Pensieve header: Mathematica notebook for Talks: Bonn-2505.

Ancestors in Talks/Geneva-2408, Talks/Beijing-2407 and in Projects/HigherRank.

exec

```
nb2tex$TeXFileName = "IType.tex";
```

```
In[∘]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
```

Preliminaries

tex

{\bf\red Implementation} (see IType.nb of \web{ap}).

pdf

```
In[∘]:= Once[<< KnotTheory`; << Rot.m];
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

pdf

```
In[∘]:= CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
   vs = Cases[ℰ, (x | p | ξ | π | g)__, ∞] ⋃ {ϵ};
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :→ Factor[c] (Times @@ vs^ps)] ];
```

tex

\vskip 1mm\rule{\linewidth}{1pt}\vspace{-2mm}

Integration

tex

{\bf\red Integration} using Picard iteration. The \myellow{core is in yellow} and \mpink{hacks are in pink}.

pdf

```
In[∘]:= 𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B];
```

pdf

```
In[∘]:= $π = Identity; (* The Wisdom Projection *)
```

pdf

In[ ]:=
```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, DDZ, FZ, a, b},
  n = Length@vs; L0 = L /. ϵ → 0;
  Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /. (p | x)__ → 0, {a, n}, {b, n}];
  If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
  Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
  FixedPoint[(DZ = Table[∂_v Z, {v, vs}];
     DDZ = Table[∂_u DZ, {u, vs}];
     FZ = Sum[G⟦a, b⟧ (DDZ⟦a, b⟧ + DZ⟦a⟧ DZ⟦b⟧), {a, n}, {b, n}] / 2;
     Z = CF[Z0 + ∫_0^λ $π[FZ] ⅆλ]) &, Z];
  PowerExpand@Factor[ω Δ^(-1/2)] 𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

tex

\parpic[r]{\parbox{0.75in}{
\includegraphics[width=0.75in]{../../Projects/Gallery/Fourier.jpg}
\footnotesize Joseph Fourier
}}
\picskip{2}

pdf

In[ ]:= $\int \mathbb{E}\left[-\mu x^2/2 + i\,\xi\,x\right] ⅆ\{x\}$

Out[ ]=
pdf

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

tex

\needspace{12mm}

pdf

In[ ]:= $\text{FofG} = \int \mathbb{E}\left[-\mu\,(x-a)^2/2 + i\,\xi\,x\right] ⅆ\{x\}$

Out[ ]=
pdf

$$\frac{\mathbb{E}\left[\frac{i\,(2\,a\,\mu + i\,\xi)\,\xi}{2\,\mu}\right]}{\sqrt{\mu}}$$

tex

\needspace{12mm}

pdf

In[∘]:= $\int \text{FofG}\, \mathbb{E}[-\mathbb{i}\, \xi\, x]\, \mathbb{d}\{\xi\}$

Out[∘]=
pdf

$$\mathbb{E}\left[-\frac{1}{2}\,(a-x)^2\,\mu\right]$$

tex

So we’ve tested and nearly proven the Fourier inversion formula!

pdf

In[∘]:= $L = -\frac{1}{2}\,\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$

tex

\parpic[r]{\parbox{0.65in}{

\includegraphics[width=0.65in]{../../Projects/Gallery/Fubini.jpg}

\scriptsize Guido Fubini

}}

\picskip{2}

pdf

In[∘]:= $Z12 = \int \mathbb{E}[L]\, \mathbb{d}\{x_1, x_2\}$

Out[∘]=
pdf

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\,(-b^2+a\,c)} + \frac{b\,\xi_1\,\xi_2}{b^2-a\,c} + \frac{a\,\xi_2^2}{2\,(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$$

pdf

In[∘]:= $\left\{Z1 = \int \mathbb{E}[L]\, \mathbb{d}\{x_1\},\ Z12 == \int Z1\, \mathbb{d}\{x_2\}\right\}$

Out[∘]=
pdf

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\,x_2^2}{2\,a} - \frac{b\,x_2\,\xi_1}{a} + \frac{\xi_1^2}{2\,a} + x_2\,\xi_2\right]}{\sqrt{a}},\ \text{True}\right\}$$

pdf

In[∘]:= $\$\pi = \text{Normal}\left[\# + O[\epsilon]^{13}\right]\&;\ \int \mathbb{E}\left[-\phi^2/2 + \epsilon\,\phi^3/6\right]\mathbb{d}\{\phi\}$

Out[∘]=
pdf

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24} + \frac{5\,\epsilon^4}{16} + \frac{1105\,\epsilon^6}{1152} + \frac{565\,\epsilon^8}{128} + \frac{82\,825\,\epsilon^{10}}{3072} + \frac{19\,675\,\epsilon^{12}}{96}\right]$$

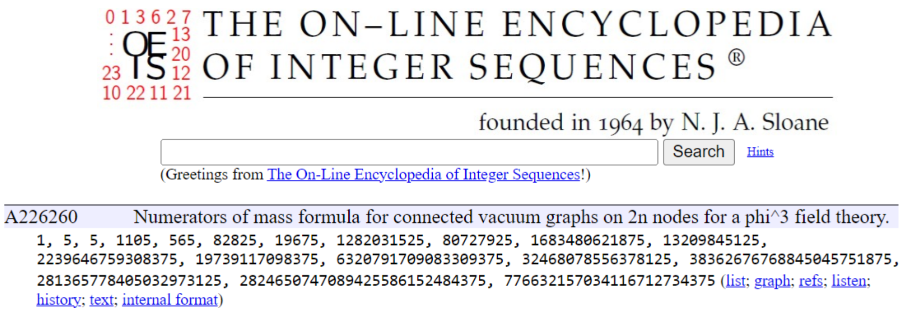
tex

\vfill

From \surl{oeis.org/A226260}:

\vskip 1mm

\includegraphics[width=\linewidth]{../Groningen-240530/OEIS.png}



THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.

1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 38362676768845045751875, 2813657784050329731 25, 28246507470894255861524843 75, 77663215703411671273437 5 (list; graph; refs; listen; history; text; internal format)

The Right-Handed Trefoil

tex

\vfill

\rule{\linewidth}{1pt}

{\bf\red The Right-Handed Trefoil.}

pdf

In[∘]:=
```
K = Mirror@Knot[3, 1]; Features[K]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$\text{Features}[7, C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]]$

pdf

In[∘]:=

$$\mathcal{L}[X_{i\_,j\_}[s\_]] := T^{s/2}\,\mathbb{E}\Big[x_i\,(p_{i+1}-p_i) + x_j\,(p_{j+1}-p_j) + \left(T^s-1\right) x_i\,(p_{i+1}-p_{j+1}) + (\epsilon\, s/2) \times \left(x_i\,(p_i-p_j)\,\left(\left(T^s-1\right) x_i\, p_j + 2\,(1-x_j\,p_j)\right) - 1\right)\Big]$$

$$\mathcal{L}[C_{i\_}[\varphi\_]] := T^{\varphi/2}\,\mathbb{E}\left[x_i\,(p_{i+1}-p_i) + \epsilon\,\varphi\left(\frac{1}{2} - x_i\,p_i\right)\right]$$

$\mathcal{L}[K\_] := \text{CF}[\mathcal{L}\ /@\ \text{Features}[K][[2]]]$

$\text{vs}[K\_] := \text{Join} @@ \text{Table}[\{p_i, x_i\}, \{i, \text{Features}[K][[1]]\}]$

exec

In[∘]:=
```
nb2tex$PDFWidth *= 1.25;
```

tex

\needspace{5cm}

pdf

In[∘]:= `{vs[K], ℒ[K]}`

Out[∘]=
pdf

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-2\in - p_1 x_1 + \in p_1 x_1 + T p_2 x_1 - \in p_5 x_1 + (1-T)\, p_6 x_1 + \frac{1}{2}\,(-1+T) \in p_1 p_5 x_1^2 + \frac{1}{2}\,(1-T) \in p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 - p_3 x_3 + \in p_3 x_3 + T p_4 x_3 - \in p_7 x_3 + (1-T)\, p_8 x_3 + \frac{1}{2}\,(-1+T) \in p_3 p_7 x_3^2 + \frac{1}{2}\,(1-T) \in p_7^2 x_3^2 - p_4 x_4 + \in p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \in p_1 p_5 x_1 x_5 + \in p_5^2 x_1 x_5 - \in p_2 x_6 + (1-T)\, p_3 x_6 - p_6 x_6 + \in p_6 x_6 + T p_7 x_6 + \in p_2^2 x_2 x_6 - \in p_2 p_6 x_2 x_6 + \frac{1}{2}\,(1-T) \in p_2^2 x_6^2 + \frac{1}{2}\,(-1+T) \in p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \in p_3 p_7 x_3 x_7 + \in p_7^2 x_3 x_7\Big]\Big\}$$

exec

In[∘]:= `nb2tex$PDFWidth /= 1.25;`

tex

\needspace{10mm}

pdf

In[∘]:= `$π = Normal[# + O[ϵ]^2] &; ∫ℒ[K] ⅆ vs[K]`

Out[∘]=
pdf

$$-\frac{\mathbb{i}\, T\, \mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^2)\in}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

In[∘]:= `∫(ℒ[K] /. x_i_ :→ 𝕚 x_i) ⅆ (vs@K)`

Out[∘]=

$$\frac{T\, \mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^2)\in}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

tex

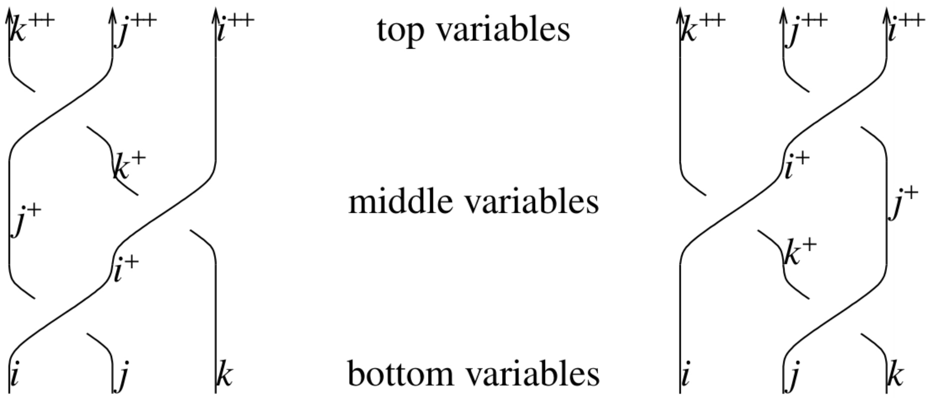
\vskip 1mm
A faster program to compute \$\rho_1\$, and more stories about it, are at~\cite{APAI}.

\rule{\linewidth}{1pt}\vspace{2mm}\vskip -2mm
%\newcolumn

Invariance Under Reidemeister 3

tex

{\bf\red Invariance Under Reidemeister 3.}
\vskip 2mm
\def\ip{{i^+}} \def\jp{{j^+}} \def\kp{{k^+}}
\def\ipp{{i^{+\!+}}} \def\jpp{{j^{+\!+}}} \def\kpp{{k^{+\!+}}}
\import{../Beijing-2407/figs}{R3.pdf_t}



pdf

In[∘]:=

$$\texttt{lhs} = \int (\mathcal{L} \;/@\; (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))\, \mathbb{d}\{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$

$$\texttt{rhs} = \int (\mathcal{L} \;/@\; (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))\, \mathbb{d}\{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$$

```
lhs === rhs
```

Out[∘]=
pdf

```
False
```

Invariance Under Reidemeister 3, Take 2

tex

\vskip 1mm\rule{\linewidth}{1pt}
\par{\bf\red Invariance Under Reidemeister 3, Take 2.}

pdf

In[∘]:=

$$\texttt{lhs} = \int (\mathcal{L} \;/@\; (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))\, \mathbb{d}\{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$

$$\texttt{rhs} = \int (\mathcal{L} \;/@\; (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))\, \mathbb{d}\{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$$

```
lhs === rhs
```

Out[∘]=
pdf

```
True
```

pdf

In[∘]:=

```
lhs
```

Out[∘]=
pdf

```
Degenerate Q!
```

tex

%\newcolumn

Invariance Under Reidemeister 3, Take 3

tex

\vskip 1mm\rule{\linewidth}{1pt}
\par{\bf\red Invariance Under Reidemeister 3, Take 3.}

exec

In[∘]:= **nb2tex$PDFWidth *= 1.25;**

pdf

In[∘]:=
$$\texttt{lhs} = \int \left(\mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j + \mathbb{i}\,\pi_k\,p_k]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1])\right) \mathbb{d}\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$
$$\texttt{rhs} = \int \left(\mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j + \mathbb{i}\,\pi_k\,p_k]\ \mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1])\right) \mathbb{d}\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$
$$\texttt{lhs} == \texttt{rhs}$$

Out[∘]=
pdf

True

tex

\needspace{20mm}

pdf

In[∘]:= **lhs**

Out[∘]=
pdf

$$\begin{aligned}
&T^{3/2}\,\mathbb{E}\Big[-\frac{3\,\in}{2} + \mathbb{i}\,T^2\,p_{2+i}\,\pi_i - \mathbb{i}\,(-1+T)\,T\,p_{2+j}\,\pi_i + \mathbb{i}\,T^2 \in p_{2+j}\,\pi_i - \mathbb{i}\,(-1+T)\,p_{2+k}\,\pi_i + \mathbb{i}\,T \in p_{2+k}\,\pi_i - \\
&\quad \frac{1}{2}\,(-1+T)\,T^3 \in p_{2+i}\,p_{2+j}\,\pi_i^2 + \frac{1}{2}\,(-1+T)\,T^3 \in p_{2+j}^2\,\pi_i^2 - \frac{1}{2}\,(-1+T)\,T^2 \in p_{2+i}\,p_{2+k}\,\pi_i^2 + \\
&\quad \frac{1}{2}\,(-1+T)^2\,T \in p_{2+j}\,p_{2+k}\,\pi_i^2 + \frac{1}{2}\,(-1+T)\,T \in p_{2+k}^2\,\pi_i^2 + \mathbb{i}\,T\,p_{2+j}\,\pi_j - \mathbb{i}\,T \in p_{2+j}\,\pi_j - \\
&\quad \mathbb{i}\,(-1+T)\,p_{2+k}\,\pi_j + \mathbb{i}\,(-1+2\,T) \in p_{2+k}\,\pi_j + T^3 \in p_{2+i}\,p_{2+j}\,\pi_i\,\pi_j - T^3 \in p_{2+j}^2\,\pi_i\,\pi_j - \\
&\quad (-1+T)\,T^2 \in p_{2+i}\,p_{2+k}\,\pi_i\,\pi_j + (-1+T)^2\,T \in p_{2+j}\,p_{2+k}\,\pi_i\,\pi_j + (-1+T)\,T \in p_{2+k}^2\,\pi_i\,\pi_j - \\
&\quad \frac{1}{2}\,(-1+T)\,T \in p_{2+j}\,p_{2+k}\,\pi_j^2 + \frac{1}{2}\,(-1+T)\,T \in p_{2+k}^2\,\pi_j^2 + \mathbb{i}\,p_{2+k}\,\pi_k - 2\,\mathbb{i} \in p_{2+k}\,\pi_k + T^2 \in p_{2+i}\,p_{2+k}\,\pi_i\,\pi_k - \\
&\quad (-1+T)\,T \in p_{2+j}\,p_{2+k}\,\pi_i\,\pi_k - T \in p_{2+k}^2\,\pi_i\,\pi_k + T \in p_{2+j}\,p_{2+k}\,\pi_j\,\pi_k - T \in p_{2+k}^2\,\pi_j\,\pi_k\Big]
\end{aligned}$$

exec

In[∘]:= **nb2tex$PDFWidth /= 1.25;**

tex

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at \web{ap}.

Invariance Under Reidemeister 3, Take 4 (just for fun)

In[∘]:=

$$\mathbf{lhs} = \int \left(\mathbb{E}[i\,\pi_i\,p_i + i\,\pi_j\,p_j + i\,\pi_k\,p_k + i\,\pi_{i+2}\,p_{i+2} + i\,\pi_{j+2}\,p_{j+2} + i\,\pi_{k+2}\,p_{k+2} + i\,\xi_{i+2}\,x_{i+2} + i\,\xi_{j+2}\,x_{j+2} + i\,\xi_{k+2}\,x_{k+2}]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1])\right) d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+2}, p_{j+2}, p_{k+2}, x_{i+2}, x_{j+2}, x_{k+2}\};$$

$$\mathbf{rhs} = \int \left(\mathbb{E}[i\,\pi_i\,p_i + i\,\pi_j\,p_j + i\,\pi_k\,p_k + i\,\pi_{i+2}\,p_{i+2} + i\,\pi_{j+2}\,p_{j+2} + i\,\pi_{k+2}\,p_{k+2} + i\,\xi_{i+2}\,x_{i+2} + i\,\xi_{j+2}\,x_{j+2} + i\,\xi_{k+2}\,x_{k+2}]\ \mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1])\right) d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+2}, p_{j+2}, p_{k+2}, x_{i+2}, x_{j+2}, x_{k+2}\};$$

$$\mathbf{lhs} == \mathbf{rhs}$$

Out[∘]=

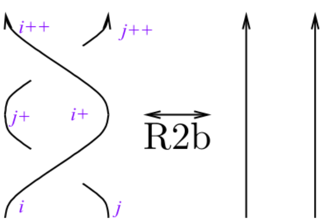
True

In[∘]:= **lhs**

Out[∘]=

Degenerate Q!

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=

$$\mathbf{lhs} = \int \mathbb{E}[i\,\pi_i\,p_i + i\,\pi_j\,p_j]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$

$$\mathbf{rhs} = \int \mathbb{E}[i\,\pi_i\,p_i + i\,\pi_j\,p_j]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\};$$

$$\mathbf{lhs} == \mathbf{rhs}$$

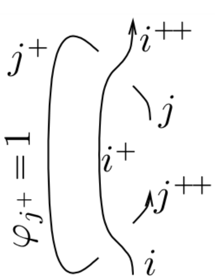
Out[∘]=

$\mathbb{E}[i\,p_{2+i}\,\pi_i + i\,p_{2+j}\,\pi_j]$

Out[∘]=

True

Invariance Under R2c

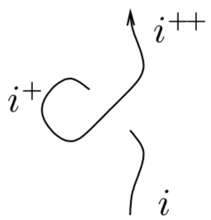


In[ ]:=
$$\mathtt{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}\ /@\ (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ \mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\}$$
$$\mathtt{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ \mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\};$$
$$\mathtt{lhs} == \mathtt{rhs}$$

Out[ ]=
$$-\mathbb{i}\,\sqrt{T}\ \mathbb{E}\left[-\frac{\epsilon}{2} + \mathbb{i}\,p_{2+i}\,\pi_i + \mathbb{i}\,p_{3+j}\,\pi_j - \mathbb{i}\,\epsilon\,p_{3+j}\,\pi_j\right]$$

Out[ ]=
True

Invariance Under R1l

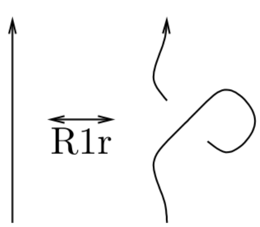


In[ ]:=
$$\mathtt{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}\ /@\ (X_{i+2,i}[1]\ C_{i+1}[1])\ \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$$
$$\mathtt{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\};$$
$$\mathtt{lhs} == \mathtt{rhs}$$

Out[ ]=
$$-\mathbb{i}\ \mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Out[ ]=
True

Invariance Under R1r



In[∘]:=
```
lhs = ∫𝔼[𝕚 π_i p_i] ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[𝕚 π_i p_i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs
```

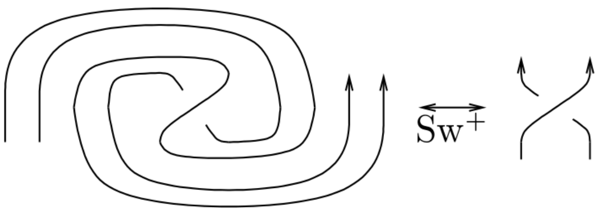
Out[∘]=

$$-i\,\mathbb{E}[i\,p_{3+i}\,\pi_i]$$

Out[∘]=

True

Invariance Under Sw



In[∘]:=
```
lhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1])
   ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0])
   ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}};
lhs == rhs
```

Out[∘]=

$$\sqrt{T}\,\mathbb{E}\Big[-\frac{\epsilon}{2}+i\,T\,p_{3+i}\,\pi_i-i\,(-1+T)\,p_{3+j}\,\pi_i+i\,T\,\epsilon\,p_{3+j}\,\pi_i-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+j}^2\,\pi_i^2+i\,p_{3+j}\,\pi_j-i\,\epsilon\,p_{3+j}\,\pi_j+T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j-T\,\epsilon\,p_{3+j}^2\,\pi_i\,\pi_j\Big]$$

Out[∘]=

True