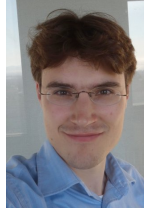




Knot Invariants from Zero-Dimensional QFT

Abstract. For the purpose of today, an “I-Type Knot Invariant” is a knot invariant computed from a knot diagram by integrating the exponential of a *perturbed Gaussian* Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.

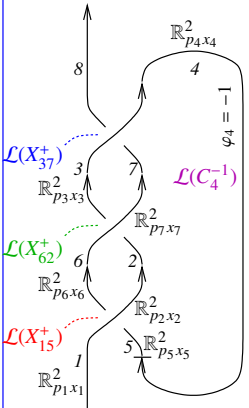


joint with R. van der Veen

- Q.** Are there any such things? **A.** Yes.
- Q.** Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.
- Q.** Didn't Witten do that back in 1988 with path integrals? **A.** No. His constructions are infinite dimensional and far from rigorous.
- Q.** But integrals belong in analysis! **A.** Ours only use squeaky-clean algebra.

Dreams.

The $sl_2^{\epsilon^2}$ Example. With T an indeterminate and with $\epsilon^2 = 0$:



$$Z = \int_{\mathbb{R}^{14}_{p_i x_i}} \mathcal{L}(X_{15}^+) \mathcal{L}(X_{62}^+) \mathcal{L}(X_{37}^+) \mathcal{L}(C_4^{-1})$$

where $\mathcal{L}(X_{ij}^s) = T^{s/2} e^{L(X_{ij}^s)}$ and $\mathcal{L}(C_i^\varphi) = T^{\varphi/2} e^{L(C_i^\varphi)}$, and

$$L(X_{ij}^s) = x_i(p_{i+1} - p_i) + x_j(p_{j+1} - p_j) + (T^s - 1)x_i(p_{i+1} - p_{j+1}) + \frac{\epsilon s}{2} \left(x_i(p_i - p_j) \left(\begin{matrix} (T^s - 1)x_i p_j \\ +2(1 - x_j p_j) \end{matrix} \right) - 1 \right)$$

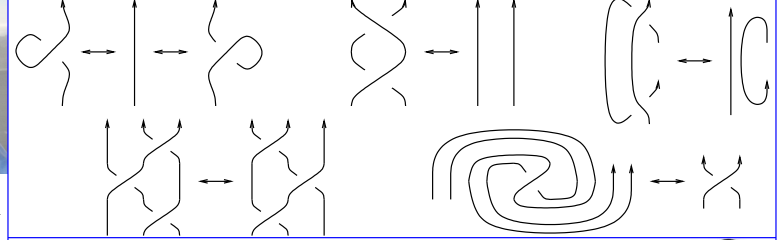
$$L(C_i^\varphi) = x_i(p_{i+1} - p_i) + \epsilon \varphi(1/2 - x_i p_i)$$

So $Z = T \int e^{L(\odot)} dp_1 \dots dp_7 dx_1 \dots dx_7$, where $L(\odot) = \sum_{i=1}^7 x_i(p_{i+1} - p_i) + (T-1)(x_1(p_2 - p_6) + x_6(p_7 - p_3) + x_3(p_4 - p_8)) + \frac{\epsilon}{2} \left(\begin{matrix} x_1(p_1 - p_5) ((T-1)x_1 p_5 + 2(1 - x_5 p_5)) - 1 \\ +x_6(p_6 - p_2) ((T-1)x_6 p_2 + 2(1 - x_2 p_2)) - 1 \\ +x_3(p_3 - p_7) ((T-1)x_3 p_7 + 2(1 - x_7 p_7)) - 1 \\ +2x_4 p_4 - 1 \end{matrix} \right)$

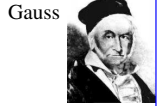
and so $Z = (T - 1 + T^{-1})^{-1} \exp\left(\epsilon \cdot \frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right) = \Delta^{-1} \exp\left(\epsilon \cdot \frac{(T-2+T^{-1})\rho_1}{\Delta^2}\right)$. Here Δ is Alexander's polynomial and ρ_1 is Rozansky-Overbay's polynomial [R1, R2, R3, Ov, BV1, BV2].



Theorem. Z is a knot invariant. **Proof.** Use Fubini (details later).



(Alternative) Gaussian Integration.



Goal. Compute $\int_{\mathbb{R}^n} dx \exp\left(-\frac{1}{2} a^{ij} x_i x_j + V(x)\right)$. (if convergent)

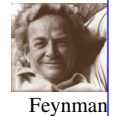
Solution. Set $\mathcal{Z}_\lambda(x) := \lambda^{n/2} \int_{\mathbb{R}^n} dy \exp\left(-\frac{1}{2\lambda} a^{ij} y_i y_j + V(x+y)\right)$. Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x) = (\det A)^{-1/2} \exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y = 0$,

$$\begin{aligned} & \int_{\mathbb{R}^n} dy g_{ij} \partial_{x_i} \partial_{x_j} \mathcal{Z}_\lambda(x) \\ &= \frac{1}{2} \int_{\mathbb{R}^n} dy g_{ij} (\partial_{x_i} - \partial_{y_i}) (\partial_{x_j} - \partial_{y_j}) \exp\left(-\frac{1}{2\lambda} a^{ij} y_i y_j + V(x+y)\right) \\ &= \frac{1}{2\lambda^2} \int_{\mathbb{R}^n} dy (g_{ij} a^{il} a^{lj} y_i y_j + \lambda g_{ij} a^{jj}) \exp\left(-\frac{1}{2\lambda} a^{ij} y_i y_j + V(x+y)\right) \\ &= \frac{1}{2\lambda^2} \int_{\mathbb{R}^n} dy (a^{ij} y_i y_j + \lambda n) \exp\left(-\frac{1}{2\lambda} a^{ij} y_i y_j + V(x+y)\right) \\ &= \partial_\lambda \mathcal{Z}_\lambda(x). \end{aligned}$$

Hence $(*) \quad \partial_\lambda \mathcal{Z}_\lambda(x) = \frac{1}{2} g_{ij} \partial_{x_i} \partial_{x_j} \mathcal{Z}_\lambda(x)$, and therefore $\mathcal{Z}_\lambda(x) = (\det A)^{-1/2} \exp\left(\frac{\lambda}{2} g_{ij} \partial_{x_i} \partial_{x_j}\right) \exp V(x)$.

We've just witnessed the birth of "Feynman Diagrams".

Even better. With $Z_\lambda := \log(\sqrt{\det A} Z_\lambda)$, by a simple substitution into (*), we get the "Synthesis Equation":



$Z_0 = V, \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j=1}^n g_{ij} (\partial_{x_i x_j} Z_\lambda + (\partial_{x_i} Z_\lambda)(\partial_{x_j} Z_\lambda)) =: F(Z_\lambda)$, an ODE (in λ) whose solution is pure algebra.

Picard Iteration (used to prove the existence and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda = F(f_\lambda)$ with a given f_0 , start with f_0 , iterate $f \mapsto f_0 + \int_0^\lambda F(f_\lambda) d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!



Definition. \oint : The result of this process, ignoring the convergence of the actual integral.

Strong. The pair (Δ, ρ_1) attains 270,316 distinct values on the 313,230 prime knots with up to 15 crossings (a deficit of 42,914), whereas the pair $(H = \text{HOMFLYPT polynomial}, Kh = \text{Khovanov Homology})$ attains only 242,985 distinct values on the same knots (a deficit of 70,245). The pair (Δ, θ) , discussed later, has a deficit of only 6,758, and the triple (Δ, θ, ρ_2) , of only 6,341.

Yet better than (H, Kh) and other Reshetikhin-Turaev-Witten invariants and knot homologies, Δ, ρ_1, ρ_2 and θ can be computed in **polynomial time** (and hence, even for very large knots). So ugly as the formulas may be (and θ 's formulas are uglier), these invariants are **the best we have!**

Acknowledgement. This work was supported by NSERC grants RGPIN-2018-04350 and RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

Implementation (see IType.nb of ωεβ/ap).

☉ Once [<< KnotTheory` ; << Rot.m] ;

☐ C:\drorbn\AcademicPensieve\Projects\KnotTheory\KnotTheory

☐ Loading KnotTheory` version

of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

☐ Loading Rot.m from

<http://drorbn.net/AP/Talks/Geneva-2408>
to compute rotation numbers.

☉ CF [ω_ . ε_ E] := CF [ω] × CF / @ ε ;

CF [ε_ List] := CF / @ ε ;

CF [ε_] := Module [{ vs, ps, c },

vs = Cases [ε, (x | p | ξ | π | g) __, ∞] ∪ { ε } ;

Total [CoefficientRules [Expand [ε], vs] / .

(ps_ → c_) ⇒ Factor [c] (Times @@ vs^ps)]] ;

Integration using Picard iteration. The **core is in yellow** and **hacks are in pink**.

☉ E / : E [A_] × E [B_] := E [A + B] ;

☉ \$π = Identity ; (* The Wisdom Projection *)

☉ Unprotect [Integrate] ;

$\int \omega_ . E [L_] d (vs_ List) :=$

Module [{ n, L0, Q, Δ, G, Z0, Z, λ, DZ, DDZ, FZ, a, b },

n = Length@vs ; L0 = L / . ε → 0 ;

Q = Table [(- ∂_{vs[[a]], vs[[b]] L0) / . Thread [vs → 0] / .}

(p | x) → 0, { a, n }, { b, n }] ;

If [(Δ = Det [Q]) == 0, Return@ "Degenerate Q!"] ;

Z = Z0 = CF@ \$π [L + vs . Q . vs / 2] ; G = Inverse [Q] ;

FixedPoint [(DZ = Table [∂_v Z, { v, vs }] ;

DDZ = Table [∂_u DZ, { u, vs }] ;

FZ = Sum [G [a, b] (DDZ [a, b] + DZ [a] × DZ [b]), { a, n }, { b, n }] / 2 ;

Z = CF [Z0 + ∫₀^λ \$π [FZ] dλ] &, Z] ;

PowerExpand@Factor [ω Δ^{-1/2}] ×

E [CF [Z / . λ → 1 / . Thread [vs → 0]]] ;

Protect [Integrate] ;

☉ $\int \mathbb{E} [- \mu x^2 / 2 + i \xi x] d \{ x \}$

$$\frac{\mathbb{E} \left[- \frac{\xi^2}{2\mu} \right]}{\sqrt{\mu}}$$

☉ FofG = $\int \mathbb{E} [- \mu (x - a)^2 / 2 + i \xi x] d \{ x \}$

$$\frac{\mathbb{E} \left[\frac{i (2 a \mu + i \xi) \xi}{2 \mu} \right]}{\sqrt{\mu}}$$



Joseph Fourier

$$\odot \int \text{FofG} \mathbb{E} [- i \xi x] d \{ \xi \}$$

$$\mathbb{E} \left[- \frac{1}{2} (a - x)^2 \mu \right]$$

So we've tested and nearly proven the Fourier inversion formula!

$$\odot L = - \frac{1}{2} \{ x_1, x_2 \} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{ x_1, x_2 \} + \{ \xi_1, \xi_2 \} \cdot \{ x_1, x_2 \} ;$$

$$Z12 = \int \mathbb{E} [L] d \{ x_1, x_2 \}$$

$$\frac{\mathbb{E} \left[\frac{c \xi_1^2}{2 (-b^2 + a c)} + \frac{b \xi_1 \xi_2}{b^2 - a c} + \frac{a \xi_2^2}{2 (-b^2 + a c)} \right]}{\sqrt{-b^2 + a c}}$$

$$\odot \{ Z1 = \int \mathbb{E} [L] d \{ x_1 \}, Z12 = \int Z1 d \{ x_2 \} \}$$

$$\frac{\mathbb{E} \left[- \frac{(-b^2 + a c) x_2^2}{2 a} - \frac{b x_2 \xi_1}{a} + \frac{\xi_1^2}{2 a} + x_2 \xi_2 \right]}{\sqrt{a}}, \text{ True} \}$$



Guido Fubini

$$\odot \$\pi = \text{Normal} [\# + 0 [\epsilon]^{13}] \& ; \int \mathbb{E} [- \phi^2 / 2 + \epsilon \phi^3 / 6] d \{ \phi \}$$

$$\frac{\mathbb{E} \left[\frac{5 \epsilon^2}{24} + \frac{5 \epsilon^4}{16} + \frac{1105 \epsilon^6}{1152} + \frac{565 \epsilon^8}{128} + \frac{82825 \epsilon^{10}}{3072} + \frac{19675 \epsilon^{12}}{96} \right]}$$

From <https://oeis.org/A226260>:

1 3 6 2 7
: 13
: 20
23 12
10 22 11 21
THE ON-LINE ENCYCLOPEDIA
OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

[Hints](#)

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.
1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125,
2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 38362676768845045751875,
281365778405032973125, 2824650747089425586152484375, 776632157034116712734375 (list; graph; refs; listen;
history; text; internal format)

The Right-Handed Trefoil.

☉ K = Mirror@Knot [3, 1] ; Features [K]

☐ Features [7, C4 [-1] X1,5 [1] X3,7 [1] X6,2 [1]]

$$\odot \mathcal{L} [X_{i,j} [s_]] := T^{s/2} \mathbb{E} [x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}) + (\epsilon s / 2) \times (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]$$

$$\mathcal{L} [C_i [\varphi_]] := T^{\varphi/2} \mathbb{E} [x_i (p_{i+1} - p_i) + \epsilon \varphi \left(\frac{1}{2} - x_i p_i \right)]$$

$$\mathcal{L} [K_] := CF [\mathcal{L} / @ Features [K] [[2]]]$$

vs [K_] :=

Join @@ Table [{ p_i, x_i }, { i, Features [K] [[1]] }]

⊙ {vs[K], L[K]}

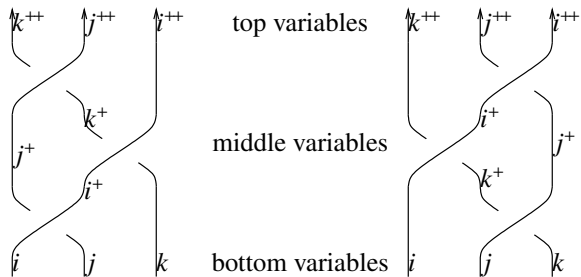
$$\begin{aligned} & \left\{ \{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\}, \right. \\ & \text{T E} \left[-2 \in -p_1 x_1 + \in p_1 x_1 + T p_2 x_1 - \in p_5 x_1 + (1-T) p_6 x_1 + \right. \\ & \quad \frac{1}{2} (-1+T) \in p_1 p_5 x_1^2 + \frac{1}{2} (1-T) \in p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 - p_3 x_3 + \\ & \quad \in p_3 x_3 + T p_4 x_3 - \in p_7 x_3 + (1-T) p_8 x_3 + \frac{1}{2} (-1+T) \in p_3 p_7 x_3^2 + \\ & \quad \frac{1}{2} (1-T) \in p_7^2 x_3^2 - p_4 x_4 + \in p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \\ & \quad \in p_1 p_5 x_1 x_5 + \in p_5^2 x_1 x_5 - \in p_2 x_6 + (1-T) p_3 x_6 - p_6 x_6 + \\ & \quad \in p_6 x_6 + T p_7 x_6 + \in p_2^2 x_2 x_6 - \in p_2 p_6 x_2 x_6 + \frac{1}{2} (1-T) \in p_2^2 x_6^2 + \\ & \quad \left. \left. \frac{1}{2} (-1+T) \in p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \in p_3 p_7 x_3 x_7 + \in p_7^2 x_3 x_7 \right] \right\} \end{aligned}$$

⊙ $\$ \pi = \text{Normal}[\# + \mathbf{0}[\epsilon]^2] \&; \int \mathcal{L}[K] \text{ d} \text{ vs}[K]$

$$\frac{\text{i T E} \left[-\frac{(-1+T)^2 (1+T^2) \epsilon}{(1-T+T^2)^2} \right]}{1-T+T^2}$$

A faster program to compute ρ_1 , and more stories about it, are at [BV2].

Invariance Under Reidemeister 3.



Reidemeister

$$\begin{aligned} \text{lhs} &= \int (\mathcal{L} / @ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])) \\ & \quad \text{d} \{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\}; \\ \text{rhs} &= \int (\mathcal{L} / @ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1])) \\ & \quad \text{d} \{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\}; \\ \text{lhs} &=== \text{rhs} \end{aligned}$$

⊠ False

Invariance Under Reidemeister 3, Take 2.

$$\begin{aligned} \text{lhs} &= \int (\mathcal{L} / @ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])) \\ & \quad \text{d} \{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\}; \\ \text{rhs} &= \int (\mathcal{L} / @ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1])) \\ & \quad \text{d} \{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\}; \\ \text{lhs} &=== \text{rhs} \end{aligned}$$

⊠ True

⊙ lhs

⊠ Degenerate Q!

Invariance Under Reidemeister 3, Take 3.

$$\begin{aligned} \text{lhs} &= \int (\text{E} [\text{i } \pi_i p_i + \text{i } \pi_j p_j + \text{i } \pi_k p_k] \times \mathcal{L} / @ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])) \\ & \quad \text{d} \{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\}; \\ \text{rhs} &= \int (\text{E} [\text{i } \pi_i p_i + \text{i } \pi_j p_j + \text{i } \pi_k p_k] \times \mathcal{L} / @ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1])) \\ & \quad \text{d} \{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\}; \\ \text{lhs} &= \text{rhs} \\ & \quad \text{⊠ True} \\ & \quad \text{⊙ lhs} \\ & \quad \text{⊠ } T^{3/2} \text{ E} \left[\right. \\ & \quad \quad -\frac{3}{2} \in + \text{i } T^2 p_{2+i} \pi_i - \text{i } (-1+T) T p_{2+j} \pi_i + \text{i } T^2 \in p_{2+j} \pi_i - \text{i } (-1+T) p_{2+k} \pi_i + \\ & \quad \quad \text{i } T \in p_{2+k} \pi_i - \frac{1}{2} (-1+T) T^3 \in p_{2+i} p_{2+j} \pi_i^2 + \frac{1}{2} (-1+T) T^3 \in p_{2+j}^2 \pi_i^2 - \\ & \quad \quad \frac{1}{2} (-1+T) T^2 \in p_{2+i} p_{2+k} \pi_i^2 + \frac{1}{2} (-1+T) T^2 \in p_{2+j} p_{2+k} \pi_i^2 + \\ & \quad \quad \frac{1}{2} (-1+T) T \in p_{2+k}^2 \pi_i^2 + \text{i } T p_{2+j} \pi_j - \text{i } T \in p_{2+j} \pi_j - \text{i } (-1+T) p_{2+k} \pi_j + \\ & \quad \quad \text{i } (-1+2T) \in p_{2+k} \pi_j + T^3 \in p_{2+i} p_{2+j} \pi_i \pi_j - T^3 \in p_{2+j}^2 \pi_i \pi_j - \\ & \quad \quad (-1+T) T^2 \in p_{2+i} p_{2+k} \pi_i \pi_j + (-1+T)^2 T \in p_{2+j} p_{2+k} \pi_i \pi_j + \\ & \quad \quad (-1+T) T \in p_{2+k}^2 \pi_i \pi_j - \frac{1}{2} (-1+T) T \in p_{2+j} p_{2+k} \pi_j^2 + \frac{1}{2} (-1+T) T \in p_{2+k}^2 \pi_j^2 + \\ & \quad \quad \text{i } p_{2+k} \pi_k - 2 \text{i } \in p_{2+k} \pi_k + T^2 \in p_{2+i} p_{2+k} \pi_i \pi_k - (-1+T) T \in p_{2+j} p_{2+k} \pi_i \pi_k - \\ & \quad \quad \left. T \in p_{2+k}^2 \pi_i \pi_k + T \in p_{2+j} p_{2+k} \pi_j \pi_k - T \in p_{2+k}^2 \pi_j \pi_k \right] \end{aligned}$$

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at [omega epsilon beta / ap](#).

There's more! To get sl_2 invariants mod ϵ^3 , add the following to $L(X_{ij}^+)$, $L(X_{ij}^-)$, and $L(C_i^\varphi)$, respectively (and see More.nb at [omega epsilon beta / ap](#) for the verifications):

⊙ $\epsilon^2 r_2[1, i, j]$

$$\begin{aligned} & \frac{1}{12} \epsilon^2 (-6 p_i x_i + 6 p_j x_i - 3 (-1+3T) p_i p_j x_i^2 + \\ & \quad 3 (-1+3T) p_j^2 x_i^2 + 4 (-1+T) p_i^2 p_j x_i^3 - 2 (-1+T) (5+T) p_i p_j^2 x_i^3 + \\ & \quad 2 (-1+T) (3+T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j - 6 p_i^2 p_j x_i^2 x_j + \\ & \quad 6 (2+T) p_i p_j^2 x_i^2 x_j - 6 (1+T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) \end{aligned}$$

⊙ $\epsilon^2 r_2[-1, i, j]$

$$\begin{aligned} & \frac{1}{12 T^2} \epsilon^2 (-6 T^2 p_i x_i + 6 T^2 p_j x_i + \\ & \quad 3 (-3+T) T p_i p_j x_i^2 - 3 (-3+T) T p_j^2 x_i^2 - 4 (-1+T) T p_i^2 p_j x_i^3 + \\ & \quad 2 (-1+T) (1+5T) p_i p_j^2 x_i^3 - 2 (-1+T) (1+3T) p_j^3 x_i^3 + \\ & \quad 18 T^2 p_i p_j x_i x_j - 18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1+2T) p_i p_j^2 x_i^2 x_j - \\ & \quad 6 T (1+T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) \end{aligned}$$

⊙ $\epsilon^2 \gamma_2[\varphi, i]$

$$\frac{1}{2} \epsilon^2 \varphi^2 p_i x_i$$

Even more! • The sl_2 formulas mod ϵ^4 are in the last page of the handout of [BN3].

- Using [GPV] we can show that every finite type invariant is I-Type.
- Probably, $\langle \text{Reshetikhin-Turaev} \rangle \subset \langle \text{I-Type} \rangle$ efficiently.
- Possibly, $\langle \text{Rozansky Polynomials} \rangle \subset \langle \text{I-Type} \rangle$ efficiently.
- Knot signatures are I-Type, at least mod 8.
- We already have some work on sl_3 , and it leads to the strongest genuinely-computable knot invariant presently known.

The $sl_3^{\epsilon^2}$ Example (continues Schaveling [Sch]). Here we have two formal variables T_1 and T_2 , we set $T_3 := T_1 T_2$, we integrate over 6 variables for each edge: $p_{1i}, p_{2i}, p_{3i}, x_{1i}, x_{2i}$, and x_{3i} .



Schaveling

```
⊙  $T_3 = T_1 T_2$ ;  $i_-^+ := i + 1$ ;
 $\$ \pi =$ 
  (CF@Normal[# + O[ $\epsilon$ ]2] /.
    { $\pi_{is\_} \rightarrow B^{-1} \pi_{is}$ ,  $x_{is\_} \rightarrow B^{-1} x_{is}$ ,
      $p_{is\_} \rightarrow B p_{is}$ } /.  $\epsilon \in B^b$  /;  $b < 0 \rightarrow 0$  /.  $B \rightarrow 1$ ) &;
```

```
⊙  $vs_{i\_} :=$  Sequence[ $p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}$ ];
 $\mathcal{F}[is\_]$  := E[Sum[ $\pi_{v,i} p_{v,i}$ , {i, {is}}, {v, 3}]];
 $\mathcal{L}[K\_]$  := CF[ $\mathcal{L}$  /@ Features[K][[2]]];
 $vs[K\_]$  :=
  Union@@Table[{ $vs_i$ }, {i, Features[K][[1]]}]
```

The Lagrangian.

```
⊙  $\mathcal{L}[X_{i,j}[s\_]] := T_3^5 E[CF@Plus[
  \sum_{v=1}^3 (x_{vi} (p_{vi^+} - p_{vi}) + x_{vj} (p_{vj^+} - p_{vj}) + (T_v^5 - 1) x_{vi} (p_{vi^+} - p_{vj^+})),
  (T_1^5 - 1) p_{3j} x_{1i} (T_2^5 x_{2i} - x_{2j}),
  \epsilon s (T_3^5 - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T_2^5 - 1),
  \epsilon s (1/2 + T_2^5 p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} -
  (T_2^5 - 1) p_{2j} p_{3i} x_{2i} x_{3i} + (T_3^5 - 1) p_{2j} p_{3j} x_{2i} x_{3i} +
  2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} -
  T_2^5 p_{2j} p_{3j} x_{2i} x_{3j} +
  ((T_1^5 - 1) p_{1j} x_{1i} (T_2^5 p_{2j} x_{2i} - T_2^5 p_{2j} x_{2j} -
  (T_2^5 + 1) (T_3^5 - 1) p_{3j} x_{3i} + T_2^5 p_{3j} x_{3j}) +
  (T_3^5 - 1) p_{3j} x_{3i} (1 - T_2^5 p_{1i} x_{1i} + p_{2i} x_{2j} + (T_2^5 - 2) p_{2j} x_{2j})) /
  (T_2^5 - 1)]]]$ 
```

```
⊙  $\mathcal{L}[C_{i\_}[\varphi\_]] := T_3^\varphi E[\sum_{v=1}^3 x_{vi} (p_{vi^+} - p_{vi}) + \epsilon \varphi (p_{3i} x_{3i} - 1/2)]$ 
```

Reidemeister 3.

```
⊙ Short[
  lhs =  $\int \mathcal{F}[i, j, k] \times \mathcal{L} /@ (X_{i,j}[1] X_{i^+,k}[1] X_{j^+,k^+}[1])$ 
  d{ $vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}$ }]
```

```
⊙  $T_1^3 T_2^3$ 
  E[ $\frac{3\epsilon}{2} + T_1^2 p_{1,2+i} \pi_{1,i} - (-1 + T_1) T_1 p_{1,2+j} \pi_{1,i} + \ll 150 \gg$ ]
```

```
⊙ rhs =  $\int \mathcal{F}[i, j, k] \times \mathcal{L} /@ (X_{j,k}[1] X_{i,k^+}[1] X_{i^+,j^+}[1])$ 
  d{ $vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}$ };
  lhs == rhs
```

True

The Trefoil.



```
⊙  $K =$  Knot[3, 1];  $\int \mathcal{L}[K] d vs[K]$ 
```

```
⊙ - ((i  $T_1^2 T_2^2$ 
  E[-(( $\epsilon (1 - T_1 + T_1^2 - T_2 - T_1^3 T_2 + T_2^2 + T_1^4 T_2^2 - T_1 T_2^3 -
  T_1^4 T_2^3 + T_1^2 T_2^4 - T_1^3 T_2^4 + T_1^4 T_2^4) / ((1 - T_1 + T_1^2)
  (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2))$ )) /
  ((1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2)))]
```

A faster program, in which the Feynman diagrams are “pre-computed” (see theta.nb at [omega-beta/alpha](#)):

```
⊙  $R_1[s_, i_, j_] =$  CF[
  s (1/2 -  $g_{3ii} + T_2^5 g_{1ii} g_{2ji} - g_{1ii} g_{2jj} - (T_2^5 - 1) g_{2ji} g_{3ii} +
  2 g_{2jj} g_{3ii} - (1 - T_3^5) g_{2ji} g_{3ji} - g_{2ii} g_{3jj} - T_2^5 g_{2ji} g_{3jj} +
  g_{1ii} g_{3jj} +
  ((T_1^5 - 1) g_{1ji} (T_2^5 g_{2ji} - T_2^5 g_{2jj} + T_2^5 g_{3jj})) +
  (T_3^5 - 1) g_{3ji} (1 - T_2^5 g_{1ii} - (T_1^5 - 1) (T_2^5 + 1) g_{1ji} +
  (T_2^5 - 2) g_{2jj} + g_{2ij})) / (T_2^5 - 1)];$ 
```

```
⊙  $\theta[\{s0_, i0_, j0_}, \{s1_, i1_, j1_}] :=$ 
  CF[ $s1 (T_1^{s0} - 1) (T_2^{s1} - 1)^{-1} (T_3^{s1} - 1) g_{1,j1,i0} g_{3,j0,i1}$ 
  (  $(T_2^{s0} g_{2,i1,i0} - g_{2,i1,j0}) - (T_2^{s0} g_{2,j1,i0} - g_{2,j1,j0})$  )]
```

```
⊙  $T_1[\varphi_, k_] = -\varphi / 2 + \varphi g_{3kk}$ ;
```

We call the invariant computed θ :

```
⊙  $\theta[K_] :=$  Module[{Cs,  $\varphi$ , n, A, s, i, j, k,  $\Delta$ , G, v,  $\alpha$ ,  $\beta$ , gEval, c, z},
  {Cs,  $\varphi$ } = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_}  $\rightarrow$ 
    (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s T^s - 1 ));];
   $\Delta = T^{(-Total[\varphi] - Total[Cs][[All, 1]]) / 2} Det[A]$ ;
  G = Inverse[A];
  gEval[ $\mathcal{L}$ ] := Factor[ $\mathcal{L}$  /.  $g_{v,\alpha,\beta} \rightarrow (G[[\alpha, \beta]] /. T \rightarrow T_v)$ ];
  z = gEval[ $\sum_{k1=1}^n \sum_{k2=1}^n \theta[Cs[[k1]], Cs[[k2]]]$ ];
  z += gEval[ $\sum_{k=1}^n R_1 @@ Cs[[k]]$ ];
  z += gEval[ $\sum_{k=1}^{12^n} T_1[\varphi[[k]], k]$ ];
  { $\Delta$ , ( $\Delta /. T \rightarrow T_1$ ) ( $\Delta /. T \rightarrow T_2$ ) ( $\Delta /. T \rightarrow T_3$ ) z} // Factor];
```

Some Knots.

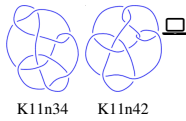
```
⊙ Expand[ $\theta$ [Knot[3, 1]]]
```

$$\left\{ -1 + \frac{1}{T} + T, -\frac{1}{T_1^2} - T_1^2 - \frac{1}{T_2^2} - \frac{1}{T_1^2 T_2^2} + \frac{1}{T_1 T_2^2} + \frac{1}{T_1^2 T_2} + \frac{T_1}{T_2} + \frac{T_2}{T_1} + T_1^2 T_2 - T_2^2 + T_1 T_2^2 - T_1^2 T_2^2 \right\}$$

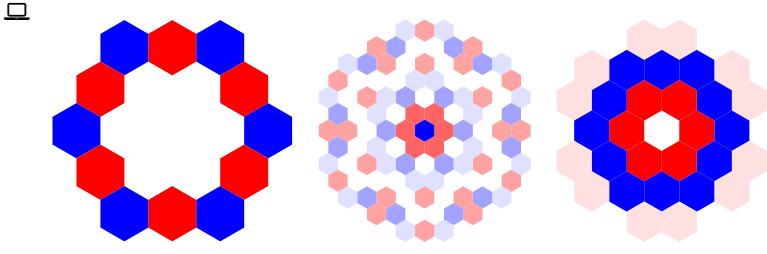
```
⊙ PolyPlot[ $\theta$ ] = Graphics[{}];
```

```
PolyPlot[p_] := Module[{crs, m1, m2, maxc, minc, s, hex},
  crs = CoefficientRules[ $T_1^{-m1} - Exponent[p, T_1, Min] T_2^{-m2} - Exponent[p, T_2, Min] p$ ,
    {T1, T2}];
  maxc = N@Log@Max@Abs[Last /@ crs];
  minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
  If[minc == maxc, s[_] = 0,
  s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
  hex = Table[{Cos[ $\alpha$ ], Sin[ $\alpha$ ]} / Cos[2  $\pi$  / 12] / 2,
    { $\alpha$ , 2  $\pi$  / 12, 2  $\pi$ , 2  $\pi$  / 6}];
  Graphics[crs /. ((x1_, x2_)  $\rightarrow$  c_)  $\rightarrow$  {
    If[c == 0, White, Lighter[If[c > 0, Red, Blue],
      0.88 s[Abs@c]]],
    Polygon[ $\left( \left( \begin{matrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{matrix} \right) \cdot \{x1 + m1, x2 + m2\} + \# \right) \& /@ hex$  ] ]];
  PolyPlot[{ $\Delta$ ,  $\theta$ }] := PolyPlot[ $\theta$ ]
```

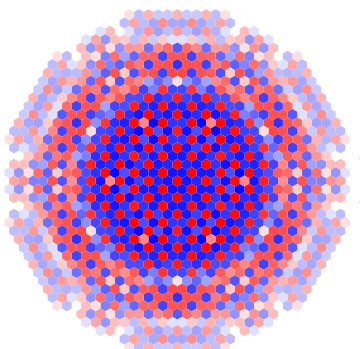

☺ GraphicsRow[PolyPlot[θ [Knot[#]]] & /@ {"3_1", "K11n34", "K11n42"}]



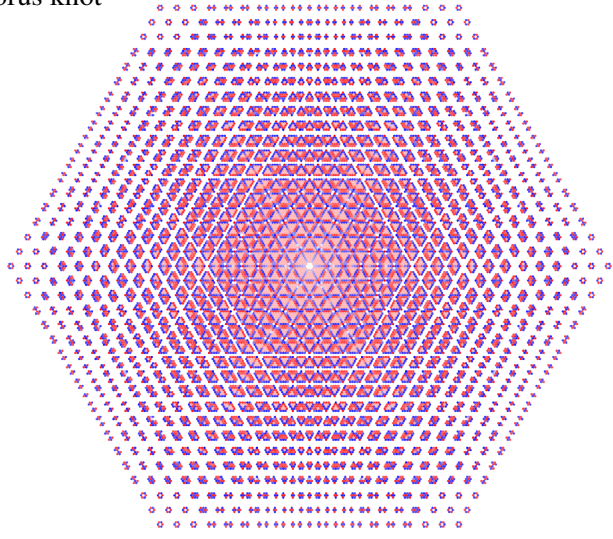
K11n34 K11n42



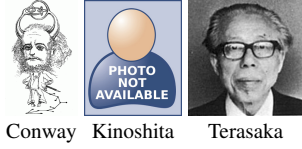
{39.0193, }



The torus knot $T_{22/7}$:

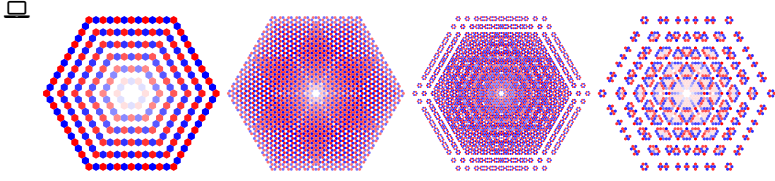


So θ detects knot mutation and separates the Conway knot K11n34 from the Kinoshita-Terasaka knot K11n42!

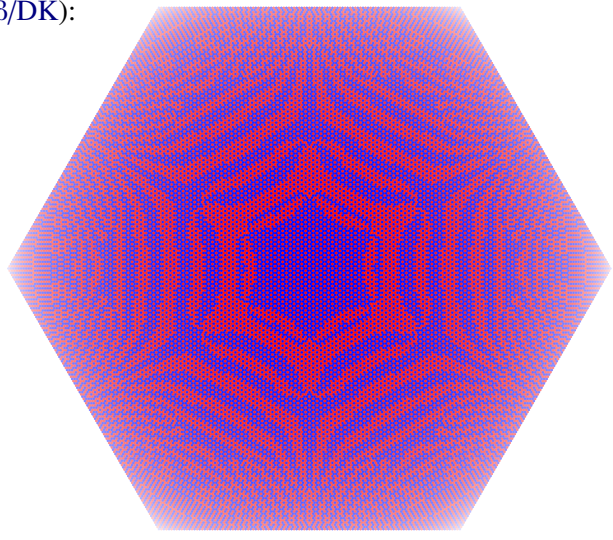


Conway Kinoshita Terasaka

☺ GraphicsRow[PolyPlot[θ [TorusKnot@@#]] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}, Spacings $\rightarrow \emptyset$]



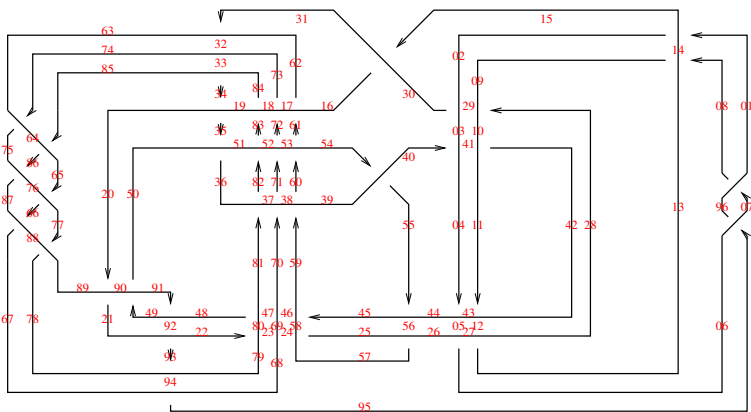
Last, a random 250 crossing knot (knot from N. Dunfield; more at $\omega\epsilon\beta/DK$):



The 48-crossing Gompf-Scharlemann-Thompson knot [GST] is significant because it may be a counterexample to the slice-ribbon conjecture:



Gompf Scharlemann Thompson



☺ AbsoluteTiming@ PolyPlot [

θ [EPD[X_{14,1}, X_{2,29}, X_{3,40}, X_{43,4}, X_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, X_{9,28}, X_{10,41}, X_{42,11}, X_{27,12}, X_{30,15}, X_{16,61}, X_{17,72}, X_{18,83}, X_{19,34}, X_{89,20}, X_{21,92}, X_{79,22}, X_{68,23}, X_{57,24}, X_{25,56}, X_{62,31}, X_{73,32}, X_{84,33}, X_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, X_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, X_{63,74}, X_{64,85}, X_{76,65}, X_{87,66}, X_{67,94}, X_{75,86}, X_{88,77}, X_{78,93}]]]

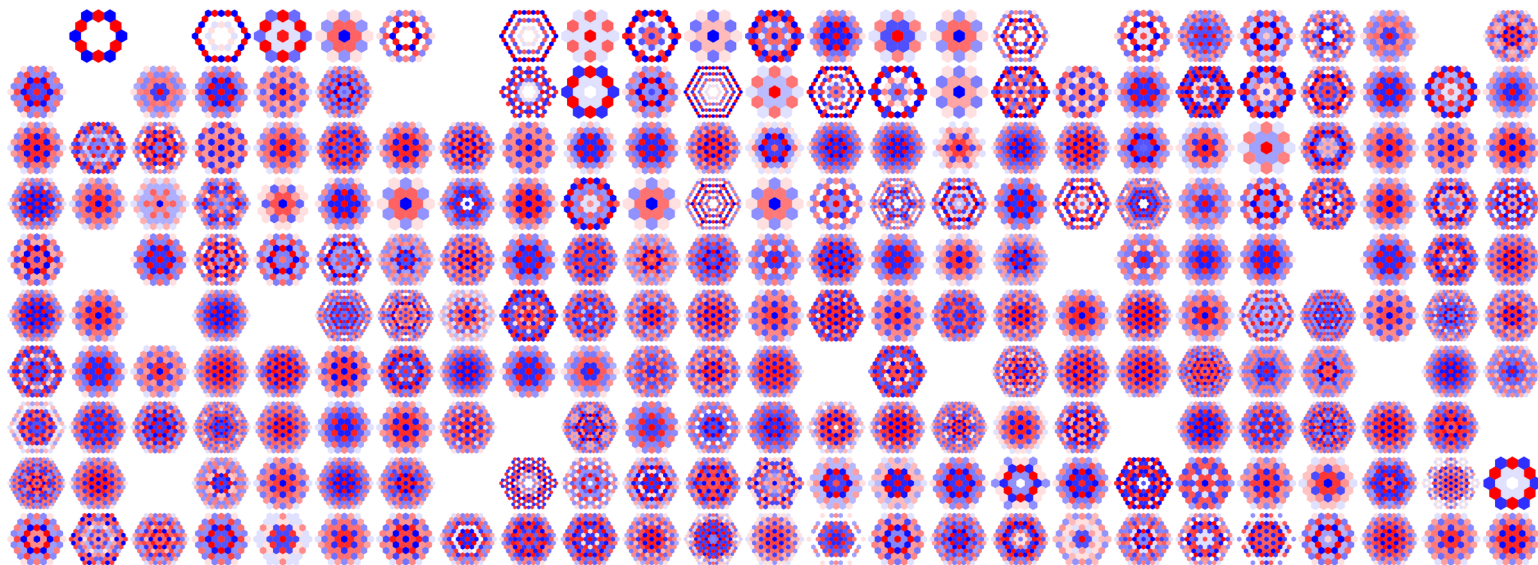
Prior Art. θ is probably equal to the “2-loop polynomial” studied by Ohtsuki at [Oh2] (at much greater difficulty, and with harder computations). θ is related, but probably not equivalent, to the invariant studied by Garoufalidis and Kashaev at [GK].



Ohtsuki Garoufalidis Kashaev

θ Sees Topology! Indeed, for a knot K , half the T_1 degree (say) of $\theta(K)$ bounds the genus of K from below, and this bound is sometimes better (and sometimes worse) than the bound coming from Δ . It is fair to hope that “anything Δ can do θ can do too” (see [BN2]), and in particular, that θ may say something about ribbon and/or slice properties.

The Rolfsen Table of Knots.



Where is it coming from? The most honest answer is “we don’t know” (and *that’s good!*). The second most, “undetermined coefficients for an ansatz that made sense”. The ansatz comes from the following principles / earlier work:

Morphisms have generating functions. Indeed, there is an isomorphism

$$\mathcal{G}: \text{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j]) \rightarrow \mathbb{Q}[y_j][[\xi_i]],$$

and by PBW, many relevant spaces are polynomial rings, though only as vector spaces.

Composition is integration. Indeed, if $f \in \text{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j])$ and $g \in \text{Hom}(\mathbb{Q}[y_j], \mathbb{Q}[z_k])$, then

$$\mathcal{G}(g \circ f) = \int e^{-y \cdot \eta} f g \, dy \, d\eta$$

Use universal invariants. These take values in a universal enveloping algebra (perhaps quantized), and thus they are expressible as long compositions of generating functions. See [La, Oh1].

“Solvable approximation” \leadsto perturbed Gaussians. Let \mathfrak{g} be a semisimple Lie algebra, let \mathfrak{h} be its Cartan subalgebra, and let \mathfrak{b}^u and \mathfrak{b}^l be its upper and lower Borel subalgebras. Then \mathfrak{b}^u has a bracket β , and as the dual of \mathfrak{b}^l it also has a cobracket δ , and in fact, $\mathfrak{g} \oplus \mathfrak{h} \cong \text{Double}(\mathfrak{b}^u, \beta, \delta)$. Let $\mathfrak{g}_\epsilon^+ := \text{Double}(\mathfrak{b}^u, \beta, \epsilon\delta) \pmod{\epsilon^{d+1}}$ it is solvable for any d . Then by [BV3, BN1] (in the case of $\mathfrak{g} = \mathfrak{sl}_2$) all the interesting tensors of $\mathcal{U}(\mathfrak{g}_\epsilon^+)$ (quantized or not) are perturbed Gaussian with perturbation parameter ϵ with understood bounds on the degrees of the perturbations.

The Philosophy Corner. “Universal invariants”, valued in universal enveloping algebra (possibly quantized) rather than in representations thereof, are a priori better than the representation theoretic ones. They are compatible with strand doubling (the Hopf coproduct), and as the knot genus and the ribbon property for knots are expressible in terms of strand doubling, universal invariants stand a chance to say something about these properties. Indeed, they sometimes do! See e.g. [BN2, Oh2, GK, LV, BG]. Representation theoretic invariants don’t do that!



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