text{j+1}}[1]\ C_{\text{j+2}}[0])\ d\{vs_{\text{i}}, vs_{\text{j}}, vs_{\text{i}^+}, vs_{\text{j}^+}, vs_{\text{j+2}}\};$$

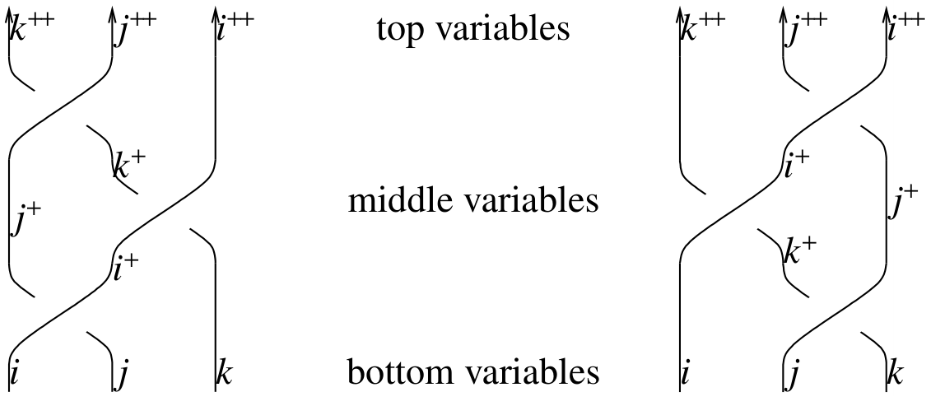
$$\text{lhs} == \text{rhs}\ \text{//}\ \text{HL}$$

(Alt) Out[ ]=

$$-\mathbb{i}\ T_1\ T_2\ \mathbb{E}\left[\frac{\in}{2} + p_{1,2+i}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j} + \in p_{3,3+j}\,\pi_{3,j}\right]$$

(Alt) Out[ ]=

True



(Alt) In[ ]:=

$$\text{Short}\Big[\text{lhs} = \int (\mathcal{F}[i, j, k]\ \$\mathcal{L} \text{ /@ } (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1]))\ \mathbb{d}\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\}\Big]$$

$$\text{rhs} = \int (\mathcal{F}[i, j, k]\ \$\mathcal{L} \text{ /@ } (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1]))\ \mathbb{d}\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\};$$

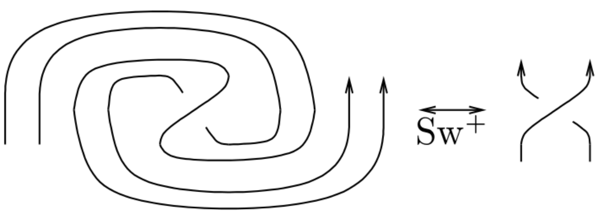
lhs == rhs // HL

(Alt) Out[ ]//Short=

$$T_1^3\, T_2^3\, \mathbb{E}\left[\frac{3\,\epsilon}{2} + \ll 153 \gg\right]$$

(Alt) Out[ ]=

True



(Alt) In[ ]:=

$$\text{Short}\Big[\text{lhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L} \text{ /@ } (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}\}\Big]$$

$$\text{rhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L} \text{ /@ } (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}\};$$

lhs == rhs // HL

(Alt) Out[ ]//Short=

$$T_1 \ll 1 \gg\ \ll 1 \gg \mathbb{E}\Big[\frac{\epsilon}{2} + T_1\, p_{1,3+i}\, \pi_{1,i} + (1 - T_1)\, p_{1 \ll 1 \gg 3 \ll 1 \gg \ll 1 \gg}\, \pi_{1,i} + \ll 35 \gg + \frac{\ll 1 \gg}{\ll 1 \gg} - \epsilon\, T_2\, p_{2,3+i}\, p_{3,3+j}\, \pi_{2,i}\, \pi_{3,j} - \epsilon\, p_{2,3+j}\, p_{3,3+j}\, \pi_{2,i}\, \pi_{3,j}\Big]$$

(Alt) Out[ ]=

True