Pensieve header: The rank 2 mod ϵ^2 invariant using integration techniques; continues Rank2.nb at pensieve://Talks/Geneva-2408/ and UC4A2.nb and Theta.nb at pensieve://Projects/HigherRank/.

Initialization

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< "IType(SimplifiedFormatting).m"];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

pdf

In[•]:=
```
T3 = T1 T2;  i_+ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /. {π_is__ :> B^-1 π_is, x_is__ :> B^-1 x_is, p_is__ :> B p_is} /. ϵ B^b_- /; b < 0 → 0 /.
    B → 1) &;
```

pdf

In[•]:=
```
vs_i_ := Sequence[p_1,i, p_2,i, p_3,i, x_1,i, x_2,i, x_3,i];
ℱ[is__] := 𝔼[Sum[π_v,i p_v,i, {i, {is}}, {v, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K][[2]]];
vs[K_] := Union @@ Table[{vs_i}, {i, Features[K][[1]]}]
```

The Lagrangian

tex

```
\needspace{30mm}
{\bf\red The Lagrangian.}
```

exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

In[•]:=
```
ℒ[X_i_,j_[s_]] := T3^s 𝔼[CF@Plus[
      Sum_{v=1}^3 (x_vi (p_vi+ - p_vi) + x_vj (p_vj+ - p_vj) + (T_v^s - 1) x_vi (p_vi+ - p_vj+)),
      (T1^s - 1) p_3j x_1i (T2^s x_2i - x_2j),
      ϵ s (T3^s - 1) p_1j (p_2i - p_2j) x_3i / (T2^s - 1),
      ϵ s (1/2 + T2^s p_1i p_2j x_1i x_2i - p_1i p_2j x_1i x_2j - p_3i x_3i - (T2^s - 1) p_2j p_3i x_2i x_3i +
          (T3^s - 1) p_2j p_3j x_2i x_3i + 2 p_2j p_3i x_2j x_3i + p_1i p_3j x_1i x_3j - p_2i p_3j x_2i x_3j - T2^s p_2j p_3j x_2i x_3j +
          ((T1^s - 1) p_1j x_1i (T2^2^s p_2j x_2i - T2^s p_2j x_2j - (T2^s + 1) (T3^s - 1) p_3j x_3i + T2^s p_3j x_3j) +
              (T3^s - 1) p_3j x_3i (1 - T2^s p_1i x_1i + p_2i x_2j + (T2^s - 2) p_2j x_2j)) / (T2^s - 1))]]
```

pdf

In[•]:=
```
ℒ[Cᵢ_[φ_]] := T₃^φ 𝔼[∑_{ν=1}^{3} x_{νi} (p_{νi⁺} - p_{νi}) + ϵ φ (p_{3i} x_{3i} - 1/2)]
```

exec

```
nb2tex$PDFWidth /= 1.25;
```

Reidemeister 3

tex

{\bf\red Reidemeister 3.}

pdf

In[•]:=
```
Short[lhs = ∫ℱ[i, j, k] ℒ /@ (X_{i,j}[1] X_{i⁺,k}[1] X_{j⁺,k⁺}[1]) 𝕕{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}}]
```

Out[•]//Short=

pdf

$$T_1^3\, T_2^3\, \mathbb{E}\left[\frac{3\,\epsilon}{2} + T_1^2\, p_{1,2+i}\, \pi_{1,i} - (-1+T_1)\, T_1\, p_{1,2+j}\, \pi_{1,i} + \text{<<150>>}\right]$$

pdf

In[•]:=
```
rhs = ∫ℱ[i, j, k] ℒ /@ (X_{j,k}[1] X_{i,k⁺}[1] X_{i⁺,j⁺}[1]) 𝕕{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}}; lhs == rhs
```

Out[•]=

pdf

```
True
```

The Trefoil

tex

\needspace{25mm}
\parpic[r]{\includegraphics[width=0.6in]{../Beijing-2407/Trefoil.jpg}}
{\bf\red The Trefoil.}

pdf

In[•]:=
```
K = Knot[3, 1]; ∫ℒ[K] 𝕕 vs[K]
```

pdf

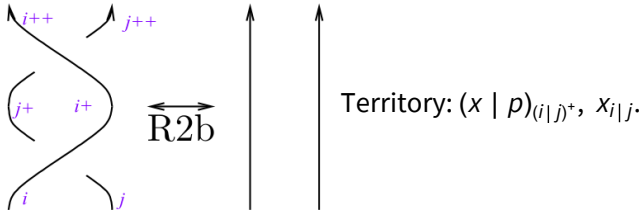
KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=

pdf

$$-\frac{i\, T_1^2\, T_2^2\, \mathbb{E}\left[-\frac{\epsilon\,\left(1-T_1+T_1^2-T_2-T_1^3\,T_2+T_2^2+T_1^4\,T_2^2-T_1\,T_2^3-T_1^4\,T_2^3+T_1^2\,T_2^4-T_1^3\,T_2^4+T_1^4\,T_2^4\right)}{\left(1-T_1+T_1^2\right)\,\left(1-T_2+T_2^2\right)\,\left(1-T_1\,T_2+T_1^2\,T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\,\left(1-T_2+T_2^2\right)\,\left(1-T_1\,T_2+T_1^2\,T_2^2\right)}$$

Invariance Under Reidemeister 2b



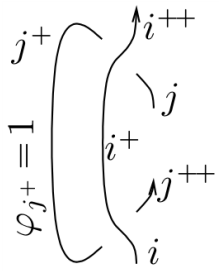
Territory: $(x \mid p)_{(i\,|\,j)^+}$, $x_{i\,|\,j}$.

```
In[•]:= lhs = ∫ℱ[i, j] ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}}
         rhs = ∫ℱ[i, j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}};
         lhs == rhs
```

```
Out[•]=
𝔼[p_{1,2+i} π_{1,i} + p_{1,2+j} π_{1,j} + p_{2,2+i} π_{2,i} + p_{2,2+j} π_{2,j} + p_{3,2+i} π_{3,i} + p_{3,2+j} π_{3,j}]
```

```
Out[•]=
True
```

Invariance Under R2c

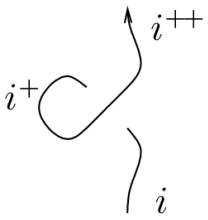



```
In[•]:= lhs = ∫ℱ[i, j] ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{j+2}}
         rhs = ∫ℱ[i, j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{j+2}};
         lhs == rhs
```

```
Out[•]=
-𝕚 T_1 T_2 𝔼[∈/2 + p_{1,2+i} π_{1,i} + p_{1,3+j} π_{1,j} + p_{2,2+i} π_{2,i} + p_{2,3+j} π_{2,j} + p_{3,2+i} π_{3,i} + p_{3,3+j} π_{3,j} + ∈ p_{3,3+j} π_{3,j}]
```

```
Out[•]=
True
```

Invariance Under R1l



```
In[•]:= lhs = ∫ℱ[i] ℒ/@(X_{i+2,i}[1] C_{i+1}[1]) 𝕕{vs_i, vs_{i⁺}, vs_{i+2}}
         rhs = ∫ℱ[i] ℒ/@(C_i[0] C_{i+1}[0] C_{i+2}[0]) 𝕕{vs_i, vs_{i⁺}, vs_{i+2}};
         lhs == rhs
```

```
Out[•]=
-𝕚 𝔼[p_{1,3+i} π_{1,i} + p_{2,3+i} π_{2,i} + p_{3,3+i} π_{3,i}]
```

```
Out[•]=
True
```

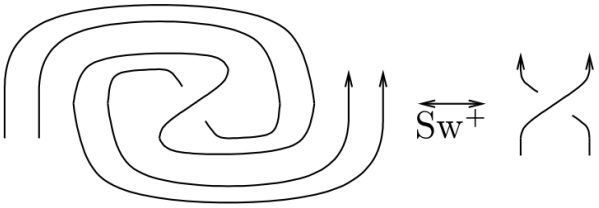
Invariance Under R1r

```
In[•]:= lhs = ∫ℱ[i] ℒ/@(X_{i,i+2}[1] C_{i+1}[-1]) 𝕕{vs_i, vs_{i⁺}, vs_{i+2}}
         rhs = ∫ℱ[i] ℒ/@(C_i[0] C_{i+1}[0] C_{i+2}[0]) 𝕕{vs_i, vs_{i⁺}, vs_{i+2}};
         lhs == rhs
```

```
Out[•]=
-𝕚 𝔼[p_{1,3+i} π_{1,i} + p_{2,3+i} π_{2,i} + p_{3,3+i} π_{3,i}]
```

```
Out[•]=
True
```

Invariance Under Sw



In[•]:=
```
lhs =
  ∫ℱ[i, j] ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{i+2}, vs_{j+2}}
rhs = ∫ℱ[i, j] ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{i+2}, vs_{j+2}};
lhs == rhs
```

Out[•]=

$$
\begin{aligned}
&T_1\,T_2\,\mathbb{E}\Big[\\
&\frac{\in}{2}+T_1\,p_{1,3+i}\,\pi_{1,i}+(1-T_1)\,p_{1,3+j}\,\pi_{1,i}+p_{1,3+j}\,\pi_{1,j}+T_2\,p_{2,3+i}\,\pi_{2,i}-\in T_2\,p_{2,3+i}\,\pi_{2,i}+(1-T_2)\,p_{2,3+j}\,\pi_{2,i}+\\
&\in T_1\,T_2\,p_{1,3+i}\,p_{2,3+j}\,\pi_{1,i}\,\pi_{2,i}+\frac{\in(-1+T_1)\,T_2\,p_{1,3+j}\,p_{2,3+j}\,\pi_{1,i}\,\pi_{2,i}}{-1+T_2}+(-1+T_1)\,T_2\,p_{3,3+j}\,\pi_{1,i}\,\pi_{2,i}+\\
&p_{2,3+j}\,\pi_{2,j}+\in p_{2,3+j}\,\pi_{2,j}-\in T_1\,p_{1,3+i}\,p_{2,3+j}\,\pi_{1,i}\,\pi_{2,j}-\frac{\in(-1+T_1)\,p_{1,3+j}\,p_{2,3+j}\,\pi_{1,i}\,\pi_{2,j}}{-1+T_2}+\\
&(1-T_1)\,p_{3,3+j}\,\pi_{1,i}\,\pi_{2,j}+\frac{\in T_2\,(-1+T_1\,T_2)\,p_{1,3+j}\,p_{2,3+i}\,\pi_{3,i}}{-1+T_2}-\frac{\in T_2\,(-1+T_1\,T_2)\,p_{1,3+j}\,p_{2,3+j}\,\pi_{3,i}}{-1+T_2}+\\
&T_1\,T_2\,p_{3,3+i}\,\pi_{3,i}+\in T_1\,T_2\,p_{3,3+i}\,\pi_{3,i}+(1-T_1\,T_2)\,p_{3,3+j}\,\pi_{3,i}-\frac{\in T_2\,(-1+T_1\,T_2)\,p_{3,3+j}\,\pi_{3,i}}{-1+T_2}-\\
&\frac{\in T_1\,T_2\,(-1+T_1\,T_2)\,p_{1,3+i}\,p_{3,3+j}\,\pi_{1,i}\,\pi_{3,i}}{-1+T_2}+\frac{\in(-1+T_1)\,T_2\,(-1+T_1\,T_2)\,p_{1,3+j}\,p_{3,3+j}\,\pi_{1,i}\,\pi_{3,i}}{-1+T_2}-\\
&\in T_1\,(-1+T_2)\,T_2\,p_{2,3+j}\,p_{3,3+i}\,\pi_{2,i}\,\pi_{3,i}+\in T_2\,(-1+T_1\,T_2)\,p_{2,3+j}\,p_{3,3+j}\,\pi_{2,i}\,\pi_{3,i}+\\
&2\in T_1\,T_2\,p_{2,3+j}\,p_{3,3+i}\,\pi_{2,j}\,\pi_{3,i}+\frac{\in T_2\,(-1+T_1\,T_2)\,p_{2,3+i}\,p_{3,3+j}\,\pi_{2,j}\,\pi_{3,i}}{-1+T_2}-\\
&\frac{\in(-1+2\,T_2)\,(-1+T_1\,T_2)\,p_{2,3+j}\,p_{3,3+j}\,\pi_{2,j}\,\pi_{3,i}}{-1+T_2}+p_{3,3+j}\,\pi_{3,j}+\in T_1\,p_{1,3+i}\,p_{3,3+j}\,\pi_{1,i}\,\pi_{3,j}+\\
&\frac{\in(-1+T_1)\,p_{1,3+j}\,p_{3,3+j}\,\pi_{1,i}\,\pi_{3,j}}{-1+T_2}-\in T_2\,p_{2,3+i}\,p_{3,3+j}\,\pi_{2,i}\,\pi_{3,j}-\in p_{2,3+j}\,p_{3,3+j}\,\pi_{2,i}\,\pi_{3,j}\Big]
\end{aligned}
$$

Out[•]=
```
True
```