Pensieve header: Experiments in 2026: Alternative integrands for ρ_1.

Initialization

```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
        Once[<< "IType(SimplifiedFormatting).m"];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

Minor utilities

```
In[•]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
        CCF[ℰ_] := Factor[ℰ];
        CF[ℰ_List] := CF /@ ℰ;
        CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
        CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, ϵ | (p | x | π | ξ | g)__, F] /. F → CCF];
        CF[ℰ_𝔼] := CF /@ ℰ;
        CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

```
In[•]:= ℒ_i_[K_] := CF[ℒ_i /@ Features[K][[2]]]
        vs[K_] := Join @@ Table[{p_i, x_i}, {i, Features[K][[1]]}]
```

```
In[•]:= {p*, x*, π*, ξ*} = {π, ξ, p, x};
        (vs_List)* := (v ↦ v*) /@ vs;
        (u_νi__)* := (u*)_νi;
```

```
In[•]:= HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, , ]];
```

g2px and px2g

Modified from pensieve://Talks/Geneva-2408/DataConversions.nb

```
In[•]:= g2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {g_i_,j_ :> λ p_i x_j}] /. {λ^k_. :> 1/k!}]
        gν2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {g_ν_,i_,j_ :> λ_ν p_ν,i x_ν,j}] /. {λ_^k_. :> 1/k!}]
```

```
In[•]:= Zip_{}[ℰ_] := ℰ;
        Zip_{ℰ_,ℰs___}[ℰ_] := (Collect[ℰ // Zip_{ℰs}, ξ] /. f_. ξ^d_. :> (D[f, {ξ*, d}])) /. ξ* → 0
```

```
In[•]:= px2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
          ps = Union[Cases[ℰ, p_, ∞]]; xs = Union[Cases[ℰ, x_, ∞]];
          Q = Sum[p0* x0* g_{p0[[2]],x0[[2]]}, {p0, ps}, {x0, xs}];
          Expand[Zip_{ps⋃xs}[ℰ e^Q]]
        ]
      pxv2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
          ps = Union[Cases[ℰ, p__, ∞]]; xs = Union[Cases[ℰ, x__, ∞]];
          Q = Sum[p0* x0* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
          Expand[Zip_{ps⋃xs}[ℰ e^Q] /. g_{α_,β_,i_,j_} :> If[α === β, g_{α,i,j}, 0]]
        ]
```

gRules

From pensieve://Talks/MonteVerita-2604/Theta.nb

```
In[•]:= gR_{s_,i_,j_} := {
          g_{jβ_} :> g_{j⁺β} + δ_{jβ}, g_{iβ_} :> T^s g_{i⁺β} + (1 - T^s) g_{j⁺β} + δ_{iβ},
          g_{α_,i⁺} :> T^s g_{αi} + δ_{αi⁺}, g_{α_,j⁺} :> g_{αj} + (1 - T^s) g_{αi} + δ_{αj⁺},
          g_{v_,jβ_} :> g_{vj⁺β} + δ_{jβ}, g_{v_,iβ_} :> T_v^s g_{vi⁺β} + (1 - T_v^s) g_{vj⁺β} + δ_{iβ},
          g_{v_,α_,i⁺} :> T_v^s g_{vαi} + δ_{αi⁺}, g_{v_,α_,j⁺} :> g_{vαj} + (1 - T_v^s) g_{vαi} + δ_{αj⁺}
        }
      gR_{i_} := {g_{i,β_} :> g_{i⁺,β} + δ_{i,β}, g_{α_,i⁺} :> g_{α,i} + δ_{α,i⁺}}
```

Inverse gRules written locally:

```
In[•]:= igR_{s_,i_,j_} := {
          g_{j⁺,β_} :> g_{j,β} - δ_{j,β}, g_{i⁺,β_} :> T^-s g_{iβ} - (T^-s - 1) g_{j⁺β} - T^-s δ_{iβ},
          g_{α_,i} :> T^-s (g_{α,i⁺} - δ_{α,i⁺}), g_{α_,j} :> g_{α,j⁺} - (1 - T^s) g_{α,i} - δ_{α,j⁺},
          g_{v_,j⁺,β_} :> g_{v,j,β} - δ_{j,β}, g_{v_,i⁺,β_} :> T_v^-s g_{viβ} - (T_v^-s - 1) g_{vj⁺β} - T_v^-s δ_{iβ},
          g_{v_,α_,i} :> T_v^-s (g_{v,α,i⁺} - δ_{α,i⁺}), g_{v_,α_,j} :> g_{v,α,j⁺} - (1 - T_v^s) g_{v,α,i} - δ_{α,j⁺}
        }
```

The Base Lagrangian

From pensieve://Talks/Bonn-2505/IType.nb.

```
In[•]:= ℒ1[X_{i_,j_}[s_]] := T^(s/2) 𝔼[
            x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
              (ϵ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
      ℒ1[C_i_[φ_]] := T^(φ/2) 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
      $ℒ = ℒ1;
```

The Trefoil

In[•]:=
```
K = Mirror@Knot[3, 1]; Features[K]
```

KnotTheory: Loading precomputed data in PD4Knots`.

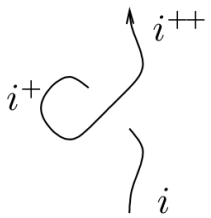
Out[•]=
$$\text{Features}[7, C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]]$$

In[•]:=
```
{vs[K], $ℒ[K]}
```

Out[•]=
$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-2\in + (-1+\in)\,p_1 x_1 + T\,p_2 x_1 - \in p_5 x_1 + (1-T)\,p_6 x_1 + \frac{1}{2}(-1+T)\in p_1 p_5 x_1^2 -$$
$$\frac{1}{2}(-1+T)\in p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 + (-1+\in)\,p_3 x_3 + T\,p_4 x_3 - \in p_7 x_3 + (1-T)\,p_8 x_3 +$$
$$\frac{1}{2}(-1+T)\in p_3 p_7 x_3^2 - \frac{1}{2}(-1+T)\in p_7^2 x_3^2 + (-1+\in)\,p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \in p_1 p_5 x_1 x_5 +$$
$$\in p_5^2 x_1 x_5 - \in p_2 x_6 + (1-T)\,p_3 x_6 + (-1+\in)\,p_6 x_6 + T\,p_7 x_6 + \in p_2^2 x_2 x_6 - \in p_2 p_6 x_2 x_6 -$$
$$\frac{1}{2}(-1+T)\in p_2^2 x_6^2 + \frac{1}{2}(-1+T)\in p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \in p_3 p_7 x_3 x_7 + \in p_7^2 x_3 x_7\Big]\Big\}$$

In[•]:=
```
$π = Normal[# + O[∈]^2] &; ∫ $ℒ[K] 𝕕 vs[K]
```

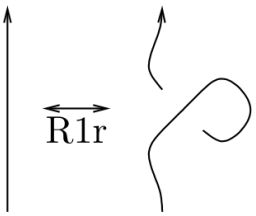
Out[•]=
$$-\frac{\mathbb{i}\,T\,\mathbb{E}\left[-\frac{(-1+T)^2\left(1+T^2\right)\in}{\left(1-T+T^2\right)^2}\right]}{1-T+T^2}$$



In[•]:=
```
lhs = ∫ 𝔼[𝕚 πᵢ pᵢ] $ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) 𝕕{xᵢ, pᵢ, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫ 𝔼[𝕚 πᵢ pᵢ] $ℒ /@ (Cᵢ[0] C_{i+1}[0] C_{i+2}[0]) 𝕕{xᵢ, pᵢ, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs // HL
```

Out[•]=
$$-\mathbb{i}\,\mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Out[•]=
True



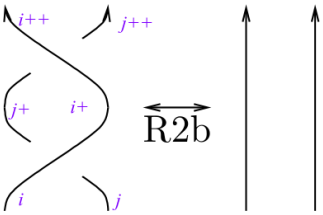
In[•]:=
```
lhs = ∫𝔼[𝕚 π_i p_i] $ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) 𝕕{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}

rhs = ∫𝔼[𝕚 π_i p_i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) 𝕕{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};

lhs == rhs // HL
```

Out[•]=
$$-\mathbb{i}\,\mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Out[•]=
True



Territory: $(x \mid p)_{(i\,|\,j)^+}$, $x_{i\,|\,j}$.

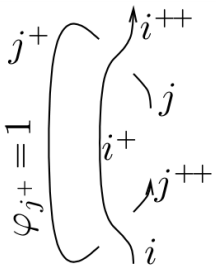
In[•]:=
```
lhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) 𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}

rhs =
  ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) 𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};

lhs == rhs // HL
```

Out[•]=
$$\mathbb{E}[\mathbb{i}\,p_{2+i}\,\pi_i + \mathbb{i}\,p_{2+j}\,\pi_j]$$

Out[•]=
True



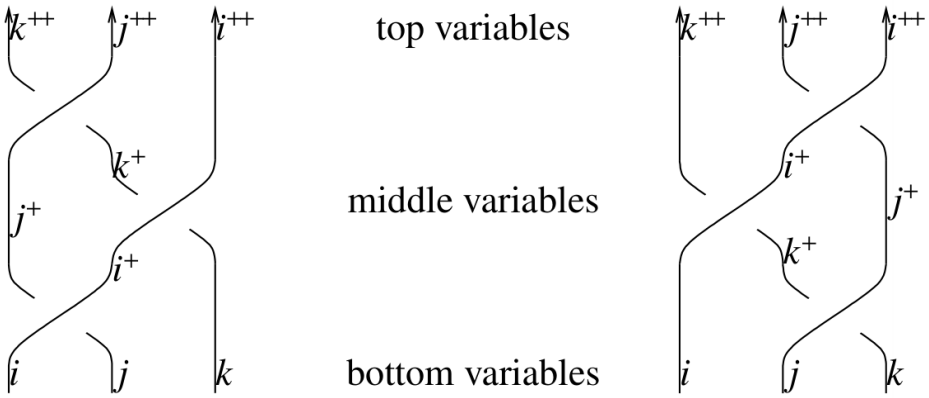
In[•]:=
```
lhs = ∫𝔼[i π_i p_i + i π_j p_j] $ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
    𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[i π_i p_i + i π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
    𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
lhs == rhs // HL
```

Out[•]=

$$-i\sqrt{T}\,\mathbb{E}\left[-\frac{\epsilon}{2}+i\,p_{2+i}\,\pi_i-i\,(-1+\epsilon)\,p_{3+j}\,\pi_j\right]$$

Out[•]=

True



In[•]:=
```
CF[$ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])]
```

Out[•]=

$$T^{3/2}\,\mathbb{E}\left[-\frac{3\epsilon}{2}-p_i\,x_i+\epsilon\,p_i\,x_i+T\,p_{1+i}\,x_i-\epsilon\,p_j\,x_i+(1-T)\,p_{1+j}\,x_i+\frac{1}{2}(-1+T)\,\epsilon\,p_i\,p_j\,x_i^2+\frac{1}{2}(1-T)\,\epsilon\,p_j^2\,x_i^2-p_{1+i}\,x_{1+i}+\epsilon\,p_{1+i}\,x_{1+i}+T\,p_{2+i}\,x_{1+i}-\epsilon\,p_k\,x_{1+i}+(1-T)\,p_{1+k}\,x_{1+i}+\frac{1}{2}(-1+T)\,\epsilon\,p_{1+i}\,p_k\,x_{1+i}^2+\frac{1}{2}(1-T)\,\epsilon\,p_k^2\,x_{1+i}^2-p_j\,x_j+p_{1+j}\,x_j-\epsilon\,p_i\,p_j\,x_i\,x_j+\epsilon\,p_j^2\,x_i\,x_j-p_{1+j}\,x_{1+j}+\epsilon\,p_{1+j}\,x_{1+j}+T\,p_{2+j}\,x_{1+j}-\epsilon\,p_{1+k}\,x_{1+j}+(1-T)\,p_{2+k}\,x_{1+j}+\frac{1}{2}(-1+T)\,\epsilon\,p_{1+j}\,p_{1+k}\,x_{1+j}^2+\frac{1}{2}(1-T)\,\epsilon\,p_{1+k}^2\,x_{1+j}^2-p_k\,x_k+p_{1+k}\,x_k-\epsilon\,p_{1+i}\,p_k\,x_{1+i}\,x_k+\epsilon\,p_k^2\,x_{1+i}\,x_k-p_{1+k}\,x_{1+k}+p_{2+k}\,x_{1+k}-\epsilon\,p_{1+j}\,p_{1+k}\,x_{1+j}\,x_{1+k}+\epsilon\,p_{1+k}^2\,x_{1+j}\,x_{1+k}\right]$$

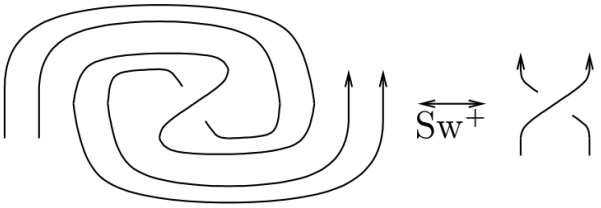
```
In[•]:= lhs = ∫ (𝔼[i πi pi + i πj pj + i πk pk] $ℒ /@ (Xi,j[1] Xi+1,k[1] Xj+1,k+1[1])) 𝕕{pi, pj, pk, xi, xj, xk, pi+1, pj+1, pk+1, xi+1, xj+1, xk+1}
        rhs = ∫ (𝔼[i πi pi + i πj pj + i πk pk] $ℒ /@ (Xj,k[1] Xi,k+1[1] Xi+1,j+1[1])) 𝕕{pi, pj, pk, xi, xj, xk, pi+1, pj+1, pk+1, xi+1, xj+1, xk+1};
        lhs == rhs // HL
```

Out[•]=

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\,\epsilon}{2}+\mathbb{i}\,T^2\,p_{2+i}\,\pi_i-\mathbb{i}\,(-1+T)\,T\,p_{2+j}\,\pi_i+\mathbb{i}\,T^2\,\epsilon\,p_{2+j}\,\pi_i-\mathbb{i}\,(-1+T)\,p_{2+k}\,\pi_i+\mathbb{i}\,T\,\epsilon\,p_{2+k}\,\pi_i-\frac{1}{2}\,(-1+T)\,T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T^3\,\epsilon\,p_{2+j}^2\,\pi_i^2-\frac{1}{2}\,(-1+T)\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i^2+\frac{1}{2}\,(-1+T)^2\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_i^2+\mathbb{i}\,T\,p_{2+j}\,\pi_j-\mathbb{i}\,T\,\epsilon\,p_{2+j}\,\pi_j-\mathbb{i}\,(-1+T)\,p_{2+k}\,\pi_j+\mathbb{i}\,(-1+2\,T)\,\epsilon\,p_{2+k}\,\pi_j+T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,\pi_j-T^3\,\epsilon\,p_{2+j}^2\,\pi_i\,\pi_j-(-1+T)\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,\pi_j+(-1+T)^2\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_j+(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_i\,\pi_j-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_j^2+\mathbb{i}\,p_{2+k}\,\pi_k-2\,\mathbb{i}\,\epsilon\,p_{2+k}\,\pi_k+T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,\pi_k-(-1+T)\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_k-T\,\epsilon\,p_{2+k}^2\,\pi_i\,\pi_k+T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j\,\pi_k-T\,\epsilon\,p_{2+k}^2\,\pi_j\,\pi_k\Big]$$

Out[•]=

True




```
In[•]:= lhs = ∫ 𝔼[i πi pi + i πj pj] $ℒ /@ (Xi+1,j+1[1] Ci[-1] Cj[-1] Ci+2[1] Cj+2[1]) 𝕕{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1, xi+2, pi+2, xj+2, pj+2}
        rhs = ∫ 𝔼[i πi pi + i πj pj] $ℒ /@ (Xi+1,j+1[1] Ci[0] Cj[0] Ci+2[0] Cj+2[0]) 𝕕{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1, xi+2, pi+2, xj+2, pj+2};
        lhs == rhs // HL
```

Out[•]=

$$\sqrt{T}\,\mathbb{E}\Big[-\frac{\epsilon}{2}+\mathbb{i}\,T\,p_{3+i}\,\pi_i+\mathbb{i}\,(1-T+T\,\epsilon)\,p_{3+j}\,\pi_i-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+j}^2\,\pi_i^2-\mathbb{i}\,(-1+\epsilon)\,p_{3+j}\,\pi_j+T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j-T\,\epsilon\,p_{3+j}^2\,\pi_i\,\pi_j\Big]$$

Out[•]=

True

The pPush Lagrangian

```
In[•]:= pPushloc__[ℰ_] := g2px[px2g[ℰ] /. gRloc /. {δα_,α_ → 1, δi,j → 0, δj,i → 0} /. α_+ :> α + 1];
```

In[•]:=
```
{pPush_{s,i,j}[x_i p_i], pPush_i[x_i p_i]}
```
Out[•]=
$$\{1+T^s\,p_{1+i}\,x_i+(1-T^s)\,p_{1+j}\,x_i,\ 1+p_{1+i}\,x_i\}$$

In[•]:=
```
ℒpPush[X_{i_,j_}[s_]] := T^(s/2) CF@𝔼[Plus[
      x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}),
      pPush_{s,i,j}[(ϵ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
    ]]
ℒpPush[C_i[φ_]] := T^(φ/2) 𝔼[x_i (p_{i+1} - p_i) + pPush_i[ϵ φ (1/2 - x_i p_i)]]
$ℒ = ℒpPush;
```

In[•]:=
```
$ℒ[X_{i,j}[s]]
$ℒ[C_i[φ]]
```
Out[•]=
$$T^{s/2}\,\mathbb{E}\Big[-\frac{s\,\epsilon}{2}-p_i\,x_i+T^s\,p_{1+i}\,x_i+(1-T^s)\,p_{1+j}\,x_i+s\,T^s\,\epsilon\,p_{1+j}\,x_i+\frac{1}{2}\,s\,T^s\,(-1+T^s)\,\epsilon\,p_{1+i}\,p_{1+j}\,x_i^2-\frac{1}{2}\,s\,T^s\,(-1+T^s)\,\epsilon\,p_{1+j}^2\,x_i^2-p_j\,x_j+p_{1+j}\,x_j-s\,\epsilon\,p_{1+j}\,x_j-s\,T^s\,\epsilon\,p_{1+i}\,p_{1+j}\,x_i\,x_j+s\,T^s\,\epsilon\,p_{1+j}^2\,x_i\,x_j\Big]$$
Out[•]=
$$T^{\varphi/2}\,\mathbb{E}\Big[-\frac{\epsilon\,\varphi}{2}-\epsilon\,\varphi\,p_{1+i}\,x_i+(-p_i+p_{1+i})\,x_i\Big]$$

In[•]:=
```
Collect[$ℒ[X_{i,j}[s]][[2, 1]], ϵ]
Collect[$ℒ[C_i[φ]][[2, 1]], ϵ]
```
Out[•]=
$$-p_i\,x_i+T^s\,p_{1+i}\,x_i+(1-T^s)\,p_{1+j}\,x_i-p_j\,x_j+p_{1+j}\,x_j+\epsilon\left(-\frac{s}{2}+s\,T^s\,p_{1+j}\,x_i+\frac{1}{2}\,s\,T^s\,(-1+T^s)\,p_{1+i}\,p_{1+j}\,x_i^2-\frac{1}{2}\,s\,T^s\,(-1+T^s)\,p_{1+j}^2\,x_i^2-s\,p_{1+j}\,x_j-s\,T^s\,p_{1+i}\,p_{1+j}\,x_i\,x_j+s\,T^s\,p_{1+j}^2\,x_i\,x_j\right)$$
Out[•]=
$$(-p_i+p_{1+i})\,x_i+\epsilon\left(-\frac{\varphi}{2}-\varphi\,p_{1+i}\,x_i\right)$$

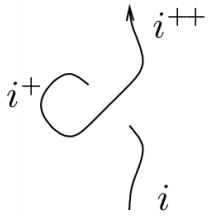
The Trefoil

In[•]:=
```
K = Mirror@Knot[3, 1]; {vs[K], $ℒ[K]}
```

Out[•]=
$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-\epsilon - p_1 x_1 + T p_2 x_1 + (1-T)\,p_6 x_1 + T\epsilon p_6 x_1 + \frac{1}{2}(-1+T)\,T\epsilon p_2 p_6 x_1^2 - \frac{1}{2}(-1+T)\,T\epsilon p_6^2 x_1^2 -$$
$$p_2 x_2 + p_3 x_2 - \epsilon p_3 x_2 - p_3 x_3 + T p_4 x_3 + (1-T)\,p_8 x_3 + T\epsilon p_8 x_3 + \frac{1}{2}(-1+T)\,T\epsilon p_4 p_8 x_3^2 -$$
$$\frac{1}{2}(-1+T)\,T\epsilon p_8^2 x_3^2 - p_4 x_4 + p_5 x_4 + \epsilon p_5 x_4 - p_5 x_5 + p_6 x_5 - \epsilon p_6 x_5 - T\epsilon p_2 p_6 x_1 x_5 +$$
$$T\epsilon p_6^2 x_1 x_5 + (1-T)\,p_3 x_6 + T\epsilon p_3 x_6 - p_6 x_6 + T p_7 x_6 + T\epsilon p_3^2 x_2 x_6 - T\epsilon p_3 p_7 x_2 x_6 -$$
$$\frac{1}{2}(-1+T)\,T\epsilon p_3^2 x_6^2 + \frac{1}{2}(-1+T)\,T\epsilon p_3 p_7 x_6^2 - p_7 x_7 + p_8 x_7 - \epsilon p_8 x_7 - T\epsilon p_4 p_8 x_3 x_7 + T\epsilon p_8^2 x_3 x_7\Big]\Big\}$$

In[•]:=
```
$π = Normal[# + O[ϵ]^2] &; ∫$ℒ[K] 𝕕 vs[K]
```

Out[•]=
$$-\frac{\mathbb{i}\,T\,\mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^2)\,\epsilon}{(1-T+T^2)^2}\right]}{1-T+T^2}$$



In[•]:=
```
lhs = ∫𝔼[𝕚 πᵢ pᵢ] $ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) 𝕕{xᵢ, pᵢ, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[𝕚 πᵢ pᵢ] $ℒ /@ (Cᵢ[0] C_{i+1}[0] C_{i+2}[0]) 𝕕{xᵢ, pᵢ, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs // HL
```

Out[•]=
$$-\mathbb{i}\,\mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Out[•]=
True

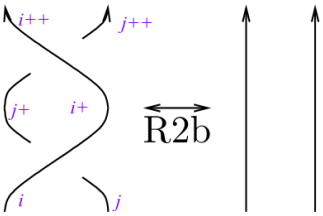
```
In[•]:= lhs = ∫ 𝔼[𝕚 π_i p_i] $ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) 𝕕{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
        rhs = ∫ 𝔼[𝕚 π_i p_i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) 𝕕{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
        lhs == rhs // HL
```

Out[•]=

$$-\mathbb{i}\,\mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Out[•]=

True



Territory: $(x \mid p)_{(i\mid j)^+}$, $x_{i\mid j}$.

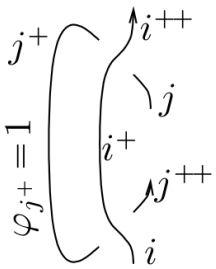
```
In[•]:= lhs = ∫ 𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) 𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
        rhs =
          ∫ 𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) 𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
        lhs == rhs // HL
```

Out[•]=

$$\mathbb{E}[\mathbb{i}\,p_{2+i}\,\pi_i + \mathbb{i}\,p_{2+j}\,\pi_j]$$

Out[•]=

True



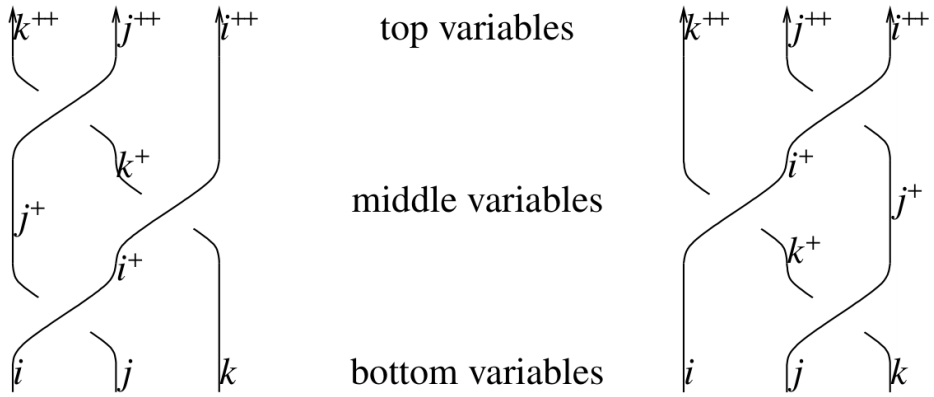

```
In[•]:= lhs = ∫ 𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
          𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
        rhs = ∫ 𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
          𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
        lhs == rhs // HL
```

Out[•]=

$$-\mathbb{i}\,\sqrt{\mathsf{T}}\,\mathbb{E}\left[-\frac{\in}{2} + \mathbb{i}\,p_{2+i}\,\pi_i + \mathbb{i}\,p_{3+j}\,\pi_j - \mathbb{i}\in p_{3+j}\,\pi_j\right]$$

Out[•]=

True



In[•]:= **$ℒ /@ (X$_{i,j}$[1] X$_{i+1,k}$[1] X$_{j+1,k+1}$[1])**

Out[•]=

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\in}{2}-p_i\,x_i+T\,p_{1+i}\,x_i+(1-T)\,p_{1+j}\,x_i+T\in p_{1+j}\,x_i+\frac{1}{2}\,(-1+T)\,T\in p_{1+i}\,p_{1+j}\,x_i^2-\frac{1}{2}\,(-1+T)\,T\in p_{1+j}^2\,x_i^2-p_{1+i}\,x_{1+i}+T\,p_{2+i}\,x_{1+i}+(1-T)\,p_{1+k}\,x_{1+i}+T\in p_{1+k}\,x_{1+i}+\frac{1}{2}\,(-1+T)\,T\in p_{2+i}\,p_{1+k}\,x_{1+i}^2-\frac{1}{2}\,(-1+T)\,T\in p_{1+k}^2\,x_{1+i}^2-p_j\,x_j+p_{1+j}\,x_j-\in p_{1+j}\,x_j-T\in p_{1+i}\,p_{1+j}\,x_i\,x_j+T\in p_{1+j}^2\,x_i\,x_j-p_{1+j}\,x_{1+j}+T\,p_{2+j}\,x_{1+j}+(1-T)\,p_{2+k}\,x_{1+j}+T\in p_{2+k}\,x_{1+j}+\frac{1}{2}\,(-1+T)\,T\in p_{2+j}\,p_{2+k}\,x_{1+j}^2-\frac{1}{2}\,(-1+T)\,T\in p_{2+k}^2\,x_{1+j}^2-p_k\,x_k+p_{1+k}\,x_k-\in p_{1+k}\,x_k-T\in p_{2+i}\,p_{1+k}\,x_{1+i}\,x_k+T\in p_{1+k}^2\,x_{1+i}\,x_k-p_{1+k}\,x_{1+k}+p_{2+k}\,x_{1+k}-\in p_{2+k}\,x_{1+k}-T\in p_{2+j}\,p_{2+k}\,x_{1+j}\,x_{1+k}+T\in p_{2+k}^2\,x_{1+j}\,x_{1+k}\Big]$$

pdf

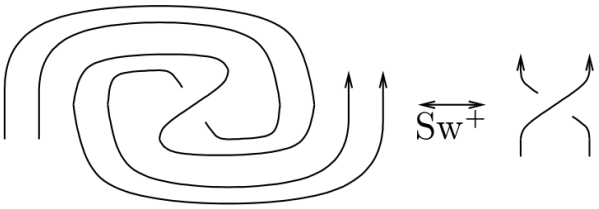
In[•]:=
```
lhs = ∫ (𝔼[i π_i p_i + i π_j p_j + i π_k p_k] $ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
    𝕕{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}}
rhs = ∫ (𝔼[i π_i p_i + i π_j p_j + i π_k p_k] $ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
    𝕕{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
lhs == rhs // HL
```

Out[•]=
pdf

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\,\epsilon}{2}+i\,T^2\,p_{2+i}\,\pi_i-i\,(-1+T)\,T\,p_{2+j}\,\pi_i+i\,T^2\,\epsilon\,p_{2+j}\,\pi_i-i\,(-1+T)\,p_{2+k}\,\pi_i+i\,T\,\epsilon\,p_{2+k}\,\pi_i-\frac{1}{2}\,(-1+T)\,T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T^3\,\epsilon\,p_{2+j}^2\,\pi_i^2-\frac{1}{2}\,(-1+T)\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i^2+\frac{1}{2}\,(-1+T)^2\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_i^2+i\,T\,p_{2+j}\,\pi_j-i\,T\,\epsilon\,p_{2+j}\,\pi_j-i\,(-1+T)\,p_{2+k}\,\pi_j+i\,(-1+2\,T)\,\epsilon\,p_{2+k}\,\pi_j+T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,\pi_j-T^3\,\epsilon\,p_{2+j}^2\,\pi_i\,\pi_j-(-1+T)\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,\pi_j+(-1+T)^2\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_j+(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_i\,\pi_j-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+k}^2\,\pi_j^2+i\,p_{2+k}\,\pi_k-2\,i\,\epsilon\,p_{2+k}\,\pi_k+T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,\pi_k-(-1+T)\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_k-T\,\epsilon\,p_{2+k}^2\,\pi_i\,\pi_k+T\,\epsilon\,p_{2+j}\,p_{2+k}\,\pi_j\,\pi_k-T\,\epsilon\,p_{2+k}^2\,\pi_j\,\pi_k\Big]$$

Out[•]=
pdf

True



In[•]:=
```
lhs = ∫ 𝔼[i π_i p_i + i π_j p_j] $ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1])
    𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}}
rhs = ∫ 𝔼[i π_i p_i + i π_j p_j] $ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0])
    𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}};
lhs == rhs // HL
```

Out[•]=

$$\sqrt{T}\,\mathbb{E}\Big[-\frac{\epsilon}{2}+i\,T\,p_{3+i}\,\pi_i-i\,(-1+T)\,p_{3+j}\,\pi_i+i\,T\,\epsilon\,p_{3+j}\,\pi_i-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+j}^2\,\pi_i^2+i\,p_{3+j}\,\pi_j-i\,\epsilon\,p_{3+j}\,\pi_j+T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j-T\,\epsilon\,p_{3+j}^2\,\pi_i\,\pi_j\Big]$$

Out[•]=

True