Pensieve header: Proof of invariance of ρ_2 using integration techniques; continues pensieve://Talks/Geneva-2408.

Initialization

In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

Initialization

In[ ]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CCF[ℰ_] := Factor[ℰ];
CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs = Cases[ℰ, (x | p | π)__, ∞] ⋃ {x, p, ϵ}, ps, c},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :→ CCF[c] (Times @@ vs^ps)]];
```

The Basic Feynman Ring

In[ ]:=
```
𝒮 = {x, x_, y, z};
q_{x_,y_}[f_] := (∂_{x,y} f) /. Thread[𝒮 → 0];
θ_{x_,y_} := x y;
f_ ≡ 0 := f === 0;
Ev_{vs_List→0}[f_] := CF[f /. Thread[vs → 0]]
```

The ϵSeries Feynman Ring

In[∘]:=
```
𝒮 = {x, y, z, ϕ, x_, p_, x̲_, p̲_};
q_{x_,y_}[ser_ϵSeries] := (∂_{x,y} ser⟦1⟧) /. Thread[𝒮 → 0];
θ_{x_,y_} := x y;
ϵSeries /: D[ser_ϵSeries, vs___] := D[#, vs] & /@ ser;
ϵSeries /: Plus[ss___ϵSeries] /; Length[{ss}] > 1 := Module[{l = Min[Length /@ {ss}]},
  ϵSeries @@ Total[Take[List @@ #, l] & /@ {ss}]]
ϵSeries /: t_ + ser_ϵSeries := MapAt[(# + t) &, ser, 1];
ϵSeries /: s1_ϵSeries * s2_ϵSeries := ϵSeries @@ Table[
    Sum[s1⟦ii + 1⟧ s2⟦kk − ii + 1⟧, {ii, 0, kk}], {kk, 0, Min[Length@s1, Length@s2] − 1}];
ϵSeries /: c_ * ser_ϵSeries := (c #) & /@ ser;
ser_ϵSeries ≡ 0 := And @@ ((# === 0) & /@ ser);
ϵSeries /: Integrate[ser_ϵSeries, pars__] := ϵSeries @@ (Integrate[#, pars] & /@ ser);
ϵSeries /: Ev_{vs_List→0}[ser_ϵSeries] := ser /. Thread[vs → 0];
CF[ser_ϵSeries] := CF /@ ser;
```

Integration

Using Picard Iteration!

In[∘]:=
```
𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B]
```

In[∘]:=
```
𝔼[sd_SeriesData] /; (List @@ sd)⟦{1, 2, 4, 6}⟧ === {ϵ, 0, 0, 1} :=
 𝔼[ϵSeries @@ PadRight[sd⟦3⟧, sd⟦5⟧, 0]]
```

Following a program in Projects/FullDoPeGDO/Engine.nb, we write $Z_\lambda = \sum Z[m]\,\lambda^m$.

In[∘]:=
```
Unprotect[Integrate];
Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, Q, Δ, G, a, b, m, m1, $m}, Clear[Z];
   n = Length@vs;
   Q = Table[q_{vs⟦a⟧,vs⟦b⟧}[L], {a, n}, {b, n}];
   If[(Δ = CF@Det[-Q]) == 0, Message[Integrate::sing]; Return[]];
   G = CF[-Inverse[Q] / 2];
   Z[] = Z[0] = CF[L - Sum[Q⟦a, b⟧ θ_{vs⟦a⟧,vs⟦b⟧}, {a, n}, {b, n}] / 2];
   Z[m_, a_] := Z[m, a] = CF@D[Z[m], vs⟦a⟧];
   Z[m_, a_, b_] /; a ≤ b := Z[m, a, b] = CF@D[Z[m, a], vs⟦b⟧];
   Z[m_, a_, b_] /; a > b := Z[m, b, a];
   For[$m = m = 0, m ≤ 2 $m, ++m,
    Z[m + 1] = CF@Sum[Sum[If[G⟦a, b⟧ === 0, 0,
          G⟦a, b⟧/(m + 1) (Z[m, a, b] + Sum[Z[m1, a] Z[m - m1, b], {m1, 0, m}])],
        {a, n}], {b, n}];
    If[! (Z[m + 1] ≡ 0), $m = m + 1; Z[] += Z[m + 1]];
   ];
   PowerExpand@Factor[ω Δ^(-1/2)] 𝔼[CF[Ev_{vs→0}[Z[]]]]
  ];
Protect[Integrate];
```

In[∘]:= $\int \mathbb{E}\left[-\mu x^2/2 + i\,\xi\,x\right] ⅆ\{x\}$

Out[∘]=

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

In[∘]:= $L = -\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$

$Z12 = \int \mathbb{E}[L]\, ⅆ\{x_1, x_2\}$

Out[∘]=

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2 - 2\,b\,\xi_1\,\xi_2 + a\,\xi_2^2}{2\,(-b^2 + a\,c)}\right]}{\sqrt{-b^2 + a\,c}}$$

In[∘]:= $\left\{Z1 = \int \mathbb{E}[L]\, ⅆ\{x_1\},\ Z12 == \int Z1\, ⅆ\{x_2\}\right\}$

Out[∘]=

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2 + a\,c)\,x_2^2}{2\,a} + \frac{\xi_1^2}{2\,a} + \frac{x_2\,(-b\,\xi_1 + a\,\xi_2)}{a}\right]}{\sqrt{a}},\ \text{True}\right\}$$

Integration of ϵSeries

In[∘]:=
$$\int \mathbb{E}\left[-x^2/2+\epsilon\, x^3/6+O[\epsilon]^{13}\right] d\{x\}$$

Out[∘]=
$$\mathbb{E}\left[\epsilon\text{Series}\left[0,0,\frac{5}{24},0,\frac{5}{16},0,\frac{1105}{1152},0,\frac{565}{128},0,\frac{82825}{3072},0,\frac{19675}{96}\right]\right]$$

The ρ_2 Integrand

Adopted from pensieve://Talks//Oaxaca-2210/Rho.nb.

In[∘]:=

$\mathcal{S}=\{x\_, p\_\};$

$q[s\_, i\_, j\_] := x_i\,(p_{i+1}-p_i)+x_j\,(p_{j+1}-p_j)+\left(T^s-1\right)x_i\,(p_{i+1}-p_{j+1});$

$r_1[s\_, i\_, j\_] := \frac{s}{2}\left(x_i\,(p_i-p_j)\left(\left(T^s-1\right)x_i\,p_j+2\,(1-p_j\,x_j)\right)-1\right);$

$r_2[1, i\_, j\_] := \left(-6\,p_i\,x_i+6\,p_j\,x_i-3\,(-1+3\,T)\,p_i\,p_j\,x_i^2+3\,(-1+3\,T)\,p_j^2\,x_i^2+4\,(-1+T)\,p_i^2\,p_j\,x_i^3-2\,(-1+T)\,(5+T)\,p_i\,p_j^2\,x_i^3+2\,(-1+T)\,(3+T)\,p_j^3\,x_i^3+18\,p_i\,p_j\,x_i\,x_j-18\,p_j^2\,x_i\,x_j-6\,p_i^2\,p_j\,x_i^2\,x_j+6\,(2+T)\,p_i\,p_j^2\,x_i^2\,x_j-6\,(1+T)\,p_j^3\,x_i^2\,x_j-6\,p_i\,p_j^2\,x_i\,x_j^2+6\,p_j^3\,x_i\,x_j^2\right)/12;$

$r_2[-1, i\_, j\_] := \left(-6\,T^2\,p_i\,x_i+6\,T^2\,p_j\,x_i+3\,(-3+T)\,T\,p_i\,p_j\,x_i^2-3\,(-3+T)\,T\,p_j^2\,x_i^2-4\,(-1+T)\,T\,p_i^2\,p_j\,x_i^3+2\,(-1+T)\,(1+5\,T)\,p_i\,p_j^2\,x_i^3-2\,(-1+T)\,(1+3\,T)\,p_j^3\,x_i^3+18\,T^2\,p_i\,p_j\,x_i\,x_j-18\,T^2\,p_j^2\,x_i\,x_j-6\,T^2\,p_i^2\,p_j\,x_i^2\,x_j+6\,T\,(1+2\,T)\,p_i\,p_j^2\,x_i^2\,x_j-6\,T\,(1+T)\,p_j^3\,x_i^2\,x_j-6\,T^2\,p_i\,p_j^2\,x_i\,x_j^2+6\,T^2\,p_j^3\,x_i\,x_j^2\right)/\left(12\,T^2\right);$

$\gamma_1[\varphi\_, k\_] := \varphi\,(1/2-x_k\,p_k);$

$\gamma_2[\varphi\_, k\_] := -\varphi^2\,p_k\,x_k/2;$

$\mathcal{L}[X_{i\_,j\_}[s\_]] := T^{s/2}\,\mathbb{E}\left[q[s,i,j]+\epsilon\,r_1[s,i,j]+\epsilon^2\,r_2[s,i,j]+O[\epsilon]^3\right];$

$\mathcal{L}[C_{k\_}[\varphi\_]] := T^{\varphi/2}\,\mathbb{E}\left[-x_k\,(p_k-p_{k+1})+\epsilon\,\gamma_1[\varphi,k]+\epsilon^2\,\gamma_2[\varphi,k]+O[\epsilon]^3\right];$

$\mathcal{L}[K\_] := (2\,\pi)^{-\text{Features}[K][[1]]}\ \text{CF}[\mathcal{L}\,/@\,\text{Features}[K][[2]]];$

$\text{vs}[K\_] := \text{Union}\;@@\;\text{Table}[\{p_i, x_i\}, \{i, \text{Features}[K][[1]]\}]$

exec

In[∘]:=
```
nb2tex$PDFWidth *= 1.25;
```

exec

TimesBy: nb2tex$PDFWidth is not a variable with a value, so its value cannot be changed.

pdf

In[∘]:= $\epsilon^2\,r_2[1, i, j]$

Out[∘]=
pdf

$$\frac{1}{12}\,\epsilon^2\left(-6\,p_i\,x_i+6\,p_j\,x_i-3\,(-1+3\,T)\,p_i\,p_j\,x_i^2+3\,(-1+3\,T)\,p_j^2\,x_i^2+4\,(-1+T)\,p_i^2\,p_j\,x_i^3-2\,(-1+T)\,(5+T)\,p_i\,p_j^2\,x_i^3+2\,(-1+T)\,(3+T)\,p_j^3\,x_i^3+18\,p_i\,p_j\,x_i\,x_j-18\,p_j^2\,x_i\,x_j-6\,p_i^2\,p_j\,x_i^2\,x_j+6\,(2+T)\,p_i\,p_j^2\,x_i^2\,x_j-6\,(1+T)\,p_j^3\,x_i^2\,x_j-6\,p_i\,p_j^2\,x_i\,x_j^2+6\,p_j^3\,x_i\,x_j^2\right)$$

pdf

In[∘]:= $\epsilon^2\, r_2[-1, i, j]$

Out[∘]=
pdf

$$\frac{1}{12\, T^2}\, \epsilon^2\, \left(-6\, T^2\, p_i\, x_i + 6\, T^2\, p_j\, x_i + 3\, (-3+T)\, T\, p_i\, p_j\, x_i^2 - 3\, (-3+T)\, T\, p_j^2\, x_i^2 - 4\, (-1+T)\, T\, p_i^2\, p_j\, x_i^3 + 2\, (-1+T)\, (1+5\, T)\, p_i\, p_j^2\, x_i^3 - 2\, (-1+T)\, (1+3\, T)\, p_j^3\, x_i^3 + 18\, T^2\, p_i\, p_j\, x_i\, x_j - 18\, T^2\, p_j^2\, x_i\, x_j - 6\, T^2\, p_i^2\, p_j\, x_i^2\, x_j + 6\, T\, (1+2\, T)\, p_i\, p_j^2\, x_i^2\, x_j - 6\, T\, (1+T)\, p_j^3\, x_i^2\, x_j - 6\, T^2\, p_i\, p_j^2\, x_i\, x_j^2 + 6\, T^2\, p_j^3\, x_i\, x_j^2\right)$$

pdf

In[∘]:= $\epsilon^2\, \gamma_2[\varphi, i]$

Out[∘]=
pdf

$$-\frac{1}{2}\, \epsilon^2\, \varphi^2\, p_i\, x_i$$

exec

In[∘]:= `nb2tex$PDFWidth /= 1.25;`

exec

DivideBy: nb2tex$PDFWidth is not a variable with a value, so its value cannot be changed.

In[∘]:= `Features[Knot[3, 1]]`

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$\text{Features}[7, C_4[-1]\, X_{2,6}[-1]\, X_{5,1}[-1]\, X_{7,3}[-1]]$

In[∘]:= `ℒ[Knot[3, 1]]`

Out[∘]=

$$
\begin{aligned}
&\frac{1}{128\,\pi^7\,T^2}\\
&\mathbb{E}\Big[\epsilon\text{Series}\Big[-p_1 x_1 + p_2 x_1 - p_2 x_2 + \frac{p_3 x_2}{T} + \frac{(-1+T)\,p_7 x_2}{T} - p_3 x_3 + p_4 x_3 - p_4 x_4 + p_5 x_4 + \frac{(-1+T)\,p_2 x_5}{T} -\\
&\quad p_5 x_5 + \frac{p_6 x_5}{T} - p_6 x_6 + p_7 x_6 + \frac{(-1+T)\,p_4 x_7}{T} - p_7 x_7 + \frac{p_8 x_7}{T},\ 1 - p_2 x_2 + p_6 x_2 + \frac{(-1+T)\,p_2 p_6 x_2^2}{2\,T} -\\
&\quad \frac{(-1+T)\,p_6^2 x_2^2}{2\,T} + p_4 x_4 + p_1 x_5 - p_5 x_5 - p_1^2 x_1 x_5 + p_1 p_5 x_1 x_5 - \frac{(-1+T)\,p_1^2 x_5^2}{2\,T} + \frac{(-1+T)\,p_1 p_5 x_5^2}{2\,T} +\\
&\quad p_2 p_6 x_2 x_6 - p_6^2 x_2 x_6 + p_3 x_7 - p_7 x_7 - p_3^2 x_3 x_7 + p_3 p_7 x_3 x_7 - \frac{(-1+T)\,p_3^2 x_7^2}{2\,T} + \frac{(-1+T)\,p_3 p_7 x_7^2}{2\,T},\\
&\quad -\frac{1}{2} p_2 x_2 + \frac{p_6 x_2}{2} + \frac{(-3+T)\,p_2 p_6 x_2^2}{4\,T} - \frac{(-3+T)\,p_6^2 x_2^2}{4\,T} - \frac{(-1+T)\,p_2^2 p_6 x_2^3}{3\,T} + \frac{(-1+T)\,(1+5\,T)\,p_2 p_6^2 x_2^3}{6\,T^2} -\\
&\quad \frac{(-1+T)\,(1+3\,T)\,p_6^3 x_2^3}{6\,T^2} - \frac{p_4 x_4}{2} + \frac{p_1 x_5}{2} - \frac{p_5 x_5}{2} - \frac{3}{2} p_1^2 x_1 x_5 + \frac{3}{2} p_1 p_5 x_1 x_5 + \frac{1}{2} p_1^3 x_1^2 x_5 -\\
&\quad \frac{1}{2} p_1^2 p_5 x_1^2 x_5 - \frac{(-3+T)\,p_1^2 x_5^2}{4\,T} + \frac{(-3+T)\,p_1 p_5 x_5^2}{4\,T} - \frac{(1+T)\,p_1^3 x_1 x_5^2}{2\,T} + \frac{(1+2\,T)\,p_1^2 p_5 x_1 x_5^2}{2\,T} -\\
&\quad \frac{1}{2} p_1 p_5^2 x_1 x_5^2 - \frac{(-1+T)\,(1+3\,T)\,p_1^3 x_5^3}{6\,T^2} + \frac{(-1+T)\,(1+5\,T)\,p_1^2 p_5 x_5^3}{6\,T^2} - \frac{(-1+T)\,p_1 p_5^2 x_5^3}{3\,T} +\\
&\quad \frac{3}{2} p_2 p_6 x_2 x_6 - \frac{3}{2} p_6^2 x_2 x_6 - \frac{1}{2} p_2^2 p_6 x_2^2 x_6 + \frac{(1+2\,T)\,p_2 p_6^2 x_2^2 x_6}{2\,T} - \frac{(1+T)\,p_6^3 x_2^2 x_6}{2\,T} -\\
&\quad \frac{1}{2} p_2 p_6^2 x_2 x_6^2 + \frac{1}{2} p_6^3 x_2 x_6^2 + \frac{p_3 x_7}{2} - \frac{p_7 x_7}{2} - \frac{3}{2} p_3^2 x_3 x_7 + \frac{3}{2} p_3 p_7 x_3 x_7 + \frac{1}{2} p_3^3 x_3^2 x_7 -\\
&\quad \frac{1}{2} p_3^2 p_7 x_3^2 x_7 - \frac{(-3+T)\,p_3^2 x_7^2}{4\,T} + \frac{(-3+T)\,p_3 p_7 x_7^2}{4\,T} - \frac{(1+T)\,p_3^3 x_3 x_7^2}{2\,T} + \frac{(1+2\,T)\,p_3^2 p_7 x_3 x_7^2}{2\,T} -\\
&\quad \frac{1}{2} p_3 p_7^2 x_3 x_7^2 - \frac{(-1+T)\,(1+3\,T)\,p_3^3 x_7^3}{6\,T^2} + \frac{(-1+T)\,(1+5\,T)\,p_3^2 p_7 x_7^3}{6\,T^2} - \frac{(-1+T)\,p_3 p_7^2 x_7^3}{3\,T}\Big]\Big]
\end{aligned}
$$

In[∘]:= `vs[Knot[3, 1]]`

Out[∘]=

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

In[∘]:= `K = Knot[3, 1]; ∫ℒ[K] d(vs@K)`

Out[∘]=

$$
-\frac{i\,T\,\mathbb{E}\Big[\epsilon\text{Series}\Big[0,\ \frac{(-1+T)^2\,(1+T^2)}{(1-T+T^2)^2},\ -\frac{T^2\,(1-4\,T^2+T^4)}{2\,(1-T+T^2)^4}\Big]\Big]}{128\,\pi^7\,(1-T+T^2)}
$$

Invariance Under Reidemeister 3

In[∘]:=

$$\mathbf{lhs} = \int \left(\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j + \pi_k\, p_k]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1])\right)\, d\{p_i,\ p_j,\ p_k,\ p_{i+1},\ p_{j+1},\ p_{k+1},\ x_i,\ x_j,\ x_k,\ x_{i+1},\ x_{j+1},\ x_{k+1}\}$$

$$\mathbf{rhs} = \int \left(\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j + \pi_k\, p_k]\ \mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1])\right)\, d\{p_i,\ p_j,\ p_k,\ p_{i+1},\ p_{j+1},\ p_{k+1},\ x_i,\ x_j,\ x_k,\ x_{i+1},\ x_{j+1},\ x_{k+1}\};$$

$$\mathbf{lhs} == \mathbf{rhs}$$

Out[∘]=

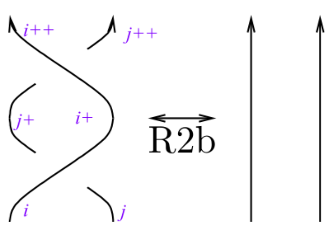
$$\begin{aligned}
&T^{3/2}\,\mathbb{E}\Big[\in\text{Series}\Big[T^2 p_{2+i} \pi_i - (-1+T)\, T\, p_{2+j} \pi_i + (1-T)\, p_{2+k} \pi_i + T\, p_{2+j} \pi_j + (1-T)\, p_{2+k} \pi_j + p_{2+k} \pi_k, \\
&\quad -\frac{3}{2} + T^2 p_{2+j} \pi_i + T\, p_{2+k} \pi_i + \frac{1}{2} (-1+T)\, T^3 p_{2+i} p_{2+j} \pi_i^2 - \frac{1}{2} (-1+T)\, T^3 p_{2+j}^2 \pi_i^2 + \\
&\quad \frac{1}{2} (-1+T)\, T^2 p_{2+i} p_{2+k} \pi_i^2 - \frac{1}{2} (-1+T)^2\, T\, p_{2+j} p_{2+k} \pi_i^2 - \frac{1}{2} (-1+T)\, T\, p_{2+k}^2 \pi_i^2 - T\, p_{2+j} \pi_j + \\
&\quad (-1+2T)\, p_{2+k} \pi_j - T^3 p_{2+i} p_{2+j} \pi_i \pi_j + T^3 p_{2+j}^2 \pi_i \pi_j + (-1+T)\, T^2 p_{2+i} p_{2+k} \pi_i \pi_j - \\
&\quad (-1+T)^2\, T\, p_{2+j} p_{2+k} \pi_i \pi_j - (-1+T)\, T\, p_{2+k}^2 \pi_i \pi_j + \frac{1}{2} (-1+T)\, T\, p_{2+j} p_{2+k} \pi_j^2 - \frac{1}{2} (-1+T)\, T\, p_{2+k}^2 \pi_j^2 - \\
&\quad 2\, p_{2+k} \pi_k - T^2 p_{2+i} p_{2+k} \pi_i \pi_k + (-1+T)\, T\, p_{2+j} p_{2+k} \pi_i \pi_k + T\, p_{2+k}^2 \pi_i \pi_k - T\, p_{2+j} p_{2+k} \pi_j \pi_k + T\, p_{2+k}^2 \pi_j \pi_k, \\
&\quad -\frac{1}{2} T^2 p_{2+j} \pi_i - \frac{1}{2} T\, p_{2+k} \pi_i - \frac{1}{4} T^3 (-1+3T)\, p_{2+i} p_{2+j} \pi_i^2 + \frac{1}{4} T^3 (-3+5T)\, p_{2+j}^2 \pi_i^2 - \\
&\quad \frac{1}{4} T^2 (-1+3T)\, p_{2+i} p_{2+k} \pi_i^2 + \frac{1}{4} (-1+T)\, T\, (-1+5T)\, p_{2+j} p_{2+k} \pi_i^2 + \\
&\quad \frac{1}{4} T\, (-3+5T)\, p_{2+k}^2 \pi_i^2 - \frac{1}{6} (-1+T)\, T^5 p_{2+i}^2 p_{2+j} \pi_i^3 + \frac{1}{6} (-1+T)\, T^4 (-1+4T)\, p_{2+i} p_{2+j}^2 \pi_i^3 - \\
&\quad \frac{1}{6} (-1+T)\, T^4 (-1+3T)\, p_{2+j}^3 \pi_i^3 - \frac{1}{6} (-1+T)\, T^4 p_{2+i}^2 p_{2+k} \pi_i^3 + \frac{5}{6} (-1+T)^2\, T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i^3 - \\
&\quad \frac{1}{6} (-1+T)^2\, T^2 (-1+4T)\, p_{2+j}^2 p_{2+k} \pi_i^3 + \frac{1}{6} (-1+T)\, T^2 (-1+4T)\, p_{2+i} p_{2+k}^2 \pi_i^3 - \\
&\quad \frac{1}{6} (-1+T)^2\, T\, (-1+4T)\, p_{2+j} p_{2+k}^2 \pi_i^3 - \frac{1}{6} (-1+T)\, T\, (-1+3T)\, p_{2+k}^3 \pi_i^3 + \frac{1}{2} T\, p_{2+j} \pi_j + \\
&\quad \frac{1}{2} (1-4T)\, p_{2+k} \pi_j + \frac{3}{2} T^3 p_{2+i} p_{2+j} \pi_i \pi_j - \frac{5}{2} T^3 p_{2+j}^2 \pi_i \pi_j - \frac{1}{2} T^2 (-3+5T)\, p_{2+i} p_{2+k} \pi_i \pi_j + \\
&\quad \frac{1}{2} (-1+T)\, T\, (-3+7T)\, p_{2+j} p_{2+k} \pi_i \pi_j + \frac{1}{2} T\, (-5+7T)\, p_{2+k}^2 \pi_i \pi_j + \frac{1}{2} T^5 p_{2+i}^2 p_{2+j} \pi_i^2 \pi_j - \\
&\quad \frac{1}{2} T^4 (-1+4T)\, p_{2+i} p_{2+j}^2 \pi_i^2 \pi_j + \frac{1}{2} T^4 (-1+3T)\, p_{2+j}^3 \pi_i^2 \pi_j - \frac{1}{2} (-1+T)\, T^4 p_{2+i}^2 p_{2+k} \pi_i^2 \pi_j + \\
&\quad \frac{1}{2} (-1+T)\, T^3 (-5+4T)\, p_{2+i} p_{2+j} p_{2+k} \pi_i^2 \pi_j - \frac{1}{2} (-1+T)\, T^2 \left(1-5T+3T^2\right) p_{2+j}^2 p_{2+k} \pi_i^2 \pi_j + \\
&\quad \frac{1}{2} (-1+T)\, T^2 (-1+4T)\, p_{2+i} p_{2+k}^2 \pi_i^2 \pi_j - \frac{1}{2} (-1+T)^2\, T\, (-1+4T)\, p_{2+j} p_{2+k}^2 \pi_i^2 \pi_j - \\
&\quad \frac{1}{2} (-1+T)\, T\, (-1+3T)\, p_{2+k}^3 \pi_i^2 \pi_j - \frac{1}{4} T\, (-5+7T)\, p_{2+j} p_{2+k} \pi_j^2 + \frac{1}{4} T\, (-7+9T)\, p_{2+k}^2 \pi_j^2 +
\end{aligned}$$

$$\frac{1}{2}\,T^4\,p_{2+i}\,p_{2+j}^2\,\pi_i\,\pi_j^2-\frac{1}{2}\,T^4\,p_{2+j}^3\,\pi_i\,\pi_j^2-2\,(-1+T)\,T^3\,p_{2+i}\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_j^2+$$
$$\frac{1}{2}\,(-1+T)\,T^2\,(-1+4\,T)\,p_{2+j}^2\,p_{2+k}\,\pi_i\,\pi_j^2+\frac{1}{2}\,(-1+T)\,T^2\,(-1+3\,T)\,p_{2+i}\,p_{2+k}^2\,\pi_i\,\pi_j^2-$$
$$\frac{1}{2}\,(-1+T)\,T\,\left(1-5\,T+3\,T^2\right)\,p_{2+j}\,p_{2+k}^2\,\pi_i\,\pi_j^2-\frac{1}{2}\,(-1+T)\,T\,(-1+3\,T)\,p_{2+k}^3\,\pi_i\,\pi_j^2-$$
$$\frac{1}{6}\,(-1+T)\,T^2\,p_{2+j}^2\,p_{2+k}\,\pi_j^3+\frac{1}{6}\,(-1+T)\,T\,(-1+4\,T)\,p_{2+j}\,p_{2+k}^2\,\pi_j^3-\frac{1}{6}\,(-1+T)\,T\,(-1+3\,T)\,p_{2+k}^3\,\pi_j^3+$$
$$2\,p_{2+k}\,\pi_k+\frac{5}{2}\,T^2\,p_{2+i}\,p_{2+k}\,\pi_i\,\pi_k-\frac{1}{2}\,T\,(-5+7\,T)\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_k-\frac{7}{2}\,T\,p_{2+k}^2\,\pi_i\,\pi_k+$$
$$\frac{1}{2}\,T^4\,p_{2+i}^2\,p_{2+k}\,\pi_i^2\,\pi_k-2\,(-1+T)\,T^3\,p_{2+i}\,p_{2+j}\,p_{2+k}\,\pi_i^2\,\pi_k+\frac{1}{2}\,(-1+T)\,T^2\,(-1+3\,T)\,p_{2+j}^2\,p_{2+k}\,\pi_i^2\,\pi_k-$$
$$\frac{1}{2}\,T^2\,(-1+4\,T)\,p_{2+i}\,p_{2+k}^2\,\pi_i^2\,\pi_k+\frac{1}{2}\,(-1+T)\,T\,(-1+4\,T)\,p_{2+j}\,p_{2+k}^2\,\pi_i^2\,\pi_k+\frac{1}{2}\,T\,(-1+3\,T)\,p_{2+k}^3\,\pi_i^2\,\pi_k+$$
$$\frac{7}{2}\,T\,p_{2+j}\,p_{2+k}\,\pi_j\,\pi_k-\frac{9}{2}\,T\,p_{2+k}^2\,\pi_j\,\pi_k+3\,T^3\,p_{2+i}\,p_{2+j}\,p_{2+k}\,\pi_i\,\pi_j\,\pi_k-T^2\,(-1+3\,T)\,p_{2+j}^2\,p_{2+k}\,\pi_i\,\pi_j\,\pi_k-$$
$$T^2\,(-1+3\,T)\,p_{2+i}\,p_{2+k}^2\,\pi_i\,\pi_j\,\pi_k+T\,\left(1-5\,T+3\,T^2\right)\,p_{2+j}\,p_{2+k}^2\,\pi_i\,\pi_j\,\pi_k+T\,(-1+3\,T)\,p_{2+k}^3\,\pi_i\,\pi_j\,\pi_k+$$
$$\frac{1}{2}\,T^2\,p_{2+j}^2\,p_{2+k}\,\pi_j^2\,\pi_k-\frac{1}{2}\,T\,(-1+4\,T)\,p_{2+j}\,p_{2+k}^2\,\pi_j^2\,\pi_k+\frac{1}{2}\,T\,(-1+3\,T)\,p_{2+k}^3\,\pi_j^2\,\pi_k+\frac{1}{2}\,T^2\,p_{2+i}\,p_{2+k}^2\,\pi_i\,\pi_k^2-$$
$$\frac{1}{2}\,(-1+T)\,T\,p_{2+j}\,p_{2+k}^2\,\pi_i\,\pi_k^2-\frac{1}{2}\,T\,p_{2+k}^3\,\pi_i\,\pi_k^2+\frac{1}{2}\,T\,p_{2+j}\,p_{2+k}^2\,\pi_j\,\pi_k^2-\frac{1}{2}\,T\,p_{2+k}^3\,\pi_j\,\pi_k^2\Big]\Big]$$

Out[*]=

```
True
```

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+},\ x_{i|j}$.

In[*]:=

$$\texttt{lhs} = \int \mathbb{E}[\pi_i\,p_i+\pi_j\,p_j]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ \mathrm{d}\{x_i,x_j,p_i,p_j,x_{i+1},x_{j+1},p_{i+1},p_{j+1}\}$$

$$\texttt{rhs} = \int \mathbb{E}[\pi_i\,p_i+\pi_j\,p_j]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ \mathrm{d}\{x_i,x_j,p_i,p_j,x_{i+1},x_{j+1},p_{i+1},p_{j+1}\};$$

$$\texttt{lhs} == \texttt{rhs}$$

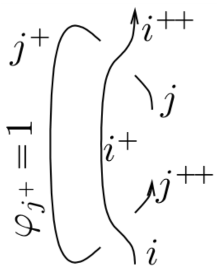
Out[*]=

$$\mathbb{E}[\epsilon\mathrm{Series}[p_{2+i}\,\pi_i+p_{2+j}\,\pi_j,\ 0,\ 0]]$$

Out[*]=

```
True
```

Invariance Under R2c



In[∘]:=

$$\mathtt{lhs} = \int \mathbb{E}[\pi_i\, p_i + \pi_j\, p_j]\ \mathcal{L}\ /@\ (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\}$$

$$\mathtt{rhs} = \int \mathbb{E}[\pi_i\, p_i + \pi_j\, p_j]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\};$$

$$\mathtt{lhs} == \mathtt{rhs}$$

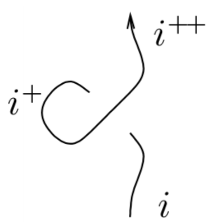
Out[∘]=

$$-i\,\sqrt{T}\ \mathbb{E}\left[\epsilon\mathrm{Series}\left[p_{2+i}\,\pi_i + p_{3+j}\,\pi_j,\ -\frac{1}{2} - p_{3+j}\,\pi_j,\ \frac{1}{2}\,p_{3+j}\,\pi_j\right]\right]$$

Out[∘]=

True

Invariance Under R1l



In[∘]:=

$$\mathtt{lhs} = \int \mathbb{E}[\pi_i\, p_i]\ \mathcal{L}\ /@\ (X_{i+2,i}[1]\ C_{i+1}[1])\ d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$$

$$\mathtt{rhs} = \int \mathbb{E}[\pi_i\, p_i]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\};$$

$$\mathtt{lhs} == \mathtt{rhs}$$

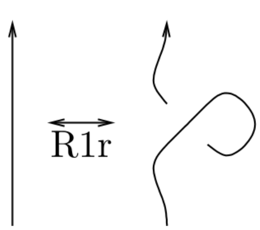
Out[∘]=

$$-i\ \mathbb{E}[\epsilon\mathrm{Series}[p_{3+i}\,\pi_i,\ 0,\ 0]]$$

Out[∘]=

True

Invariance Under R1r



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\pi_i\, p_i]\ \mathcal{L}\ /@\ (X_{i,i+2}[1]\ C_{i+1}[-1])\ d\{x_i,\ p_i,\ x_{i+1},\ p_{i+1},\ x_{i+2},\ p_{i+2}\}$$

$$\text{rhs} = \int \mathbb{E}[\pi_i\, p_i]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{x_i,\ p_i,\ x_{i+1},\ p_{i+1},\ x_{i+2},\ p_{i+2}\};$$

$$\text{lhs} == \text{rhs}$$

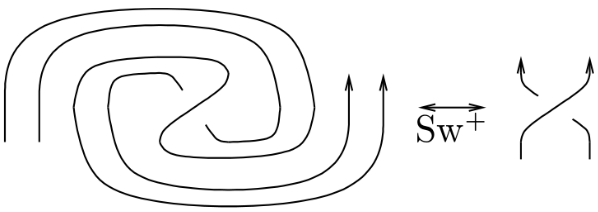
Out[∘]=

$$-i\,\mathbb{E}[\epsilon\text{Series}[p_{3+i}\,\pi_i,\ 0,\ 0]]$$

Out[∘]=

True

Invariance Under Sw



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\pi_i\, p_i + \pi_j\, p_j]\ \mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])\ d\{x_i,\ x_j,\ p_i,\ p_j,\ x_{i+1},\ x_{j+1},\ p_{i+1},\ p_{j+1},\ x_{i+2},\ p_{i+2},\ x_{j+2},\ p_{j+2}\}$$

$$\text{rhs} = \int \mathbb{E}[\pi_i\, p_i + \pi_j\, p_j]\ \mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])\ d\{x_i,\ x_j,\ p_i,\ p_j,\ x_{i+1},\ x_{j+1},\ p_{i+1},\ p_{j+1},\ x_{i+2},\ p_{i+2},\ x_{j+2},\ p_{j+2}\};$$

$$\text{lhs} == \text{rhs}$$

Out[∘]=

$$\sqrt{T}\ \mathbb{E}\Big[\epsilon\text{Series}\Big[T\, p_{3+i}\,\pi_i + (1-T)\, p_{3+j}\,\pi_i + p_{3+j}\,\pi_j,\ -\frac{1}{2} + T\, p_{3+j}\,\pi_i + \frac{1}{2}(-1+T)\,T\, p_{3+i}\, p_{3+j}\,\pi_i^2 - \frac{1}{2}(-1+T)\,T\, p_{3+j}^2\,\pi_i^2 - p_{3+j}\,\pi_j - T\, p_{3+i}\, p_{3+j}\,\pi_i\,\pi_j + T\, p_{3+j}^2\,\pi_i\,\pi_j,\ -\frac{1}{2}T\, p_{3+j}\,\pi_i - \frac{1}{4}T(-1+3T)\, p_{3+i}\, p_{3+j}\,\pi_i^2 + \frac{1}{4}T(-3+5T)\, p_{3+j}^2\,\pi_i^2 - \frac{1}{6}(-1+T)\,T^2\, p_{3+i}^2\, p_{3+j}\,\pi_i^3 + \frac{1}{6}(-1+T)\,T\,(-1+4T)\, p_{3+i}\, p_{3+j}^2\,\pi_i^3 - \frac{1}{6}(-1+T)\,T\,(-1+3T)\, p_{3+j}^3\,\pi_i^3 + \frac{1}{2}p_{3+j}\,\pi_j + \frac{3}{2}T\, p_{3+i}\, p_{3+j}\,\pi_i\,\pi_j - \frac{5}{2}T\, p_{3+j}^2\,\pi_i\,\pi_j + \frac{1}{2}T^2\, p_{3+i}^2\, p_{3+j}\,\pi_i^2\,\pi_j - \frac{1}{2}T(-1+4T)\, p_{3+i}\, p_{3+j}^2\,\pi_i^2\,\pi_j + \frac{1}{2}T(-1+3T)\, p_{3+j}^3\,\pi_i^2\,\pi_j + \frac{1}{2}T\, p_{3+i}\, p_{3+j}^2\,\pi_i\,\pi_j^2 - \frac{1}{2}T\, p_{3+j}^3\,\pi_i\,\pi_j^2\Big]\Big]$$

Out[∘]=

True