

Pensieve header: May 14, 2026 experiments: R3 fails without the wisdom projection - probably meaning that not every to- ϵ solution extends to a to- ϵ^2 solution.

Initialization

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< KnotTheory` ; << Rot.m];
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/AP/Talks/Bonn-2505> to compute rotation numbers.

pdf

```
In[*]:= CF[ $\omega$  .  $\mathcal{E}$   $\mathbb{E}$ ] := CF[ $\omega$ ] CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$  List] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ ] := Module[{vs, ps, c},
  vs = Cases[ $\mathcal{E}$ , (x | p |  $\xi$  |  $\pi$  | g)_,  $\infty$ ] U { $\epsilon$ };
  Total[CoefficientRules[Expand[ $\mathcal{E}$ ], vs] /. (ps_ -> c_) => Factor[c] (Times @@ vsps) ]];
```

pdf

```
In[*]:=  $\mathbb{E} /: \mathbb{E}[A] \mathbb{E}[B] := \mathbb{E}[A + B];$ 
```

pdf

```
In[*]:=  $\$ \pi = \text{Identity};$  (* The Wisdom Projection *)
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In[*]:= Unprotect[Integrate];
 $\int \omega$  .  $\mathbb{E}[L]$   $d$ (vs List) := Module[{n, L0, Q,  $\Delta$ , G, Z0, Z,  $\lambda$ , DZ, DDZ, FZ, a, b},
  n = Length@vs; L0 = L /.  $\epsilon$  -> 0;
  Q = Table[(- $\partial_{vs[[a]], vs[[b]]$  L0) /. Thread[vs -> 0] /. (p | x) -> 0, {a, n}, {b, n}];
  If[( $\Delta = \text{Det}[Q]$ ) == 0, Return["Degenerate Q!"];
  Z = Z0 = CF@ $\$ \pi$ [L + vs.Q.vs / 2]; G = Inverse[Q];
  FixedPoint[
    {
      DZ = Table[ $\partial_v$  Z, {v, vs}];
      DDZ = Table[ $\partial_u$  DZ, {u, vs}];
      FZ = Sum[G[[a, b]] (DDZ[[a, b]] + DZ[[a]] DZ[[b]]), {a, n}, {b, n}] / 2;
      Z = CF[Z0 +  $\int_0^\lambda$   $\$ \pi$ [FZ]  $d\lambda$ ] &, Z];
  PowerExpand@Factor[ $\omega \Delta^{-1/2}$ ]  $\mathbb{E}$ [CF[Z /.  $\lambda$  -> 1 /. Thread[vs -> 0]]];
Protect[Integrate];
```

pdf

```
In[*]:= T3 = T1 T2; i_+ := i + 1;
$π =
(CF@Normal[# + O[ε]^2] /. {πis__ => B-1 πis, xis__ => B-1 xis, pis__ => B pis} /. ε Bb /. b < 0 → 0 /.
B → 1) &;
```

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```
In[*]:= vsi := Sequence[p1,i, p2,i, p3,i, x1,i, x2,i, x3,i];
F[is__] := E[Sum[πv,i pv,i, {i, {is}}, {v, 3}]];
L[K_] := CF[L/@Features[K][[2]]];
vs[K_] := Union@@Table[{vsi}, {i, Features[K][[1]]}]
```

The Lagrangian

tex

```
\needspace{30mm}
{\bf\red The Lagrangian.}
```

exec

```
nb2tex$PDFWidth *= 1.25;
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```
In[*]:= L[Xi,j[s_]] := T3s E[CF@Plus[
  Sumv=13 (xvi (pvi+ - pvi) + xvj (pvj+ - pvj) + (Tvs - 1) xvi (pvi+ - pvj+)),
  (T1s - 1) p3j x1i (T2s x2i - x2j),
  ε s (T3s - 1) p1j (p2i - p2j) x3i / (T2s - 1),
  ε s (1/2 + T2s p1i p2j x1i x2i - p1i p2j x1i x2j - p3i x3i - (T2s - 1) p2j p3i x2i x3i +
  (T3s - 1) p2j p3j x2i x3i + 2 p2j p3i x2j x3i + p1i p3j x1i x3j - p2i p3j x2i x3j - T2s p2j p3j x2i x3j +
  ((T1s - 1) p1j x1i (T22s p2j x2i - T2s p2j x2j - (T2s + 1) (T3s - 1) p3j x3i + T2s p3j x3j) +
  (T3s - 1) p3j x3i (1 - T2s p1i x1i + p2i x2j + (T2s - 2) p2j x2j)) / (T2s - 1)]]]
```

pdf

```
In[*]:= L[Ci[φ_]] := T3φ E[Sumv=13 xvi (pvi+ - pvi) + ε φ (p3i x3i - 1/2)]
```

exec

```
nb2tex$PDFWidth /= 1.25;
```

Reidemeister 3

tex

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{\bf\red Reidemeister 3.}
```

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$$\text{In[*]:= Short [lhs = \int \mathcal{F}[i, j, k] \mathcal{L} / @ (X_{i,j}[1] X_{i^*,k}[1] X_{j^*,k^*}[1]) \mathcal{d}\{v_{s_i}, v_{s_j}, v_{s_k}, v_{s_{i^*}}, v_{s_{j^*}}, v_{s_{k^*}}\}]$$

Out[*]//Short=

pdf

$$T_1^3 T_2^3 E \left[\frac{3 \in}{2} + \ll 153 \gg \right]$$

pdf

$$\text{In[*]:= rhs = \int \mathcal{F}[i, j, k] \mathcal{L} / @ (X_{j,k}[1] X_{i,k^*}[1] X_{i^*,j^*}[1]) \mathcal{d}\{v_{s_i}, v_{s_j}, v_{s_k}, v_{s_{i^*}}, v_{s_{j^*}}, v_{s_{k^*}}\}; lhs == rhs$$

Out[*]=

pdf

True

$$\text{In[*]:= } \pi = (\text{CF} @ \text{Normal} [\# + 0[\epsilon]^2]) \&;$$

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$$\text{In[*]:= Short [lhs = \int \mathcal{F}[i, j, k] \mathcal{L} / @ (X_{i,j}[1] X_{i^*,k}[1] X_{j^*,k^*}[1]) \mathcal{d}\{v_{s_i}, v_{s_j}, v_{s_k}, v_{s_{i^*}}, v_{s_{j^*}}, v_{s_{k^*}}\}]$$

Out[*]//Short=

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$$T_1^3 T_2^3 E \left[\frac{3 \in}{2} + \ll 311 \gg + \in (1 - T_1) p_{3,2+k}^2 \pi_{1,j} \pi_{2,k} \pi_{3,k} \right]$$

pdf

$$\text{In[*]:= rhs = \int \mathcal{F}[i, j, k] \mathcal{L} / @ (X_{j,k}[1] X_{i,k^*}[1] X_{i^*,j^*}[1]) \mathcal{d}\{v_{s_i}, v_{s_j}, v_{s_k}, v_{s_{i^*}}, v_{s_{j^*}}, v_{s_{k^*}}\}; lhs == rhs$$

Out[*]=

pdf



$$\text{In[*]:= diff = CF[lhs[[3, 1]] - rhs[[3, 1]]] /. \{\pi_{is_} \to B \pi_{is}, x_{is_} \to B x_{is}, p_{is_} \to B^{-1} p_{is}\}$$

Out[*]=

$$\begin{aligned} & 2 B \in (-1 + T_1) T_1 T_2^2 p_{3,2+j} \pi_{1,i} \pi_{2,i} - B \in (-1 + T_1) (1 - T_1 - T_2 + T_1 T_2 + 2 T_1 T_2^2) p_{3,2+k} \pi_{1,i} \pi_{2,i} + \\ & B \in (-1 + T_1) (-1 + T_2) p_{3,2+k} \pi_{1,j} \pi_{2,i} - B \in (-1 + T_1) T_1 T_2^4 p_{2,2+k} p_{3,2+j} \pi_{1,i} \pi_{2,i}^2 + \\ & B \in (-1 + T_1) T_2^3 p_{2,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,i}^2 + B \in (-1 + T_1) T_2^3 (-1 + T_1 T_2) p_{2,2+k} p_{3,2+k} \pi_{1,i} \pi_{2,i}^2 - \\ & B^2 \in (-1 + T_1)^2 T_1 T_2^2 (1 - T_1 - T_2 - T_2^2 + 2 T_1 T_2^2) p_{3,2+j} p_{3,2+k} \pi_{1,i}^2 \pi_{2,i}^2 + \\ & B^2 \in (-1 + T_1)^2 T_2^2 (1 - T_1 + T_2 - 2 T_1 T_2 - T_1 T_2^2 + 2 T_1^2 T_2^2) p_{3,2+k}^2 \pi_{1,i}^2 \pi_{2,i}^2 + \\ & B^2 \in (-1 + T_1)^2 T_1 (-1 + T_2) T_2^2 (1 + T_2) p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{1,j} \pi_{2,i}^2 - \\ & B^2 \in (-1 + T_1)^2 T_2^2 (-1 - T_2 + T_1 T_2^2) p_{3,2+k}^2 \pi_{1,i} \pi_{1,j} \pi_{2,i}^2 - 2 B \in (-1 + T_1) T_1 T_2 p_{3,2+j} \pi_{1,i} \pi_{2,j} + \\ & B \in (-1 + T_1) (-1 + T_1 + T_1 T_2) p_{3,2+k} \pi_{1,i} \pi_{2,j} + B \in (-1 + T_1)^2 T_1^2 T_2 p_{1,2+i} p_{3,2+k} \pi_{1,i}^2 \pi_{2,j} - \\ & B \in (-1 + T_1)^2 T_1^2 T_2 p_{1,2+j} p_{3,2+k} \pi_{1,i}^2 \pi_{2,j} + B \in (-1 + T_1) T_2 p_{3,2+k} \pi_{1,j} \pi_{2,j} - \\ & 2 B \in (-1 + T_1) T_1^2 T_2 p_{1,2+i} p_{3,2+k} \pi_{1,i} \pi_{1,j} \pi_{2,j} + 2 B \in (-1 + T_1) T_1^2 T_2 p_{1,2+j} p_{3,2+k} \pi_{1,i} \pi_{1,j} \pi_{2,j} + \\ & B \in (-1 + T_1) T_1 T_2^2 p_{2,2+k} p_{3,2+j} \pi_{1,i} \pi_{2,i} \pi_{2,j} - B \in (-1 + T_1)^2 T_2^3 p_{2,2+i} p_{3,2+k} \pi_{1,i} \pi_{2,i} \pi_{2,j} + \\ & B \in (-1 + T_1) T_2^2 (2 - 3 T_2 + T_1 T_2) p_{2,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,i} \pi_{2,j} \\ & \hline & -1 + T_2 \\ & B \in (-1 + T_1) (-1 + 2 T_1) T_2^2 p_{2,2+k} p_{3,2+k} \pi_{1,i} \pi_{2,i} \pi_{2,j} + \\ & B^2 \in (-1 + T_1)^2 T_1 T_2 (-1 + 2 T_2) (-1 + T_1 + T_1 T_2) p_{3,2+j} p_{3,2+k} \pi_{1,i}^2 \pi_{2,i} \pi_{2,j} - \\ & B^2 \in (-1 + T_1)^2 T_2 (1 - T_1 + T_2 - 3 T_1 T_2 + T_1^2 T_2 + T_1^2 T_2^2) p_{3,2+k}^2 \pi_{1,i}^2 \pi_{2,i} \pi_{2,j} + \\ & B \in (-1 + T_1) T_2^3 p_{2,2+i} p_{3,2+k} \pi_{1,j} \pi_{2,i} \pi_{2,j} + B \in (-1 + T_1) T_2^2 p_{2,2+k} p_{3,2+k} \pi_{1,j} \pi_{2,i} \pi_{2,j} - \end{aligned}$$

$$\begin{aligned}
 & B^2 \in (-1 + T_1)^2 T_1 T_2 (-1 + T_2 + T_2^2) p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{1,j} \pi_{2,i} \pi_{2,j} + \\
 & B^2 \in (-1 + T_1)^2 T_2 (-1 - 2 T_2 + T_1 T_2) p_{3,2+k}^2 \pi_{1,i} \pi_{1,j} \pi_{2,i} \pi_{2,j} + \\
 & B \in (-1 + T_1) T_1 (-1 + T_2) T_2 p_{2,2+k} p_{3,2+j} \pi_{1,i} \pi_{2,j}^2 - \\
 & B \in (-1 + T_1) T_2 (1 - 3 T_2 + T_1 T_2 + T_2^2) p_{2,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,j}^2 - \\
 & \qquad \qquad \qquad -1 + T_2 \\
 & B \in (-1 + T_1) T_1 (-1 + T_2) T_2 p_{2,2+k} p_{3,2+k} \pi_{1,i} \pi_{2,j}^2 - \\
 & B^2 \in (-1 + T_1)^3 T_1 T_2 p_{3,2+j} p_{3,2+k} \pi_{1,i}^2 \pi_{2,j}^2 - B^2 \in (-1 + T_1)^3 T_1 T_2^2 p_{3,2+k}^2 \pi_{1,i}^2 \pi_{2,j}^2 + \\
 & B^2 \in (-1 + T_1)^2 T_1 T_2 p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{1,j} \pi_{2,j}^2 + B^2 \in (-1 + T_1)^2 T_1 T_2^2 p_{3,2+k}^2 \pi_{1,i} \pi_{1,j} \pi_{2,j}^2 + \\
 & B \in (-1 + T_1) T_1 p_{3,2+k} \pi_{1,i} \pi_{2,k} - B \in (-1 + T_1)^2 T_1^2 p_{1,2+i} p_{3,2+k} \pi_{1,i}^2 \pi_{2,k} + \\
 & B \in (-1 + T_1)^2 T_1^2 p_{1,2+j} p_{3,2+k} \pi_{1,i}^2 \pi_{2,k} + B \in (1 - T_1) p_{3,2+k} \pi_{1,j} \pi_{2,k} + \\
 & 2 B \in (-1 + T_1) T_1^2 p_{1,2+i} p_{3,2+k} \pi_{1,i} \pi_{1,j} \pi_{2,k} - 2 B \in (-1 + T_1) T_1^2 p_{1,2+j} p_{3,2+k} \pi_{1,i} \pi_{1,j} \pi_{2,k} + \\
 & 3 B \in (-1 + T_1) T_1 T_2^2 p_{2,2+k} p_{3,2+j} \pi_{1,i} \pi_{2,i} \pi_{2,k} + B \in (-1 + T_1)^2 T_2^2 p_{2,2+i} p_{3,2+k} \pi_{1,i} \pi_{2,i} \pi_{2,k} - \\
 & B \in (-1 + T_1) T_2 (1 + T_1 - 2 T_2 - 3 T_1 T_2 + 3 T_1 T_2^2) p_{2,2+k} p_{3,2+k} \pi_{1,i} \pi_{2,i} \pi_{2,k} + \\
 & \qquad \qquad \qquad -1 + T_2 \\
 & B^2 \in (-1 + T_1)^2 T_1 (-3 + 2 T_1) T_2^2 p_{3,2+j} p_{3,2+k} \pi_{1,i}^2 \pi_{2,i} \pi_{2,k} - \\
 & B^2 \in (-1 + T_1)^3 T_2 (T_1 - 2 T_2 - 2 T_1 T_2 + 3 T_1 T_2^2) p_{3,2+k}^2 \pi_{1,i}^2 \pi_{2,i} \pi_{2,k} - \\
 & \qquad \qquad \qquad -1 + T_2 \\
 & B \in (-1 + T_1) T_2^2 p_{2,2+i} p_{3,2+k} \pi_{1,j} \pi_{2,i} \pi_{2,k} - B \in (-1 + T_1) T_2 p_{2,2+k} p_{3,2+k} \pi_{1,j} \pi_{2,i} \pi_{2,k} - \\
 & 2 B^2 \in (-1 + T_1)^2 T_1 T_2^2 p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{1,j} \pi_{2,i} \pi_{2,k} + \\
 & B^2 \in (-1 + T_1)^2 T_2 (-1 + T_1 - T_2 - 2 T_1 T_2 + 3 T_1 T_2^2) p_{3,2+k}^2 \pi_{1,i} \pi_{1,j} \pi_{2,i} \pi_{2,k} - \\
 & \qquad \qquad \qquad -1 + T_2 \\
 & 3 B \in (-1 + T_1) T_1 T_2 p_{2,2+k} p_{3,2+j} \pi_{1,i} \pi_{2,j} \pi_{2,k} + B \in (-1 + T_1) T_2 p_{2,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,j} \pi_{2,k} + \\
 & B \in (-1 + T_1) (1 - 2 T_2 - 2 T_1 T_2 + 3 T_1 T_2^2) p_{2,2+k} p_{3,2+k} \pi_{1,i} \pi_{2,j} \pi_{2,k} - \\
 & \qquad \qquad \qquad -1 + T_2 \\
 & 2 B^2 \in (-1 + T_1)^3 T_1 T_2 p_{3,2+j} p_{3,2+k} \pi_{1,i}^2 \pi_{2,j} \pi_{2,k} + \\
 & B^2 \in (-1 + T_1)^3 T_2 (-2 - T_1 + 3 T_1 T_2) p_{3,2+k}^2 \pi_{1,i}^2 \pi_{2,j} \pi_{2,k} + \\
 & \qquad \qquad \qquad -1 + T_2 \\
 & 2 B^2 \in (-1 + T_1)^2 T_1 T_2 p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{1,j} \pi_{2,j} \pi_{2,k} - \\
 & B^2 \in (-1 + T_1)^2 T_2 (-2 - T_1 + 3 T_1 T_2) p_{3,2+k}^2 \pi_{1,i} \pi_{1,j} \pi_{2,j} \pi_{2,k} + \\
 & \qquad \qquad \qquad -1 + T_2 \\
 & \frac{1}{-1 + T_2} B \in (-1 + T_1) T_1 T_2 (-1 + T_1 T_2) (T_1 + T_2 - 2 T_1 T_2 - 2 T_2^2 + 2 T_1 T_2^3) p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,i} \pi_{3,i} - \\
 & \frac{1}{-1 + T_2} B \in (-1 + T_1) T_2 (-1 + T_1 T_2) (-1 + 2 T_1 + T_2 - T_1 T_2 - T_1^2 T_2 - 2 T_1 T_2^2 + 2 T_1^2 T_2^3) p_{3,2+k}^2 \pi_{1,i} \pi_{2,i} \pi_{3,i} + \\
 & B \in (-1 + T_1) T_1 T_2 (-1 + T_1 T_2) p_{3,2+j} p_{3,2+k} \pi_{1,j} \pi_{2,i} \pi_{3,i} - \\
 & B \in (-1 + T_1) T_2 (-1 + T_1 T_2) p_{3,2+k}^2 \pi_{1,j} \pi_{2,i} \pi_{3,i} - \\
 & B \in (-1 + T_1) T_1 T_2 (-1 + T_1 T_2) (-1 - T_1 - 2 T_2 + 4 T_1 T_2) p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,j} \pi_{3,i} + \\
 & \qquad \qquad \qquad -1 + T_2 \\
 & B \in (-1 + T_1) T_2 (-1 + T_1 T_2) (1 - 3 T_1 + T_1 T_2 + T_1^2 T_2) p_{3,2+k}^2 \pi_{1,i} \pi_{2,j} \pi_{3,i} + \\
 & \qquad \qquad \qquad -1 + T_2
 \end{aligned}$$

$$\begin{aligned}
 & 2 \mathbf{B} \in (-1 + T_1) T_1^2 T_2^3 p_{3,2+i} p_{3,2+k} \pi_{1,j} \pi_{2,j} \pi_{3,i} - \\
 & \mathbf{B} \in \frac{(-1 + T_1) T_1 T_2 (-1 + T_1 T_2) (1 - 2 T_2 + 2 T_2^2) p_{3,2+j} p_{3,2+k} \pi_{1,j} \pi_{2,j} \pi_{3,i}}{-1 + T_2} + \\
 & \mathbf{B} \in \frac{(-1 + T_1) T_2 (-1 + T_1 T_2) (1 - 2 T_2 + T_1 T_2) p_{3,2+k}^2 \pi_{1,j} \pi_{2,j} \pi_{3,i}}{-1 + T_2} - \\
 & 2 \mathbf{B} \in (-1 + T_1) T_1^2 T_2 (-1 + T_1 T_2) p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,k} \pi_{3,i} + \\
 & \mathbf{B} \in \frac{(-1 + T_1) T_1 T_2 (-1 + T_1 T_2) (-3 + T_1 + 2 T_1 T_2) p_{3,2+k}^2 \pi_{1,i} \pi_{2,k} \pi_{3,i}}{-1 + T_2} - \\
 & 2 \mathbf{B} \in (-1 + T_1) T_1^2 T_2^2 p_{3,2+i} p_{3,2+k} \pi_{1,j} \pi_{2,k} \pi_{3,i} + \\
 & 2 \mathbf{B} \in (-1 + T_1) T_1 T_2 (-1 + T_1 T_2) p_{3,2+j} p_{3,2+k} \pi_{1,j} \pi_{2,k} \pi_{3,i} - \\
 & \mathbf{B} \in \frac{(-2 + T_1) (-1 + T_1) T_2 (-1 + T_1 T_2) p_{3,2+k}^2 \pi_{1,j} \pi_{2,k} \pi_{3,i}}{-1 + T_2} - \\
 & \mathbf{B} \in (-1 + T_1) T_1 T_2 (1 - T_1 - 2 T_2 + 2 T_1 T_2^2) p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,i} \pi_{3,j} + \\
 & \mathbf{B} \in (-1 + T_1) T_2 (-1 + T_1 T_2) (-1 + T_1 + T_1 T_2) p_{3,2+k}^2 \pi_{1,i} \pi_{2,i} \pi_{3,j} - \\
 & \mathbf{B} \in (-1 + T_1) T_1 T_2 p_{3,2+j} p_{3,2+k} \pi_{1,j} \pi_{2,i} \pi_{3,j} - \mathbf{B} \in (-1 + T_1) T_2 (-1 + T_1 T_2) p_{3,2+k}^2 \pi_{1,j} \pi_{2,i} \pi_{3,j} + \\
 & \mathbf{B} \in \frac{(-1 + T_1)^2 T_1 T_2 (-1 + 2 T_2) p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,j} \pi_{3,j}}{-1 + T_2} + \\
 & \mathbf{B} \in (-1 + T_1) T_1 T_2 (-1 + T_1 T_2) p_{3,2+k}^2 \pi_{1,i} \pi_{2,j} \pi_{3,j} + 2 \mathbf{B} \in (-1 + T_1) T_1^2 T_2 p_{3,2+j} p_{3,2+k} \pi_{1,i} \pi_{2,k} \pi_{3,j} - \\
 & \mathbf{B} \in \frac{(-1 + T_1) (-1 + T_1 T_2) (-1 + 2 T_1 T_2) p_{3,2+k}^2 \pi_{1,i} \pi_{2,k} \pi_{3,j}}{-1 + T_2} + \\
 & \mathbf{B} \in (-1 + T_1) T_1 (-1 + T_2) T_2 p_{3,2+k}^2 \pi_{1,i} \pi_{2,i} \pi_{3,k} + \\
 & \mathbf{B} \in (-1 + T_1) T_2 p_{3,2+k}^2 \pi_{1,j} \pi_{2,i} \pi_{3,k} - \mathbf{B} \in (-1 + T_1) T_1 T_2 p_{3,2+k}^2 \pi_{1,i} \pi_{2,j} \pi_{3,k}
 \end{aligned}$$

In[*]:= {diff /. e -> 0, diff /. B -> 0}

Out[*]=
{0, 0}

The Trefoil

tex

```

\needspace{25mm}
\parpic[r]{\includegraphics[width=0.6in]{../Beijing-2407/Trefoil.jpg}}
{\bf\red The Trefoil.}
    
```

pdf

In[*]:= $\mathbf{K} = \text{Knot}[3, 1]; \int \mathcal{L}[\mathbf{K}] \mathbf{d} \mathbf{vs}[\mathbf{K}]$

pdf

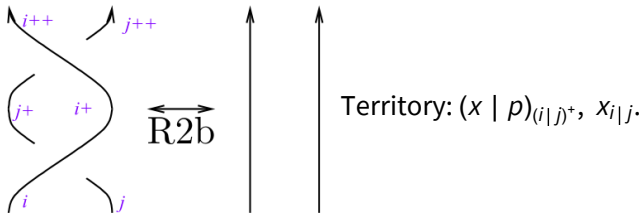
 KnotTheory: Loading precomputed data in PD4Knots`

Out[*]=

pdf

$$\frac{i T_1^2 T_2^2 \mathbb{E} \left[- \frac{\epsilon (1 - T_1 + T_1^2 - T_2 - T_1^3 T_2 + T_2^2 + T_1^4 T_2^2 - T_1 T_2^3 - T_1^2 T_2^4 + T_1^2 T_2^4 - T_1^3 T_2^4 + T_1^4 T_2^4)}{(1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2)} \right]}{(1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2)}$$

Invariance Under Reidemeister 2b



$$\text{In[*]} := \text{lhs} = \int \mathcal{F}[i, j] \mathcal{L} / @ (X_{i,j}[1] X_{i+1,j+1}[-1]) \mathcal{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\}$$

$$\text{rhs} = \int \mathcal{F}[i, j] \mathcal{L} / @ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) \mathcal{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\};$$

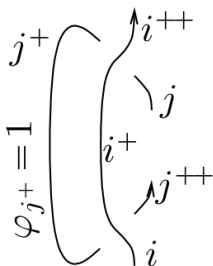
lhs == rhs

Out[*]=

$$\mathbb{E} [p_{1,2+i} \pi_{1,i} + p_{1,2+j} \pi_{1,j} + p_{2,2+i} \pi_{2,i} + p_{2,2+j} \pi_{2,j} + p_{3,2+i} \pi_{3,i} + p_{3,2+j} \pi_{3,j}]$$

Out[*]=
True

Invariance Under R2c



$$\text{In[*]} := \text{lhs} = \int \mathcal{F}[i, j] \mathcal{L} / @ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1]) \mathcal{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\}$$

$$\text{rhs} = \int \mathcal{F}[i, j] \mathcal{L} / @ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0]) \mathcal{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\};$$

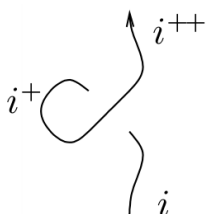
lhs == rhs

Out[*]=

$$-\frac{i}{2} T_1 T_2 \mathbb{E} \left[\frac{\epsilon}{2} + p_{1,2+i} \pi_{1,i} + p_{1,3+j} \pi_{1,j} + p_{2,2+i} \pi_{2,i} + p_{2,3+j} \pi_{2,j} + p_{3,2+i} \pi_{3,i} + p_{3,3+j} \pi_{3,j} + \epsilon p_{3,3+j} \pi_{3,j} \right]$$

Out[*]=
True

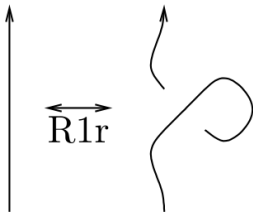
Invariance Under R1l



```
In[*]:= lhs = ∫ ℱ[i] ℒ /@ (Xi+2,i[1] Ci+1[1]) d{vsi, vsi+, vsi+2}
rhs = ∫ ℱ[i] ℒ /@ (Ci[0] Ci+1[0] Ci+2[0]) d{vsi, vsi+, vsi+2};
lhs == rhs
```

```
Out[*]= -i E [p1,3+i π1,i + p2,3+i π2,i + p3,3+i π3,i]
Out[*]= True
```

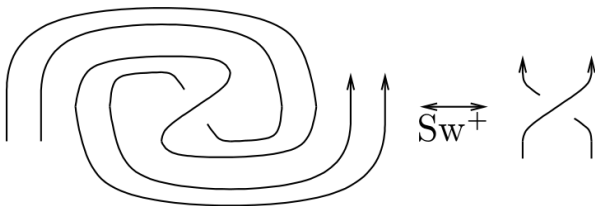
Invariance Under R1r



```
In[*]:= lhs = ∫ ℱ[i] ℒ /@ (Xi,i+2[1] Ci+1[-1]) d{vsi, vsi+, vsi+2}
rhs = ∫ ℱ[i] ℒ /@ (Ci[0] Ci+1[0] Ci+2[0]) d{vsi, vsi+, vsi+2};
lhs == rhs
```

```
Out[*]= -i E [p1,3+i π1,i + p2,3+i π2,i + p3,3+i π3,i]
Out[*]= True
```

Invariance Under Sw



In[*]:= lhs =

$$\int \mathcal{F}[i, j] \mathcal{L} / @ (X_{i+1, j+1} [1] C_i [-1] C_j [-1] C_{i+2} [1] C_{j+2} [1]) \, d\{vs_i, vs_j, vs_{i^*}, vs_{j^*}, vs_{i+2}, vs_{j+2}\}$$

$$rhs = \int \mathcal{F}[i, j] \mathcal{L} / @ (X_{i+1, j+1} [1] C_i [0] C_j [0] C_{i+2} [0] C_{j+2} [0]) \, d\{vs_i, vs_j, vs_{i^*}, vs_{j^*}, vs_{i+2}, vs_{j+2}\};$$

lhs == rhs

Out[*]=

$$T_1 T_2 E \left[\begin{aligned} & \frac{\epsilon}{2} + T_1 p_{1,3+i} \pi_{1,i} + (1 - T_1) p_{1,3+j} \pi_{1,i} + p_{1,3+j} \pi_{1,j} + T_2 p_{2,3+i} \pi_{2,i} - \epsilon T_2 p_{2,3+i} \pi_{2,i} + (1 - T_2) p_{2,3+j} \pi_{2,i} + \\ & \epsilon T_1 T_2 p_{1,3+i} p_{2,3+j} \pi_{1,i} \pi_{2,i} + \frac{\epsilon (-1 + T_1) T_2 p_{1,3+j} p_{2,3+j} \pi_{1,i} \pi_{2,i}}{-1 + T_2} + (-1 + T_1) T_2 p_{3,3+j} \pi_{1,i} \pi_{2,i} + \\ & p_{2,3+j} \pi_{2,j} + \epsilon p_{2,3+j} \pi_{2,j} - \epsilon T_1 p_{1,3+i} p_{2,3+j} \pi_{1,i} \pi_{2,j} - \frac{\epsilon (-1 + T_1) p_{1,3+j} p_{2,3+j} \pi_{1,i} \pi_{2,j}}{-1 + T_2} + \\ & (1 - T_1) p_{3,3+j} \pi_{1,i} \pi_{2,j} + \frac{\epsilon T_2 (-1 + T_1 T_2) p_{1,3+j} p_{2,3+i} \pi_{3,i}}{-1 + T_2} - \frac{\epsilon T_2 (-1 + T_1 T_2) p_{1,3+j} p_{2,3+j} \pi_{3,i}}{-1 + T_2} + \\ & T_1 T_2 p_{3,3+i} \pi_{3,i} + \epsilon T_1 T_2 p_{3,3+i} \pi_{3,i} + (1 - T_1 T_2) p_{3,3+j} \pi_{3,i} - \frac{\epsilon T_2 (-1 + T_1 T_2) p_{3,3+j} \pi_{3,i}}{-1 + T_2} - \\ & \frac{\epsilon T_1 T_2 (-1 + T_1 T_2) p_{1,3+i} p_{3,3+j} \pi_{1,i} \pi_{3,i}}{-1 + T_2} + \frac{\epsilon (-1 + T_1) T_2 (-1 + T_1 T_2) p_{1,3+j} p_{3,3+j} \pi_{1,i} \pi_{3,i}}{-1 + T_2} - \\ & \epsilon T_1 (-1 + T_2) T_2 p_{2,3+j} p_{3,3+i} \pi_{2,i} \pi_{3,i} + \epsilon T_2 (-1 + T_1 T_2) p_{2,3+j} p_{3,3+j} \pi_{2,i} \pi_{3,i} + \\ & 2 \epsilon T_1 T_2 p_{2,3+j} p_{3,3+i} \pi_{2,j} \pi_{3,i} + \frac{\epsilon T_2 (-1 + T_1 T_2) p_{2,3+i} p_{3,3+j} \pi_{2,j} \pi_{3,i}}{-1 + T_2} - \\ & \frac{\epsilon (-1 + 2 T_2) (-1 + T_1 T_2) p_{2,3+j} p_{3,3+j} \pi_{2,j} \pi_{3,i}}{-1 + T_2} + p_{3,3+j} \pi_{3,j} + \epsilon T_1 p_{1,3+i} p_{3,3+j} \pi_{1,i} \pi_{3,j} + \\ & \frac{\epsilon (-1 + T_1) p_{1,3+j} p_{3,3+j} \pi_{1,i} \pi_{3,j}}{-1 + T_2} - \epsilon T_2 p_{2,3+i} p_{3,3+j} \pi_{2,i} \pi_{3,j} - \epsilon p_{2,3+j} p_{3,3+j} \pi_{2,i} \pi_{3,j} \end{aligned} \right]$$

Out[*]=

True