(Alternative) Gaussian Integration.

Gauss

Goal. Compute (if convergent)
$$\int_{\mathbb{R}^n} dx \exp\left(-\frac{1}{2}a^{ij}x_i x_j + V(x)\right).$$

Solution. Set
$$\mathcal{Z}_\lambda(x) := \lambda^{\cancel{n/2}\ -N/2}\int_{\mathbb{R}^n} dy \exp\left(-\frac{1}{2\lambda}a^{ij}y_i y_j + V(x+y)\right).$$

Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x) = (\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y = 0$,

$$\frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)$$
$$= \lambda^{-n/2}\frac{1}{2}\int_{\mathbb{R}^n} dy\, g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_i y_j + V(x+y)\right)$$
$$= \lambda^{-n/2}\frac{1}{2\lambda^2}\int_{\mathbb{R}^n} dy\left(g_{ij}a^{ii'}a^{jj'}y_{i'}y_{j'} + \lambda g_{ij}a^{ji}\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_i y_j + V(x+y)\right)$$
$$= \frac{1}{2\lambda^2}\lambda^{-n/2}\int_{\mathbb{R}^n} dy\left(a^{ij}y_i y_j + \lambda n\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_i y_j + V(x+y)\right)$$
$$= \partial_\lambda \mathcal{Z}_\lambda(x).$$

Hence
$$(*)\quad \partial_\lambda\mathcal{Z}_\lambda(x) = \frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x),$$

and therefore
$$\mathcal{Z}_\lambda(x) = (\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$

We've just witnessed the birth of "Feynman Diagrams".

Feynman

Even better. With $Z_\lambda := \log(\sqrt{\det A}\,\mathcal{Z}_\lambda)$, by a simple substitution into (*), we get the "Synthesis Equation":

$$Z_0 = V, \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}\left(\partial_{x_i,x_j}Z_\lambda + (\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right) =: F(Z_\lambda),$$

an ODE (in λ) whose solution is pure algebra.