Pensieve header: The rank 2 mod ϵ^2 invariant using integration techniques; continues Rank2.nb at pensieve://Talks/Geneva-2408/ and UC4A2.nb and Theta.nb at pensieve://Projects/HigherRank/.

Initialization

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< "IType.m"];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

pdf

```
In[∘]:= T₃ = T₁ T₂;  i_⁺ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /. {π_{is__} :→ B^-1 π_{is}, x_{is__} :→ B^-1 x_{is}, p_{is__} :→ B p_{is}} /. ϵ B^b_ /; b < 0 → 0 /.
     B → 1) &;
```

pdf

```
In[∘]:= vs_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
𝓕[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧];
vs[K_] := Union @@ Table[{vs_i}, {i, Features[K]⟦1⟧}]
```

The Lagrangian

tex

\needspace{30mm}
{\bf\red The Lagrangian.}

exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

```
In[∘]:= ℒ[X_{i_,j_}[s_]] := T₃^s 𝔼[CF@Plus[
      ∑_{ν=1}^{3} (x_{νi} (p_{νi⁺} - p_{νi}) + x_{νj} (p_{νj⁺} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi⁺} - p_{νj⁺})),
      (T₁^s - 1) p_{3j} x_{1i} (T₂^s x_{2i} - x_{2j}),
      ϵ s (T₃^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T₂^s - 1),
      ϵ s (1/2 + T₂^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} - (T₂^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} +
         (T₃^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} + 2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} - T₂^s p_{2j} p_{3j} x_{2i} x_{3j} +
         ((T₁^s - 1) p_{1j} x_{1i} (T₂^{2s} p_{2j} x_{2i} - T₂^s p_{2j} x_{2j} - (T₂^s + 1) (T₃^s - 1) p_{3j} x_{3i} + T₂^s p_{3j} x_{3j}) +
            (T₃^s - 1) p_{3j} x_{3i} (1 - T₂^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T₂^s - 2) p_{2j} x_{2j})) / (T₂^s - 1))]]
```

pdf

In[∘]:= $\mathcal{L}[C_{i\_}[\varphi\_]] := T_3^{\varphi}\,\mathbb{E}\left[\sum_{\nu=1}^{3} x_{\nu i}\,(p_{\nu i^+} - p_{\nu i}) + \epsilon\,\varphi\,(p_{3i}\,x_{3i} - 1/2)\right]$

exec

```
nb2tex$PDFWidth /= 1.25;
```

Reidemeister 3

tex

{\bf\red Reidemeister 3.}

pdf

In[∘]:= $\text{Short}\left[\text{lhs} = \int \mathcal{F}[i, j, k]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i^+,k}[1]\ X_{j^+,k^+}[1])\ d\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\}\right]$

Out[∘]//Short=
pdf

$$T_1^3\,T_2^3\,\mathbb{E}\left[\frac{3\,\epsilon}{2} + T_1^2\,p_{1,2+i}\,\pi_{1,i} - (-1 + T_1)\,T_1\,p_{1,2+j}\,\pi_{1,i} + \ll 150 \gg\right]$$

pdf

In[∘]:= $\text{rhs} = \int \mathcal{F}[i, j, k]\ \mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k^+}[1]\ X_{i^+,j^+}[1])\ d\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\};\ \text{lhs} == \text{rhs}$

Out[∘]=
pdf

```
True
```

The Trefoil

tex

\needspace{25mm}

\parpic[r]{\includegraphics[width=0.6in]{../Beijing-2407/Trefoil.jpg}}

{\bf\red The Trefoil.}

pdf

In[∘]:= $K = \text{Knot}[3, 1];\ \int \mathcal{L}[K]\ d\ vs[K]$

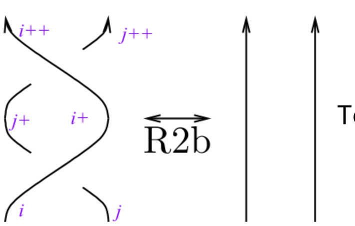
pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$$-\frac{i\,T_1^2\,T_2^2\,\mathbb{E}\left[-\frac{\epsilon\,(1 - T_1 + T_1^2 - T_2 - T_1^3\,T_2 + T_2^2 + T_1^4\,T_2^2 - T_1\,T_2^3 - T_1^4\,T_2^3 + T_1^2\,T_2^4 - T_1^3\,T_2^4 + T_1^4\,T_2^4)}{(1 - T_1 + T_1^2)\,(1 - T_2 + T_2^2)\,(1 - T_1\,T_2 + T_1^2\,T_2^2)}\right]}{(1 - T_1 + T_1^2)\,(1 - T_2 + T_2^2)\,(1 - T_1\,T_2 + T_1^2\,T_2^2)}$$

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+},\ x_{i|j}$.

In[∘]:=
$\text{lhs} = \int \mathcal{F}[i, j]\ \mathcal{L} \text{ /@ } (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\}$

$\text{rhs} = \int \mathcal{F}[i, j]\ \mathcal{L} \text{ /@ } (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\};$

$\text{lhs} == \text{rhs}$

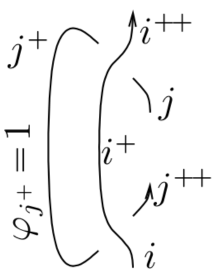
Out[∘]=

$\mathbb{E}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}]$

Out[∘]=

True

Invariance Under R2c



In[∘]:=
$\text{lhs} = \int \mathcal{F}[i, j]\ \mathcal{L} \text{ /@ } (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\}$

$\text{rhs} = \int \mathcal{F}[i, j]\ \mathcal{L} \text{ /@ } (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\};$

$\text{lhs} == \text{rhs}$

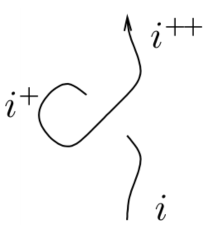
Out[∘]=

$-\mathbb{i}\, T_1\, T_2\, \mathbb{E}\left[\frac{\epsilon}{2} + p_{1,2+i}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j} + \epsilon\, p_{3,3+j}\,\pi_{3,j}\right]$

Out[∘]=

True

Invariance Under R1l



In[∘]:=

$$\text{lhs} = \int \mathcal{F}[i]\ \mathcal{L}\ /@\ (X_{i+2,i}[1]\ C_{i+1}[1])\ \mathbb{d}\{vs_i,\ vs_{i^+},\ vs_{i+2}\}$$

$$\text{rhs} = \int \mathcal{F}[i]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ \mathbb{d}\{vs_i,\ vs_{i^+},\ vs_{i+2}\};$$

$$\text{lhs} == \text{rhs}$$

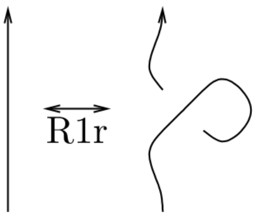
Out[∘]=

$$-\mathbb{i}\ \mathbb{E}[p_{1,3+i}\ \pi_{1,i} + p_{2,3+i}\ \pi_{2,i} + p_{3,3+i}\ \pi_{3,i}]$$

Out[∘]=

True

Invariance Under R1r



In[∘]:=

$$\text{lhs} = \int \mathcal{F}[i]\ \mathcal{L}\ /@\ (X_{i,i+2}[1]\ C_{i+1}[-1])\ \mathbb{d}\{vs_i,\ vs_{i^+},\ vs_{i+2}\}$$

$$\text{rhs} = \int \mathcal{F}[i]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ \mathbb{d}\{vs_i,\ vs_{i^+},\ vs_{i+2}\};$$

$$\text{lhs} == \text{rhs}$$

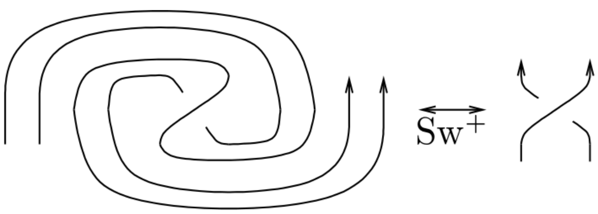
Out[∘]=

$$-\mathbb{i}\ \mathbb{E}[p_{1,3+i}\ \pi_{1,i} + p_{2,3+i}\ \pi_{2,i} + p_{3,3+i}\ \pi_{3,i}]$$

Out[∘]=

True

Invariance Under Sw



In[∘]:=

```
lhs =
  ∫𝓕[i, j] ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1]) ⅆ{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}}
rhs = ∫𝓕[i, j] ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0]) ⅆ{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}};
lhs == rhs
```

Out[∘]=

$$
\begin{aligned}
&T_1 T_2 \mathbb{E}\Big[\\
&\frac{\epsilon}{2} + T_1 p_{1,3+i}\pi_{1,i} + (1-T_1) p_{1,3+j}\pi_{1,i} + p_{1,3+j}\pi_{1,j} + T_2 p_{2,3+i}\pi_{2,i} - \epsilon T_2 p_{2,3+i}\pi_{2,i} + (1-T_2) p_{2,3+j}\pi_{2,i} +\\
&\epsilon T_1 T_2 p_{1,3+i} p_{2,3+j}\pi_{1,i}\pi_{2,i} + \frac{\epsilon(-1+T_1)T_2 p_{1,3+j}p_{2,3+j}\pi_{1,i}\pi_{2,i}}{-1+T_2} + (-1+T_1)T_2 p_{3,3+j}\pi_{1,i}\pi_{2,i} +\\
&p_{2,3+j}\pi_{2,j} + \epsilon p_{2,3+j}\pi_{2,j} - \epsilon T_1 p_{1,3+i}p_{2,3+j}\pi_{1,i}\pi_{2,j} - \frac{\epsilon(-1+T_1)p_{1,3+j}p_{2,3+j}\pi_{1,i}\pi_{2,j}}{-1+T_2} +\\
&(1-T_1)p_{3,3+j}\pi_{1,i}\pi_{2,j} + \frac{\epsilon T_2(-1+T_1T_2)p_{1,3+j}p_{2,3+i}\pi_{3,i}}{-1+T_2} - \frac{\epsilon T_2(-1+T_1T_2)p_{1,3+j}p_{2,3+j}\pi_{3,i}}{-1+T_2} +\\
&T_1T_2 p_{3,3+i}\pi_{3,i} + \epsilon T_1T_2 p_{3,3+i}\pi_{3,i} + (1-T_1T_2)p_{3,3+j}\pi_{3,i} - \frac{\epsilon T_2(-1+T_1T_2)p_{3,3+j}\pi_{3,i}}{-1+T_2} -\\
&\frac{\epsilon T_1T_2(-1+T_1T_2)p_{1,3+i}p_{3,3+j}\pi_{1,i}\pi_{3,i}}{-1+T_2} + \frac{\epsilon(-1+T_1)T_2(-1+T_1T_2)p_{1,3+j}p_{3,3+j}\pi_{1,i}\pi_{3,i}}{-1+T_2} -\\
&\epsilon T_1(-1+T_2)T_2 p_{2,3+j}p_{3,3+i}\pi_{2,i}\pi_{3,i} + \epsilon T_2(-1+T_1T_2)p_{2,3+j}p_{3,3+j}\pi_{2,i}\pi_{3,i} +\\
&2\epsilon T_1T_2 p_{2,3+j}p_{3,3+i}\pi_{2,j}\pi_{3,i} + \frac{\epsilon T_2(-1+T_1T_2)p_{2,3+i}p_{3,3+j}\pi_{2,j}\pi_{3,i}}{-1+T_2} -\\
&\frac{\epsilon(-1+2T_2)(-1+T_1T_2)p_{2,3+j}p_{3,3+j}\pi_{2,j}\pi_{3,i}}{-1+T_2} + p_{3,3+j}\pi_{3,j} + \epsilon T_1 p_{1,3+i}p_{3,3+j}\pi_{1,i}\pi_{3,j} +\\
&\frac{\epsilon(-1+T_1)p_{1,3+j}p_{3,3+j}\pi_{1,i}\pi_{3,j}}{-1+T_2} - \epsilon T_2 p_{2,3+i}p_{3,3+j}\pi_{2,i}\pi_{3,j} - \epsilon p_{2,3+j}p_{3,3+j}\pi_{2,i}\pi_{3,j}\Big]
\end{aligned}
$$

Out[∘]=

True