Pensieve header: Experiments in 2026: Alternative integrands for θ.

Initialization

(Alt) In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< "IType(SimplifiedFormatting).m"];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

Minor utilities

(Alt) In[ ]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CCF[ℰ_] := Factor[ℰ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, ϵ | (p | x | π | ξ | g)__, F] /. F → CCF];
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

(Alt) In[ ]:=
```
ℒ_i_[K_] := CF[ℒ_i /@ Features[K][[2]]]
vs[K_] := Flatten@Table[{p_ν,i, x_ν,i}, {ν, 3}, {i, Features[K][[1]]}]
```

(Alt) In[ ]:=
```
{p*, x*, π*, ξ*} = {π, ξ, p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_vi__)* := (u*)_vi;
```

(Alt) In[ ]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

g2px and px2g

Modified from pensieve://Talks/Geneva-2408/DataConversions.nb

(Alt) In[ ]:=
```
g2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {g_{i_,j_} :> λ p_i x_j}] /. {λ^k_. :> 1 / k!}]
gν2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {g_{ν_,i_,j_} :> λ_ν p_{ν,i} x_{ν,j}}] /. {λ_^k_. :> 1 / k!}]
```

(Alt) In[ ]:=
```
Zip_{}[ℰ_] := ℰ;
Zip_{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip_{ξs}, ξ] /. f_. ξ^d_. :> (D[f, {ξ*, d}])) /. ξ* → 0
```

(Alt) In[ ]:=

```
px2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
   ps = Union[Cases[ℰ, p_, ∞]]; xs = Union[Cases[ℰ, x_, ∞]];
   Q = Sum[p0* x0* g_{p0[[2]],x0[[2]]}, {p0, ps}, {x0, xs}];
   Expand[Zip_{ps⋃xs}[ℰ ⅇ^Q]]
  ]
pxv2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
   ps = Union[Cases[ℰ, p__, ∞]]; xs = Union[Cases[ℰ, x__, ∞]];
   Q = Sum[p0* x0* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
   Expand[Zip_{ps⋃xs}[ℰ ⅇ^Q] /. g_{α_,β_,i_,j_} :→ If[α === β, g_{α,i,j}, 0]]
  ]
```

gRules

From pensieve://Talks/MonteVerita-2604/Theta.nb

(Alt) In[ ]:=

```
gR_{s_,i_,j_} := {
   g_{jβ_} :→ g_{j⁺β} + δ_{jβ}, g_{iβ_} :→ T^s g_{i⁺β} + (1 - T^s) g_{j⁺β} + δ_{iβ},
   g_{α_i⁺} :→ T^s g_{αi} + δ_{αi⁺}, g_{α_j⁺} :→ g_{αj} + (1 - T^s) g_{αi} + δ_{αj⁺},
   g_{ν_jβ_} :→ g_{νj⁺β} + δ_{jβ}, g_{ν_iβ_} :→ T_ν^s g_{νi⁺β} + (1 - T_ν^s) g_{νj⁺β} + δ_{iβ},
   g_{ν_α_i⁺} :→ T_ν^s g_{ναi} + δ_{αi⁺}, g_{ν_α_j⁺} :→ g_{ναj} + (1 - T_ν^s) g_{ναi} + δ_{αj⁺}
  }
gR_{i_} := {g_{i,β_} :→ g_{i⁺,β} + δ_{i,β}, g_{α_,i⁺} :→ g_{α,i} + δ_{α,i⁺},
   g_{ν_,i,β_} :→ g_{ν,i⁺,β} + δ_{i,β}, g_{ν_,α_,i⁺} :→ g_{ν,α,i} + δ_{α,i⁺}
  }
```

Inverse gRules written locally:

(Alt) In[ ]:=

```
igR_{s_,i_,j_} := {
   g_{j⁺,β_} :→ g_{j,β} - δ_{j,β},  g_{i⁺,β_} :→ T^-s g_{iβ} - (T^-s - 1) g_{j⁺β} - T^-s δ_{iβ},
   g_{α_,i} :→ T^-s (g_{α,i⁺} - δ_{α,i⁺}), g_{α_,j} :→ g_{α,j⁺} - (1 - T^s) g_{α,i} - δ_{α,j⁺},
   g_{ν_,j⁺,β_} :→ g_{ν,j,β} - δ_{j,β},  g_{ν_,i⁺,β_} :→ T_ν^-s g_{νiβ} - (T_ν^-s - 1) g_{νj⁺β} - T_ν^-s δ_{iβ},
   g_{ν_,α_,i} :→ T_ν^-s (g_{ν,α,i⁺} - δ_{α,i⁺}), g_{ν_,α_,j} :→ g_{ν,α,j⁺} - (1 - T_ν^s) g_{ν,α,i} - δ_{α,j⁺}
  }
```

The Base Lagrangian

From pensieve://Talks/Rank2.nb.

(Alt) In[∘]:=
pdf

```
T3 = T1 T2;  i_+ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /. {π_{is__} :→ B^-1 π_{is}, x_{is__} :→ B^-1 x_{is}, p_{is__} :→ B p_{is}} /. ϵ B^{b_} /; b < 0 → 0 /.
    B → 1) &;
```

(Alt) In[∘]:=

```
ℒ1[X_{i_,j_}[s_]] := T3^s 𝔼[CF@Plus[
    Sum_{ν=1}^{3} (x_{νi} (p_{νi+} - p_{νi}) + x_{νj} (p_{νj+} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi+} - p_{νj+})),
    (T1^s - 1) p_{3j} x_{1i} (T2^s x_{2i} - x_{2j}),
    ϵ s (T3^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T2^s - 1),
    ϵ s (1/2 + T2^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} - (T2^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} +
      (T3^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} + 2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} - T2^s p_{2j} p_{3j} x_{2i} x_{3j} +
      ((T1^s - 1) p_{1j} x_{1i} (T2^{2s} p_{2j} x_{2i} - T2^s p_{2j} x_{2j} - (T2^s + 1) (T3^s - 1) p_{3j} x_{3i} + T2^s p_{3j} x_{3j}) +
        (T3^s - 1) p_{3j} x_{3i} (1 - T2^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T2^s - 2) p_{2j} x_{2j})) / (T2^s - 1))]]
ℒ1[C_{i_}[φ_]] := T3^φ 𝔼[Sum_{ν=1}^{3} x_{νi} (p_{νi+} - p_{νi}) + ϵ φ (p_{3i} x_{3i} - 1/2)]
```

In[∘]:= `$ℒ = ℒ1;`

The Trefoil

In[∘]:= `K = Mirror@Knot[3, 1]; Features[K]`

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$\text{Features}[7, C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]]$

In[∘]:= `{vs[K], $ℒ[K]}`

Out[∘]=

$\{\{p_{1,1}, x_{1,1}, p_{1,2}, x_{1,2}, p_{1,3}, x_{1,3}, p_{1,4}, x_{1,4}, p_{1,5}, x_{1,5}, p_{1,6}, x_{1,6}, p_{1,7}, x_{1,7}, p_{2,1}, x_{2,1}, p_{2,2}, x_{2,2}, p_{2,3}, x_{2,3}, p_{2,4}, x_{2,4}, p_{2,5}, x_{2,5}, p_{2,6}, x_{2,6}, p_{2,7}, x_{2,7}, p_{3,1}, x_{3,1}, p_{3,2}, x_{3,2}, p_{3,3}, x_{3,3}, p_{3,4}, x_{3,4}, p_{3,5}, x_{3,5}, p_{3,6}, x_{3,6}, p_{3,7}, x_{3,7}\},$

$T_3^2\,\mathbb{E}\Big[2\epsilon - p_{1,1} x_{1,1} + T_1 p_{1,1^+} x_{1,1} + (1-T_1) p_{1,5^+} x_{1,1} - p_{1,2} x_{1,2} + p_{1,2^+} x_{1,2} - p_{1,3} x_{1,3} + T_1 p_{1,3^+} x_{1,3} + (1-T_1) p_{1,7^+} x_{1,3} - p_{1,4} x_{1,4} + p_{1,4^+} x_{1,4} - p_{1,5} x_{1,5} + p_{1,5^+} x_{1,5} - p_{1,6} x_{1,6} + (1-T_1) p_{1,2^+} x_{1,6} + T_1 p_{1,6^+} x_{1,6} - p_{1,7} x_{1,7} + p_{1,7^+} x_{1,7} - p_{2,1} x_{2,1} + T_2 p_{2,1^+} x_{2,1} + (1-T_2) p_{2,5^+} x_{2,1} + \epsilon T_2 p_{1,1} p_{2,5} x_{1,1} x_{2,1} + \frac{\epsilon (-1+T_1) T_2^2 p_{1,5} p_{2,5} x_{1,1} x_{2,1}}{-1+T_2} + (-1+T_1) T_2 p_{3,5} x_{1,1} x_{2,1} - p_{2,2} x_{2,2} + p_{2,2^+} x_{2,2} - \frac{\epsilon (-1+T_1) T_2 p_{1,2} p_{2,2} x_{1,6} x_{2,2}}{-1+T_2} - \epsilon p_{1,6} p_{2,2} x_{1,6} x_{2,2} + (1-T_1) p_{3,2} x_{1,6} x_{2,2} - p_{2,3} x_{2,3} + T_2 p_{2,3^+} x_{2,3} + (1-T_2) p_{2,7^+} x_{2,3} + \epsilon T_2 p_{1,3} p_{2,7} x_{1,3} x_{2,3} + \frac{\epsilon (-1+T_1) T_2^2 p_{1,7} p_{2,7} x_{1,3} x_{2,3}}{-1+T_2} + (-1+T_1) T_2 p_{3,7} x_{1,3} x_{2,3} - p_{2,4} x_{2,4} + p_{2,4^+} x_{2,4} - p_{2,5} x_{2,5} + p_{2,5^+} x_{2,5} - \epsilon p_{1,1} p_{2,5} x_{1,1} x_{2,5} - \frac{\epsilon (-1+T_1) T_2 p_{1,5} p_{2,5} x_{1,1} x_{2,5}}{-1+T_2} + (1-T_1) p_{3,5} x_{1,1} x_{2,5} - p_{2,6} x_{2,6} + (1-T_2) p_{2,2^+} x_{2,6} +$

$$T_2\, p_{2,6^+}\, x_{2,6} + \frac{\in\ (-1+T_1)\ T_2^2\, p_{1,2}\, p_{2,2}\, x_{1,6}\, x_{2,6}}{-1+T_2} + \in T_2\, p_{1,6}\, p_{2,2}\, x_{1,6}\, x_{2,6} + (-1+T_1)\ T_2\, p_{3,2}\, x_{1,6}\, x_{2,6} -$$
$$p_{2,7}\, x_{2,7} + p_{2,7^+}\, x_{2,7} - \in p_{1,3}\, p_{2,7}\, x_{1,3}\, x_{2,7} - \frac{\in\ (-1+T_1)\ T_2\, p_{1,7}\, p_{2,7}\, x_{1,3}\, x_{2,7}}{-1+T_2} +$$
$$(1-T_1)\ p_{3,7}\, x_{1,3}\, x_{2,7} + \frac{\in\ (-1+T_3)\ p_{1,5}\, p_{2,1}\, x_{3,1}}{-1+T_2} - \frac{\in\ (-1+T_3)\ p_{1,5}\, p_{2,5}\, x_{3,1}}{-1+T_2} -$$
$$p_{3,1}\, x_{3,1} - \in p_{3,1}\, x_{3,1} + \frac{\in\ (-1+T_3)\ p_{3,5}\, x_{3,1}}{-1+T_2} + T_3\, p_{3,1^+}\, x_{3,1} + (1-T_3)\ p_{3,5^+}\, x_{3,1} -$$
$$\frac{\in T_2\ (-1+T_3)\ p_{1,1}\, p_{3,5}\, x_{1,1}\, x_{3,1}}{-1+T_2} - \frac{\in\ (-1+T_1)\ (1+T_2)\ (-1+T_3)\ p_{1,5}\, p_{3,5}\, x_{1,1}\, x_{3,1}}{-1+T_2} +$$
$$\in\ (1-T_2)\ p_{2,5}\, p_{3,1}\, x_{2,1}\, x_{3,1} + \in\ (-1+T_3)\ p_{2,5}\, p_{3,5}\, x_{2,1}\, x_{3,1} + 2 \in p_{2,5}\, p_{3,1}\, x_{2,5}\, x_{3,1} +$$
$$\frac{\in\ (-1+T_3)\ p_{2,1}\, p_{3,5}\, x_{2,5}\, x_{3,1}}{-1+T_2} + \frac{\in\ (-2+T_2)\ (-1+T_3)\ p_{2,5}\, p_{3,5}\, x_{2,5}\, x_{3,1}}{-1+T_2} - p_{3,2}\, x_{3,2} +$$
$$p_{3,2^+}\, x_{3,2} + \frac{\in\ (-1+T_1)\ T_2\, p_{1,2}\, p_{3,2}\, x_{1,6}\, x_{3,2}}{-1+T_2} + \in p_{1,6}\, p_{3,2}\, x_{1,6}\, x_{3,2} - \in T_2\, p_{2,2}\, p_{3,2}\, x_{2,6}\, x_{3,2} -$$
$$\in p_{2,6}\, p_{3,2}\, x_{2,6}\, x_{3,2} + \frac{\in\ (-1+T_3)\ p_{1,7}\, p_{2,3}\, x_{3,3}}{-1+T_2} - \frac{\in\ (-1+T_3)\ p_{1,7}\, p_{2,7}\, x_{3,3}}{-1+T_2} - p_{3,3}\, x_{3,3} - \in p_{3,3}\, x_{3,3} +$$
$$\frac{\in\ (-1+T_3)\ p_{3,7}\, x_{3,3}}{-1+T_2} + T_3\, p_{3,3^+}\, x_{3,3} + (1-T_3)\ p_{3,7^+}\, x_{3,3} - \frac{\in T_2\ (-1+T_3)\ p_{1,3}\, p_{3,7}\, x_{1,3}\, x_{3,3}}{-1+T_2} -$$
$$\frac{\in\ (-1+T_1)\ (1+T_2)\ (-1+T_3)\ p_{1,7}\, p_{3,7}\, x_{1,3}\, x_{3,3}}{-1+T_2} + \in\ (1-T_2)\ p_{2,7}\, p_{3,3}\, x_{2,3}\, x_{3,3} +$$
$$\in\ (-1+T_3)\ p_{2,7}\, p_{3,7}\, x_{2,3}\, x_{3,3} + 2 \in p_{2,7}\, p_{3,3}\, x_{2,7}\, x_{3,3} + \frac{\in\ (-1+T_3)\ p_{2,3}\, p_{3,7}\, x_{2,7}\, x_{3,3}}{-1+T_2} +$$
$$\frac{\in\ (-2+T_2)\ (-1+T_3)\ p_{2,7}\, p_{3,7}\, x_{2,7}\, x_{3,3}}{-1+T_2} - p_{3,4}\, x_{3,4} - \in p_{3,4}\, x_{3,4} + p_{3,4^+}\, x_{3,4} - p_{3,5}\, x_{3,5} +$$
$$p_{3,5^+}\, x_{3,5} + \in p_{1,1}\, p_{3,5}\, x_{1,1}\, x_{3,5} + \frac{\in\ (-1+T_1)\ T_2\, p_{1,5}\, p_{3,5}\, x_{1,1}\, x_{3,5}}{-1+T_2} - \in p_{2,1}\, p_{3,5}\, x_{2,1}\, x_{3,5} -$$
$$\in T_2\, p_{2,5}\, p_{3,5}\, x_{2,1}\, x_{3,5} - \frac{\in\ (-1+T_3)\ p_{1,2}\, p_{2,2}\, x_{3,6}}{-1+T_2} + \frac{\in\ (-1+T_3)\ p_{1,2}\, p_{2,6}\, x_{3,6}}{-1+T_2} +$$
$$\frac{\in\ (-1+T_3)\ p_{3,2}\, x_{3,6}}{-1+T_2} - p_{3,6}\, x_{3,6} - \in p_{3,6}\, x_{3,6} + (1-T_3)\ p_{3,2^+}\, x_{3,6} + T_3\, p_{3,6^+}\, x_{3,6} -$$
$$\frac{\in\ (-1+T_1)\ (1+T_2)\ (-1+T_3)\ p_{1,2}\, p_{3,2}\, x_{1,6}\, x_{3,6}}{-1+T_2} - \frac{\in T_2\ (-1+T_3)\ p_{1,6}\, p_{3,2}\, x_{1,6}\, x_{3,6}}{-1+T_2} +$$
$$\frac{\in\ (-2+T_2)\ (-1+T_3)\ p_{2,2}\, p_{3,2}\, x_{2,2}\, x_{3,6}}{-1+T_2} + \frac{\in\ (-1+T_3)\ p_{2,6}\, p_{3,2}\, x_{2,2}\, x_{3,6}}{-1+T_2} + 2 \in p_{2,2}\, p_{3,6}\, x_{2,2}\, x_{3,6} +$$
$$\in\ (-1+T_3)\ p_{2,2}\, p_{3,2}\, x_{2,6}\, x_{3,6} + \in\ (1-T_2)\ p_{2,2}\, p_{3,6}\, x_{2,6}\, x_{3,6} - p_{3,7}\, x_{3,7} + p_{3,7^+}\, x_{3,7} +$$
$$\in p_{1,3}\, p_{3,7}\, x_{1,3}\, x_{3,7} + \frac{\in\ (-1+T_1)\ T_2\, p_{1,7}\, p_{3,7}\, x_{1,3}\, x_{3,7}}{-1+T_2} - \in p_{2,3}\, p_{3,7}\, x_{2,3}\, x_{3,7} - \in T_2\, p_{2,7}\, p_{3,7}\, x_{2,3}\, x_{3,7} \Big]\Big\}$$

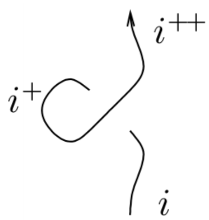
In[•]:= $\int \$\mathcal{L}[K]\ d\mathbb{1}\ \mathrm{vs}[K]$

Out[∘]=

$$-\frac{\mathbb{i}\, T_1^2\, T_2^2\, \mathbb{E}\left[\frac{\epsilon\, \left(1-T_1+T_1^2-T_2-T_1^3\, T_2+T_2^2+T_1^4\, T_2^2-T_1\, T_2^3-T_1^4\, T_2^3+T_1^2\, T_2^4-T_1^3\, T_2^4+T_1^4\, T_2^4\right)}{\left(1-T_1+T_1^2\right)\, \left(1-T_2+T_2^2\right)\, \left(1-T_1\, T_2+T_1^2\, T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\, \left(1-T_2+T_2^2\right)\, \left(1-T_1\, T_2+T_1^2\, T_2^2\right)}$$

(Alt) In[∘]:=

```
𝓕[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
vs_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
```



In[∘]:=

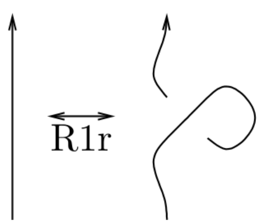
```
lhs = ∫𝓕[i] $ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{vs_i, vs_{i+}, vs_{i+2}}
rhs = ∫𝓕[i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{vs_i, vs_{i+}, vs_{i+2}};
lhs == rhs // HL
```

Out[∘]=

$-\mathbb{i}\, \mathbb{E}[p_{1,3+i}\, \pi_{1,i} + p_{2,3+i}\, \pi_{2,i} + p_{3,3+i}\, \pi_{3,i}]$

Out[∘]=

True



In[∘]:=

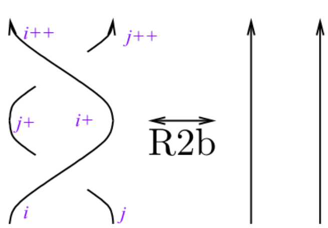
```
lhs = ∫𝓕[i] $ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) ⅆ{vs_i, vs_{i+}, vs_{i+2}}
rhs = ∫𝓕[i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{vs_i, vs_{i+}, vs_{i+2}};
lhs == rhs // HL
```

Out[∘]=

$-\mathbb{i}\, \mathbb{E}[p_{1,3+i}\, \pi_{1,i} + p_{2,3+i}\, \pi_{2,i} + p_{3,3+i}\, \pi_{3,i}]$

Out[∘]=

True



Territory: $(x \mid p)_{(i\mid j)^+}$, $x_{i\mid j}$.

In[∘]:=

$\mathtt{lhs} = \int \mathcal{F}[\mathtt{i}, \mathtt{j}]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ d\!\!l\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\}$

$\mathtt{rhs} = \int \mathcal{F}[\mathtt{i}, \mathtt{j}]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ d\!\!l\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\};$

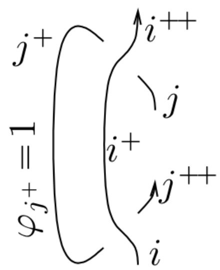
`lhs == rhs // HL`

Out[∘]=

$\mathbb{E}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}]$

Out[∘]=

True



In[∘]:=

$\mathtt{lhs} = \int \mathcal{F}[\mathtt{i}, \mathtt{j}]\ \$\mathcal{L}\ /@\ (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ d\!\!l\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\}$

$\mathtt{rhs} = \int \mathcal{F}[\mathtt{i}, \mathtt{j}]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ d\!\!l\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\};$

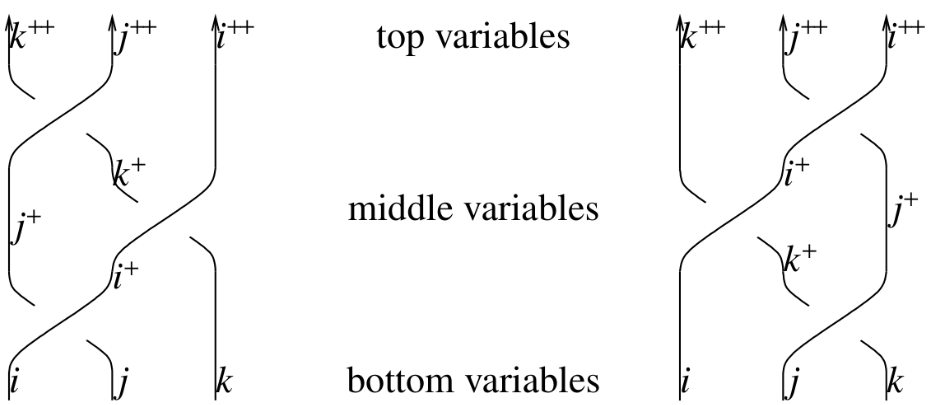
`lhs == rhs // HL`

Out[∘]=

$-\mathbb{i}\ T_1\ T_2\ \mathbb{E}\left[\frac{\epsilon}{2} + p_{1,2+i}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j} + \epsilon\ p_{3,3+j}\,\pi_{3,j}\right]$

Out[∘]=

True

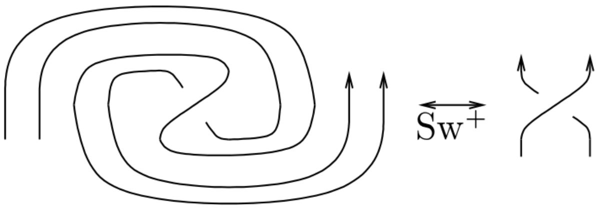


In[∘]:=

$\mathtt{Short}\left[\mathtt{lhs} = \int (\mathcal{F}[\mathtt{i}, \mathtt{j}, \mathtt{k}]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1]))\ d\!\!l\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\}\right]$

$\mathtt{rhs} = \int (\mathcal{F}[\mathtt{i}, \mathtt{j}, \mathtt{k}]\ \$\mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1]))\ d\!\!l\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\};$

`lhs == rhs // HL`




```
Short[lhs =
  ∫𝓕[i, j] $ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{i+2}, vs_{j+2}}]
rhs =
  ∫𝓕[i, j] $ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{i+2}, vs_{j+2}};
lhs == rhs // HL
```

The pPush Lagrangian

(Alt) In[ ]:=

```
pPush[ω_. 𝔼[L_]] := Module[{B, L1, Q, P0, P1a, P1b, θ0},
   L1 = CoefficientRules[B L /. {p_{αβ__} :→ B p_{αβ}, x_{αβ__} :→ B^-1 x_{αβ}}, {ϵ, B}];
   {ω, Q = {0, 1} /. L1, P0 = {0, 0} /. L1, P1a = {1, 2} /. L1, P1b = {1, 1} /. L1};
   θ0 = CF[
     px∨2g[CF[(P0 /. Thread[{s, i, j} → {s0, i0, j0}])
          (P1a /. Thread[{s, i, j} → {s1, i1, j1}])
        ]] /. gR_{s0,i0,j0} /. gR_{s1,i1,j1} /. {δ_{j1,i0} | δ_{j0,i1} | δ_{i1,j0} → 0, δ_{i1,i0} | δ_{j1,j0} → B}
    ];
   θ1 = List @@ Factor[g∨2px[θ0 /. B → 0]];
   {Y0, Y1} = {Times @@ Select[θ1, FreeQ[s1 | i1 | j1]],
      Times @@ Select[θ1, FreeQ[s0 | i0 | j0]]} /. {s0 | s1 → s, i0 | i1 → i, j0 | j1 → j};
   Times @@ θ1 == Y0 Y1;
   degθ = g∨2px[Coefficient[θ0, B] /. {s0 | s1 → s, i0 | i1 → i, j0 | j1 → j}];
   ω 𝔼[CF@Plus[
       Q, Y0, ϵ Y1, ϵ degθ,
       ϵ g∨2px[px∨2g[P1b] /. gR_{s,i,j} /. {δ_{i,j} | δ_{j,i} → 0, δ_{i,i} | δ_{j,j} → 1}]
      ]]
  ];
pPush@ℒ_1[X_{i,j}[s]]
```

(Alt) Out[]=

$$(T_1 T_2)^s \mathbb{E}\Big[\frac{s\in}{2} - p_{1,i} x_{1,i} + T_1^s p_{1,1+i} x_{1,i} + (1-T_1^s) p_{1,1+j} x_{1,i} - p_{1,j} x_{1,j} + p_{1,1+j} x_{1,j} - p_{2,i} x_{2,i} + T_2^s p_{2,1+i} x_{2,i} - s\in T_2^s p_{2,1+i} x_{2,i} + (1-T_2^s) p_{2,1+j} x_{2,i} + s\in T_1^s T_2^s p_{1,1+i} p_{2,1+j} x_{1,i} x_{2,i} + \frac{s\in(-1+T_1^s) T_2^s p_{1,1+j} p_{2,1+j} x_{1,i} x_{2,i}}{-1+T_2^s} + (-1+T_1^s) T_2^s p_{3,1+j} x_{1,i} x_{2,i} - p_{2,j} x_{2,j} + p_{2,1+j} x_{2,j} + s\in p_{2,1+j} x_{2,j} - s\in T_1^s p_{1,1+i} p_{2,1+j} x_{1,i} x_{2,j} - \frac{s\in(-1+T_1^s) p_{1,1+j} p_{2,1+j} x_{1,i} x_{2,j}}{-1+T_2^s} + (1-T_1^s) p_{3,1+j} x_{1,i} x_{2,j} + \frac{s\in T_2^s(-1+(T_1 T_2)^s) p_{1,1+j} p_{2,1+i} x_{3,i}}{-1+T_2^s} - \frac{s\in T_2^s(-1+(T_1 T_2)^s) p_{1,1+j} p_{2,1+j} x_{3,i}}{-1+T_2^s} - p_{3,i} x_{3,i} + (T_1 T_2)^s p_{3,1+i} x_{3,i} + s\in (T_1 T_2)^s p_{3,1+i} x_{3,i} + (1-(T_1 T_2)^s) p_{3,1+j} x_{3,i} - \frac{s\in T_2^s(-1+(T_1 T_2)^s) p_{3,1+j} x_{3,i}}{-1+T_2^s} - \frac{s\in T_1^s T_2^s(-1+(T_1 T_2)^s) p_{1,1+i} p_{3,1+j} x_{1,i} x_{3,i}}{-1+T_2^s} + \frac{s\in(-1+T_1^s) T_2^s(-1+(T_1 T_2)^s) p_{1,1+j} p_{3,1+j} x_{1,i} x_{3,i}}{-1+T_2^s} - s\in (T_1 T_2)^s(-1+T_2^s) p_{2,1+j} p_{3,1+i} x_{2,i} x_{3,i} + s\in T_2^s(-1+(T_1 T_2)^s) p_{2,1+j} p_{3,1+j} x_{2,i} x_{3,i} + 2 s\in (T_1 T_2)^s p_{2,1+j} p_{3,1+i} x_{2,j} x_{3,i} + \frac{s\in T_2^s(-1+(T_1 T_2)^s) p_{2,1+i} p_{3,1+j} x_{2,j} x_{3,i}}{-1+T_2^s} - \frac{s\in(-1+2T_2^s)(-1+(T_1 T_2)^s) p_{2,1+j} p_{3,1+j} x_{2,j} x_{3,i}}{-1+T_2^s} - p_{3,j} x_{3,j} + p_{3,1+j} x_{3,j} + s\in T_1^s p_{1,1+i} p_{3,1+j} x_{1,i} x_{3,j} + \frac{s\in(-1+T_1^s) p_{1,1+j} p_{3,1+j} x_{1,i} x_{3,j}}{-1+T_2^s} - s\in T_2^s p_{2,1+i} p_{3,1+j} x_{2,i} x_{3,j} - s\in p_{2,1+j} p_{3,1+j} x_{2,i} x_{3,j}\Big]$$

(Alt) In[]:=

$$\mathcal{L}_1[C_i[\varphi]]$$

(Alt) Out[]=

$$(T_1 T_2)^{\varphi} \mathbb{E}\Big[(-p_{1,i}+p_{1,1+i}) x_{1,i} + (-p_{2,i}+p_{2,1+i}) x_{2,i} + (-p_{3,i}+p_{3,1+i}) x_{3,i} + \in\varphi\left(-\frac{1}{2} + p_{3,i} x_{3,i}\right)\Big]$$

(Alt) In[]:=

$$\text{g}\nu\text{2px}\left[\text{px}\nu\text{2g}\left[-\frac{1}{2} + p_{3,i} x_{3,i}\right] \text{ /. gR}_i \text{ /. } \{\delta_{i,i} \to 1\}\right]$$

(Alt) Out[]=

$$\frac{1}{2} + p_{3,1+i} x_{3,i}$$

(Alt) In[]:=

```
𝓛_pPush[X_{i_,j_}[s_]] = pPush@𝓛_1[X_{i,j}[s]];
𝓛_pPush[C_{i_}[φ_]] := T_3^φ 𝔼[Sum[x_{ν,i} (p_{ν,i+1} − p_{ν,i}), {ν, 3}] + ∈ φ (1/2 + p_{3,i+1} x_{3,i})];
$𝓛 = 𝓛_pPush;
```

(Alt) In[ ]:=

```
$ℒ[X_{i,j}[s]]
$ℒ[C_i[φ]]
```

(Alt) Out[ ]=

$$(T_1 T_2)^s \mathbb{E}\Big[\frac{s \in}{2} - p_{1,i} x_{1,i} + T_1^s p_{1,1+i} x_{1,i} + (1 - T_1^s) p_{1,1+j} x_{1,i} - p_{1,j} x_{1,j} + p_{1,1+j} x_{1,j} - p_{2,i} x_{2,i} + T_2^s p_{2,1+i} x_{2,i} - s \in T_2^s p_{2,1+i} x_{2,i} + (1 - T_2^s) p_{2,1+j} x_{2,i} + s \in T_1^s T_2^s p_{1,1+i} p_{2,1+j} x_{1,i} x_{2,i} + \frac{s \in (-1 + T_1^s) T_2^s p_{1,1+j} p_{2,1+j} x_{1,i} x_{2,i}}{-1 + T_2^s} + (-1 + T_1^s) T_2^s p_{3,1+j} x_{1,i} x_{2,i} - p_{2,j} x_{2,j} + p_{2,1+j} x_{2,j} + s \in p_{2,1+j} x_{2,j} - s \in T_1^s p_{1,1+i} p_{2,1+j} x_{1,i} x_{2,j} - \frac{s \in (-1 + T_1^s) p_{1,1+j} p_{2,1+j} x_{1,i} x_{2,j}}{-1 + T_2^s} + (1 - T_1^s) p_{3,1+j} x_{1,i} x_{2,j} + \frac{s \in T_2^s (-1 + (T_1 T_2)^s) p_{1,1+j} p_{2,1+i} x_{3,i}}{-1 + T_2^s} - \frac{s \in T_2^s (-1 + (T_1 T_2)^s) p_{1,1+j} p_{2,1+j} x_{3,i}}{-1 + T_2^s} - p_{3,i} x_{3,i} + (T_1 T_2)^s p_{3,1+i} x_{3,i} + s \in (T_1 T_2)^s p_{3,1+i} x_{3,i} + (1 - (T_1 T_2)^s) p_{3,1+j} x_{3,i} - \frac{s \in T_2^s (-1 + (T_1 T_2)^s) p_{3,1+j} x_{3,i}}{-1 + T_2^s} - \frac{s \in T_1^s T_2^s (-1 + (T_1 T_2)^s) p_{1,1+i} p_{3,1+j} x_{1,i} x_{3,i}}{-1 + T_2^s} + \frac{s \in (-1 + T_1^s) T_2^s (-1 + (T_1 T_2)^s) p_{1,1+j} p_{3,1+j} x_{1,i} x_{3,i}}{-1 + T_2^s} - s \in (T_1 T_2)^s (-1 + T_2^s) p_{2,1+j} p_{3,1+i} x_{2,i} x_{3,i} + s \in T_2^s (-1 + (T_1 T_2)^s) p_{2,1+j} p_{3,1+j} x_{2,i} x_{3,i} + 2 s \in (T_1 T_2)^s p_{2,1+j} p_{3,1+i} x_{2,j} x_{3,i} + \frac{s \in T_2^s (-1 + (T_1 T_2)^s) p_{2,1+i} p_{3,1+j} x_{2,j} x_{3,i}}{-1 + T_2^s} - \frac{s \in (-1 + 2 T_2^s) (-1 + (T_1 T_2)^s) p_{2,1+j} p_{3,1+j} x_{2,j} x_{3,i}}{-1 + T_2^s} - p_{3,j} x_{3,j} + p_{3,1+j} x_{3,j} + s \in T_1^s p_{1,1+i} p_{3,1+j} x_{1,i} x_{3,j} + \frac{s \in (-1 + T_1^s) p_{1,1+j} p_{3,1+j} x_{1,i} x_{3,j}}{-1 + T_2^s} - s \in T_2^s p_{2,1+i} p_{3,1+j} x_{2,i} x_{3,j} - s \in p_{2,1+j} p_{3,1+j} x_{2,i} x_{3,j}\Big]$$

(Alt) Out[ ]=

$$(T_1 T_2)^\varphi \mathbb{E}\Big[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-p_{2,i} + p_{2,1+i}) x_{2,i} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + \in \varphi \left(\frac{1}{2} + p_{3,1+i} x_{3,i}\right)\Big]$$

(Alt) In[ ]:=

```
Collect[$ℒ[X_{i,j}[s]][[2, 1]], ∈]
Collect[$ℒ[C_i[φ]][[2, 1]], ∈]
```

(Alt) Out[ ]=

$$-p_{1,i}\,x_{1,i}+T_1^s\,p_{1,1+i}\,x_{1,i}+(1-T_1^s)\,p_{1,1+j}\,x_{1,i}-p_{1,j}\,x_{1,j}+p_{1,1+j}\,x_{1,j}-p_{2,i}\,x_{2,i}+T_2^s\,p_{2,1+i}\,x_{2,i}+(1-T_2^s)\,p_{2,1+j}\,x_{2,i}+(-1+T_1^s)\,T_2^s\,p_{3,1+j}\,x_{1,i}\,x_{2,i}-p_{2,j}\,x_{2,j}+p_{2,1+j}\,x_{2,j}+(1-T_1^s)\,p_{3,1+j}\,x_{1,i}\,x_{2,j}-p_{3,i}\,x_{3,i}+(T_1T_2)^s\,p_{3,1+i}\,x_{3,i}+(1-(T_1T_2)^s)\,p_{3,1+j}\,x_{3,i}-p_{3,j}\,x_{3,j}+p_{3,1+j}\,x_{3,j}+$$
$$\in\Big(\frac{s}{2}-s\,T_2^s\,p_{2,1+i}\,x_{2,i}+s\,T_1^s\,T_2^s\,p_{1,1+i}\,p_{2,1+j}\,x_{1,i}\,x_{2,i}+\frac{s\,(-1+T_1^s)\,T_2^s\,p_{1,1+j}\,p_{2,1+j}\,x_{1,i}\,x_{2,i}}{-1+T_2^s}+s\,p_{2,1+j}\,x_{2,j}-s\,T_1^s\,p_{1,1+i}\,p_{2,1+j}\,x_{1,i}\,x_{2,j}-\frac{s\,(-1+T_1^s)\,p_{1,1+j}\,p_{2,1+j}\,x_{1,i}\,x_{2,j}}{-1+T_2^s}+\frac{s\,T_2^s\,(-1+(T_1T_2)^s)\,p_{1,1+j}\,p_{2,1+i}\,x_{3,i}}{-1+T_2^s}-\frac{s\,T_2^s\,(-1+(T_1T_2)^s)\,p_{1,1+j}\,p_{2,1+j}\,x_{3,i}}{-1+T_2^s}+s\,(T_1T_2)^s\,p_{3,1+i}\,x_{3,i}-\frac{s\,T_2^s\,(-1+(T_1T_2)^s)\,p_{3,1+j}\,x_{3,i}}{-1+T_2^s}-\frac{s\,T_1^s\,T_2^s\,(-1+(T_1T_2)^s)\,p_{1,1+i}\,p_{3,1+j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s}+\frac{s\,(-1+T_1^s)\,T_2^s\,(-1+(T_1T_2)^s)\,p_{1,1+j}\,p_{3,1+j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s}-s\,(T_1T_2)^s\,(-1+T_2^s)\,p_{2,1+j}\,p_{3,1+i}\,x_{2,i}\,x_{3,i}+s\,T_2^s\,(-1+(T_1T_2)^s)\,p_{2,1+j}\,p_{3,1+j}\,x_{2,i}\,x_{3,i}+2\,s\,(T_1T_2)^s\,p_{2,1+j}\,p_{3,1+i}\,x_{2,j}\,x_{3,i}+\frac{s\,T_2^s\,(-1+(T_1T_2)^s)\,p_{2,1+i}\,p_{3,1+j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s}-\frac{s\,(-1+2\,T_2^s)\,(-1+(T_1T_2)^s)\,p_{2,1+j}\,p_{3,1+j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s}+s\,T_1^s\,p_{1,1+i}\,p_{3,1+j}\,x_{1,i}\,x_{3,j}+\frac{s\,(-1+T_1^s)\,p_{1,1+j}\,p_{3,1+j}\,x_{1,i}\,x_{3,j}}{-1+T_2^s}-s\,T_2^s\,p_{2,1+i}\,p_{3,1+j}\,x_{2,i}\,x_{3,j}-s\,p_{2,1+j}\,p_{3,1+j}\,x_{2,i}\,x_{3,j}\Big)$$

(Alt) Out[ ]=

$$(-p_{1,i}+p_{1,1+i})\,x_{1,i}+(-p_{2,i}+p_{2,1+i})\,x_{2,i}+(-p_{3,i}+p_{3,1+i})\,x_{3,i}+\in\varphi\left(\frac{1}{2}+p_{3,1+i}\,x_{3,i}\right)$$

The Trefoil

(Alt) In[ ]:=

```
K = Mirror@Knot[3, 1]; {vs[K], $ℒ[K]}
```

KnotTheory: Loading precomputed data in PD4Knots`.

(Alt) Out[ ]=

$$\{\{p_{1,1},x_{1,1},p_{1,2},x_{1,2},p_{1,3},x_{1,3},p_{1,4},x_{1,4},p_{1,5},x_{1,5},p_{1,6},x_{1,6},p_{1,7},x_{1,7},p_{2,1},x_{2,1},p_{2,2},x_{2,2},p_{2,3},x_{2,3},p_{2,4},x_{2,4},p_{2,5},x_{2,5},p_{2,6},x_{2,6},p_{2,7},x_{2,7},p_{3,1},x_{3,1},p_{3,2},x_{3,2},p_{3,3},x_{3,3},p_{3,4},x_{3,4},p_{3,5},x_{3,5},p_{3,6},x_{3,6},p_{3,7},x_{3,7}\},$$
$$T_1^2\,T_2^2\,\mathbb{E}\Big[\in-p_{1,1}\,x_{1,1}+T_1\,p_{1,2}\,x_{1,1}+(1-T_1)\,p_{1,6}\,x_{1,1}-p_{1,2}\,x_{1,2}+p_{1,3}\,x_{1,2}-p_{1,3}\,x_{1,3}+T_1\,p_{1,4}\,x_{1,3}+(1-T_1)\,p_{1,8}\,x_{1,3}-p_{1,4}\,x_{1,4}+p_{1,5}\,x_{1,4}-p_{1,5}\,x_{1,5}+p_{1,6}\,x_{1,5}+(1-T_1)\,p_{1,3}\,x_{1,6}-p_{1,6}\,x_{1,6}+T_1\,p_{1,7}\,x_{1,6}-p_{1,7}\,x_{1,7}+p_{1,8}\,x_{1,7}-p_{2,1}\,x_{2,1}+T_2\,p_{2,2}\,x_{2,1}-\in T_2\,p_{2,2}\,x_{2,1}+(1-T_2)\,p_{2,6}\,x_{2,1}+\in T_1\,T_2\,p_{1,2}\,p_{2,6}\,x_{1,1}\,x_{2,1}+\frac{\in(-1+T_1)\,T_2\,p_{1,6}\,p_{2,6}\,x_{1,1}\,x_{2,1}}{-1+T_2}+(-1+T_1)\,T_2\,p_{3,6}\,x_{1,1}\,x_{2,1}-p_{2,2}\,x_{2,2}+p_{2,3}\,x_{2,2}+\in p_{2,3}\,x_{2,2}-\frac{\in(-1+T_1)\,p_{1,3}\,p_{2,3}\,x_{1,6}\,x_{2,2}}{-1+T_2}-\in T_1\,p_{1,7}\,p_{2,3}\,x_{1,6}\,x_{2,2}+(1-T_1)\,p_{3,3}\,x_{1,6}\,x_{2,2}-p_{2,3}\,x_{2,3}+T_2\,p_{2,4}\,x_{2,3}-\in T_2\,p_{2,4}\,x_{2,3}+(1-T_2)\,p_{2,8}\,x_{2,3}+$$

$$\in T_1 T_2 p_{1,4} p_{2,8} x_{1,3} x_{2,3} + \frac{\in (-1+T_1) T_2 p_{1,8} p_{2,8} x_{1,3} x_{2,3}}{-1+T_2} + (-1+T_1) T_2 p_{3,8} x_{1,3} x_{2,3} - p_{2,4} x_{2,4} +$$

$$p_{2,5} x_{2,4} - p_{2,5} x_{2,5} + p_{2,6} x_{2,5} + \in p_{2,6} x_{2,5} - \in T_1 p_{1,2} p_{2,6} x_{1,1} x_{2,5} - \frac{\in (-1+T_1) p_{1,6} p_{2,6} x_{1,1} x_{2,5}}{-1+T_2} +$$

$$(1-T_1) p_{3,6} x_{1,1} x_{2,5} + (1-T_2) p_{2,3} x_{2,6} - p_{2,6} x_{2,6} + T_2 p_{2,7} x_{2,6} - \in T_2 p_{2,7} x_{2,6} +$$

$$\frac{\in (-1+T_1) T_2 p_{1,3} p_{2,3} x_{1,6} x_{2,6}}{-1+T_2} + \in T_1 T_2 p_{1,7} p_{2,3} x_{1,6} x_{2,6} + (-1+T_1) T_2 p_{3,3} x_{1,6} x_{2,6} -$$

$$p_{2,7} x_{2,7} + p_{2,8} x_{2,7} + \in p_{2,8} x_{2,7} - \in T_1 p_{1,4} p_{2,8} x_{1,3} x_{2,7} - \frac{\in (-1+T_1) p_{1,8} p_{2,8} x_{1,3} x_{2,7}}{-1+T_2} +$$

$$(1-T_1) p_{3,8} x_{1,3} x_{2,7} + \frac{\in T_2 (-1+T_1 T_2) p_{1,6} p_{2,2} x_{3,1}}{-1+T_2} - \frac{\in T_2 (-1+T_1 T_2) p_{1,6} p_{2,6} x_{3,1}}{-1+T_2} -$$

$$p_{3,1} x_{3,1} + T_1 T_2 p_{3,2} x_{3,1} + \in T_1 T_2 p_{3,2} x_{3,1} + (1-T_1 T_2) p_{3,6} x_{3,1} - \frac{\in T_2 (-1+T_1 T_2) p_{3,6} x_{3,1}}{-1+T_2} -$$

$$\frac{\in T_1 T_2 (-1+T_1 T_2) p_{1,2} p_{3,6} x_{1,1} x_{3,1}}{-1+T_2} + \frac{\in (-1+T_1) T_2 (-1+T_1 T_2) p_{1,6} p_{3,6} x_{1,1} x_{3,1}}{-1+T_2} -$$

$$\in T_1 (-1+T_2) T_2 p_{2,6} p_{3,2} x_{2,1} x_{3,1} + \in T_2 (-1+T_1 T_2) p_{2,6} p_{3,6} x_{2,1} x_{3,1} + 2 \in T_1 T_2 p_{2,6} p_{3,2} x_{2,5} x_{3,1} +$$

$$\frac{\in T_2 (-1+T_1 T_2) p_{2,2} p_{3,6} x_{2,5} x_{3,1}}{-1+T_2} - \frac{\in (-1+2 T_2) (-1+T_1 T_2) p_{2,6} p_{3,6} x_{2,5} x_{3,1}}{-1+T_2} - p_{3,2} x_{3,2} +$$

$$p_{3,3} x_{3,2} + \frac{\in (-1+T_1) p_{1,3} p_{3,3} x_{1,6} x_{3,2}}{-1+T_2} + \in T_1 p_{1,7} p_{3,3} x_{1,6} x_{3,2} - \in p_{2,3} p_{3,3} x_{2,6} x_{3,2} -$$

$$\in T_2 p_{2,7} p_{3,3} x_{2,6} x_{3,2} + \frac{\in T_2 (-1+T_1 T_2) p_{1,8} p_{2,4} x_{3,3}}{-1+T_2} - \frac{\in T_2 (-1+T_1 T_2) p_{1,8} p_{2,8} x_{3,3}}{-1+T_2} -$$

$$p_{3,3} x_{3,3} + T_1 T_2 p_{3,4} x_{3,3} + \in T_1 T_2 p_{3,4} x_{3,3} + (1-T_1 T_2) p_{3,8} x_{3,3} - \frac{\in T_2 (-1+T_1 T_2) p_{3,8} x_{3,3}}{-1+T_2} -$$

$$\frac{\in T_1 T_2 (-1+T_1 T_2) p_{1,4} p_{3,8} x_{1,3} x_{3,3}}{-1+T_2} + \frac{\in (-1+T_1) T_2 (-1+T_1 T_2) p_{1,8} p_{3,8} x_{1,3} x_{3,3}}{-1+T_2} -$$

$$\in T_1 (-1+T_2) T_2 p_{2,8} p_{3,4} x_{2,3} x_{3,3} + \in T_2 (-1+T_1 T_2) p_{2,8} p_{3,8} x_{2,3} x_{3,3} + 2 \in T_1 T_2 p_{2,8} p_{3,4} x_{2,7} x_{3,3} +$$

$$\frac{\in T_2 (-1+T_1 T_2) p_{2,4} p_{3,8} x_{2,7} x_{3,3}}{-1+T_2} - \frac{\in (-1+2 T_2) (-1+T_1 T_2) p_{2,8} p_{3,8} x_{2,7} x_{3,3}}{-1+T_2} - p_{3,4} x_{3,4} -$$

$$\in p_{3,4} x_{3,4} + p_{3,5} x_{3,4} - p_{3,5} x_{3,5} + p_{3,6} x_{3,5} + \in T_1 p_{1,2} p_{3,6} x_{1,1} x_{3,5} + \frac{\in (-1+T_1) p_{1,6} p_{3,6} x_{1,1} x_{3,5}}{-1+T_2} -$$

$$\in T_2 p_{2,2} p_{3,6} x_{2,1} x_{3,5} - \in p_{2,6} p_{3,6} x_{2,1} x_{3,5} - \frac{\in T_2 (-1+T_1 T_2) p_{1,3} p_{2,3} x_{3,6}}{-1+T_2} +$$

$$\frac{\in T_2 (-1+T_1 T_2) p_{1,3} p_{2,7} x_{3,6}}{-1+T_2} + (1-T_1 T_2) p_{3,3} x_{3,6} - \frac{\in T_2 (-1+T_1 T_2) p_{3,3} x_{3,6}}{-1+T_2} -$$

$$p_{3,6} x_{3,6} + T_1 T_2 p_{3,7} x_{3,6} + \in T_1 T_2 p_{3,7} x_{3,6} + \frac{\in (-1+T_1) T_2 (-1+T_1 T_2) p_{1,3} p_{3,3} x_{1,6} x_{3,6}}{-1+T_2} -$$

$$\frac{\in T_1 T_2 (-1+T_1 T_2) p_{1,7} p_{3,3} x_{1,6} x_{3,6}}{-1+T_2} - \frac{\in (-1+2 T_2) (-1+T_1 T_2) p_{2,3} p_{3,3} x_{2,2} x_{3,6}}{-1+T_2} +$$

$$\frac{\in T_2\,(-1+T_1 T_2)\,p_{2,7}\,p_{3,3}\,x_{2,2}\,x_{3,6}}{-1+T_2} + 2\in T_1 T_2\,p_{2,3}\,p_{3,7}\,x_{2,2}\,x_{3,6} + \in T_2\,(-1+T_1 T_2)\,p_{2,3}\,p_{3,3}\,x_{2,6}\,x_{3,6} -$$
$$\in T_1\,(-1+T_2)\,T_2\,p_{2,3}\,p_{3,7}\,x_{2,6}\,x_{3,6} - p_{3,7}\,x_{3,7} + p_{3,8}\,x_{3,7} + \in T_1\,p_{1,4}\,p_{3,8}\,x_{1,3}\,x_{3,7} +$$
$$\left.\left.\frac{\in(-1+T_1)\,p_{1,8}\,p_{3,8}\,x_{1,3}\,x_{3,7}}{-1+T_2} - \in T_2\,p_{2,4}\,p_{3,8}\,x_{2,3}\,x_{3,7} - \in p_{2,8}\,p_{3,8}\,x_{2,3}\,x_{3,7}\right]\right\}$$

(Alt) In[ ]:=

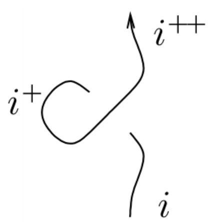
```
∫$ℒ[K] ⅆ vs[K]
```

(Alt) Out[ ]=

$$-\frac{\mathbb{i}\,T_1^2\,T_2^2\,\mathbb{E}\left[\frac{\in\left(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}$$

(Alt) In[ ]:=

```
ℱ[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
vs_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
```



(Alt) In[ ]:=

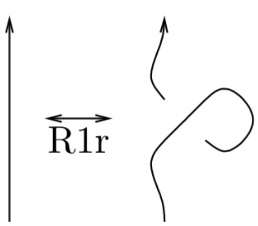
```
lhs = ∫ℱ[i] $ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{vs_i, vs_{i⁺}, vs_{i+2}}
rhs = ∫ℱ[i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{vs_i, vs_{i⁺}, vs_{i+2}};
lhs == rhs // HL
```

(Alt) Out[ ]=

$$-\mathbb{i}\,\mathbb{E}\left[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}\right]$$

(Alt) Out[ ]=

True



(Alt) In[ ]:=

$$\mathtt{lhs} = \int \mathcal{F}[\mathtt{i}]\ \$\mathcal{L}\ /@\ (X_{i,i+2}[1]\ C_{i+1}[-1])\ d\{vs_i,\ vs_{i^+},\ vs_{i+2}\}$$

$$\mathtt{rhs} = \int \mathcal{F}[\mathtt{i}]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{vs_i,\ vs_{i^+},\ vs_{i+2}\};$$

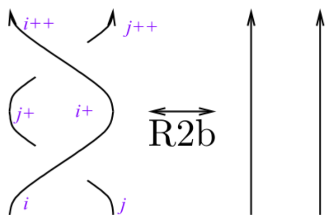
$$\mathtt{lhs} == \mathtt{rhs}\ //\ \mathtt{HL}$$

(Alt) Out[ ]=

$$-\mathbb{i}\ \mathbb{E}[p_{1,3+i}\ \pi_{1,i} + p_{2,3+i}\ \pi_{2,i} + p_{3,3+i}\ \pi_{3,i}]$$

(Alt) Out[ ]=

True



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

(Alt) In[ ]:=

$$\mathtt{lhs} = \int \mathcal{F}[\mathtt{i},\ \mathtt{j}]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ d\{vs_i,\ vs_j,\ vs_{i^+},\ vs_{j^+}\}$$

$$\mathtt{rhs} = \int \mathcal{F}[\mathtt{i},\ \mathtt{j}]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ d\{vs_i,\ vs_j,\ vs_{i^+},\ vs_{j^+}\};$$

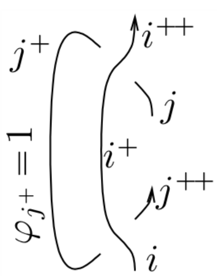
$$\mathtt{lhs} == \mathtt{rhs}\ //\ \mathtt{HL}$$

(Alt) Out[ ]=

$$\mathbb{E}[p_{1,2+i}\ \pi_{1,i} + p_{1,2+j}\ \pi_{1,j} + p_{2,2+i}\ \pi_{2,i} + p_{2,2+j}\ \pi_{2,j} + p_{3,2+i}\ \pi_{3,i} + p_{3,2+j}\ \pi_{3,j}]$$

(Alt) Out[ ]=

True



(Alt) In[ ]:=

$$\mathtt{lhs} = \int \mathcal{F}[\mathtt{i},\ \mathtt{j}]\ \$\mathcal{L}\ /@\ (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ d\{vs_i,\ vs_j,\ vs_{i^+},\ vs_{j^+},\ vs_{j+2}\}$$

$$\mathtt{rhs} = \int \mathcal{F}[\mathtt{i},\ \mathtt{j}]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ d\{vs_i,\ vs_j,\ vs_{i^+},\ vs_{j^+},\ vs_{j+2}\};$$

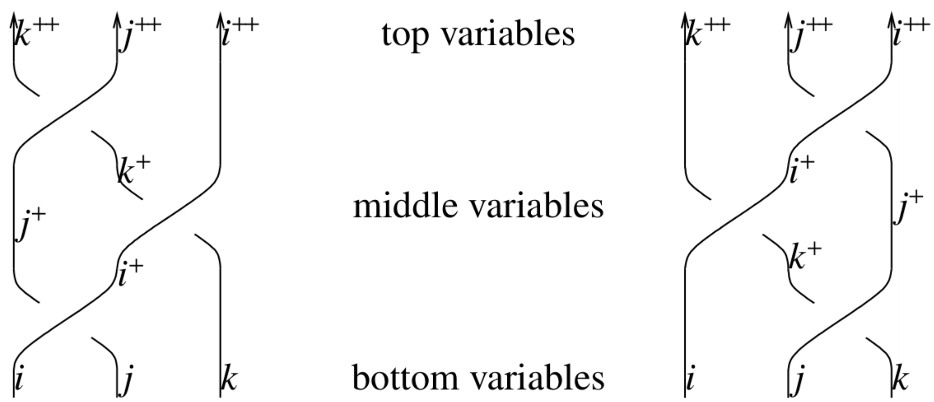
$$\mathtt{lhs} == \mathtt{rhs}\ //\ \mathtt{HL}$$

(Alt) Out[ ]=

$$-\mathbb{i}\ T_1\ T_2\ \mathbb{E}\left[\frac{\in}{2} + p_{1,2+i}\ \pi_{1,i} + p_{1,3+j}\ \pi_{1,j} + p_{2,2+i}\ \pi_{2,i} + p_{2,3+j}\ \pi_{2,j} + p_{3,2+i}\ \pi_{3,i} + p_{3,3+j}\ \pi_{3,j} + \in p_{3,3+j}\ \pi_{3,j}\right]$$

(Alt) Out[ ]=

True



(Alt) In[ ]:=

$$\text{Short}\Big[\text{lhs} = \int (\mathcal{F}[i, j, k]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1]))\ \mathbb{d}\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\}\Big]$$

$$\text{rhs} = \int (\mathcal{F}[i, j, k]\ \$\mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1]))\ \mathbb{d}\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\};$$

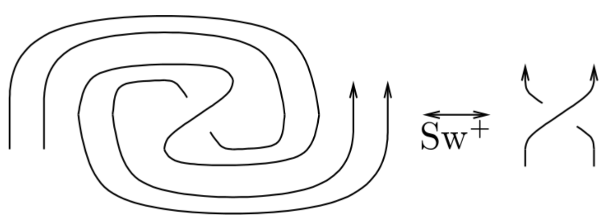
$$\text{lhs} == \text{rhs}\ //\ \text{HL}$$

(Alt) Out[ ]//Short=

$$T_1^3\, T_2^3\, \mathbb{E}\left[\frac{3\,\epsilon}{2} + \ll 153 \gg\right]$$

(Alt) Out[ ]=

True



(Alt) In[ ]:=

$$\text{Short}\Big[\text{lhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}\}\Big]$$

$$\text{rhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}\};$$

$$\text{lhs} == \text{rhs}\ //\ \text{HL}$$

(Alt) Out[ ]//Short=

$$T_1 \ll 1 \gg\ \ll 1 \gg \mathbb{E}\Big[\frac{\epsilon}{2} + T_1\, p_{1,3+i}\, \pi_{1,i} + (1 - T_1)\, p_{1 \ll 1 \gg 3 \ll 1 \gg \ll 1 \gg}\, \pi_{1,i} + \ll 35 \gg + \frac{\ll 1 \gg}{\ll 1 \gg} - \epsilon\, T_2\, p_{2,3+i}\, p_{3,3+j}\, \pi_{2,i}\, \pi_{3,j} - \epsilon\, p_{2,3+j}\, p_{3,3+j}\, \pi_{2,i}\, \pi_{3,j}\Big]$$

(Alt) Out[ ]=

True