In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Beijing-2407"];
Once[<< IType.m];
T3 = T1 T2;
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/AP/Talks/Beijing-2407 to compute rotation numbers.

exec
In[∘]:=
```
nb2tex$PDFWidth *= 1.25;
```

## The Programs

tex

{\red\bf A faster program,} in which the Feynman diagrams are ``pre-computed'' (see theta.nb at \web{ap}):

pdf

$$R_1[1, i\_, j\_] = CF[1/2 - T_3\, g_{1ji}\, g_{2ji} - g_{3ii} + g_{2jj}\, g_{3ii} + T_1\,(T_3 - 1)\, g_{1ji}\, g_{3ji} + T_2\,(T_3 - 1)\, g_{2ji}\, g_{3ji} - T_2\, g_{2ji}\, g_{3jj} + (g_{1jj}\, g_{2ii} + (T_3 - 1)\, g_{1jj}\, g_{2ji} - T_1\, g_{1ii}\, g_{2jj} - g_{1jj}\, g_{3ii} - T_1\,(T_3 - 1)\, g_{1jj}\, g_{3ji} + T_1\, g_{1ii}\, g_{3jj}) / (T_1 - 1)];$$

In[∘]:=

$$\text{Simplify}\Big[R_1[1, i, j] == \frac{1}{2} + \frac{g_{1,j,j}\, g_{2,i,i}}{-1 + T_1} - T_1 T_2\, g_{1,j,i}\, g_{2,j,i} + \frac{(-1 + T_1 T_2)\, g_{1,j,j}\, g_{2,j,i}}{-1 + T_1} - \frac{T_1\, g_{1,i,i}\, g_{2,j,j}}{-1 + T_1} - g_{3,i,i} - \frac{g_{1,j,j}\, g_{3,i,i}}{-1 + T_1} + g_{2,j,j}\, g_{3,i,i} + T_1\,(-1 + T_1 T_2)\, g_{1,j,i}\, g_{3,j,i} - \frac{T_1\,(-1 + T_1 T_2)\, g_{1,j,j}\, g_{3,j,i}}{-1 + T_1} + T_2\,(-1 + T_1 T_2)\, g_{2,j,i}\, g_{3,j,i} + \frac{T_1\, g_{1,i,i}\, g_{3,j,j}}{-1 + T_1} - T_2\, g_{2,j,i}\, g_{3,j,j}\Big]$$

Out[∘]=

True

pdf

$$R_1[-1, i\_, j\_] = CF\Big[-1/2 - T_1^{-1}\, g_{1ji}\, g_{2ii} - \left(1 - T_1^{-1} - T_2^{-1}\right) g_{1ji}\, g_{2ji} - g_{1jj}\, g_{2ji} - g_{1ji}\, g_{2jj} + g_{3ii} + T_1^{-1}\, g_{1ji}\, g_{3ii} - \left(1 - T_2^{-1}\right) g_{2ji}\, g_{3ii} - g_{2jj}\, g_{3ii} + \left(1 - T_3^{-1}\right) g_{1ji}\, g_{3ji} - \left(1 - T_3^{-1}\right) g_{2ii}\, g_{3ji} + \left(2 - T_2^{-1}\right)\left(1 - T_3^{-1}\right) g_{2ji}\, g_{3ji} + \left(1 - T_3^{-1}\right) g_{2jj}\, g_{3ji} + g_{1ji}\, g_{3jj} + g_{2ji}\, g_{3jj} + \left(T_1\left(1 - T_2^{-1}\right) g_{1ii}\, g_{2ji} - g_{1jj}\, g_{2ii} + T_1\, g_{1ii}\, g_{2jj} + g_{1jj}\, g_{3ii} - T_2^{-1}\,(T_3 - 1)\, g_{1ii}\, g_{3ji} - T_1\, g_{1ii}\, g_{3jj}\right) / (T_1 - 1)\Big];$$

In[∘]:= **$R_1[-1, i, j]$**

Out[∘]=

$$-\frac{1}{2}-\frac{g_{1,j,i}\,g_{2,i,i}}{T_1}-\frac{g_{1,j,j}\,g_{2,i,i}}{-1+T_1}+\frac{T_1\,(-1+T_2)\,g_{1,i,i}\,g_{2,j,i}}{(-1+T_1)\,T_2}-\frac{(-T_1-T_2+T_1T_2)\,g_{1,j,i}\,g_{2,j,i}}{T_1T_2}-g_{1,j,j}\,g_{2,j,i}+\frac{T_1\,g_{1,i,i}\,g_{2,j,j}}{-1+T_1}-g_{1,j,i}\,g_{2,j,j}+g_{3,i,i}+\frac{g_{1,j,i}\,g_{3,i,i}}{T_1}+\frac{g_{1,j,j}\,g_{3,i,i}}{-1+T_1}-\frac{(-1+T_2)\,g_{2,j,i}\,g_{3,i,i}}{T_2}-g_{2,j,j}\,g_{3,i,i}-\frac{(-1+T_1T_2)\,g_{1,i,i}\,g_{3,j,i}}{(-1+T_1)\,T_2}+\frac{(-1+T_1T_2)\,g_{1,j,i}\,g_{3,j,i}}{T_1T_2}-\frac{(-1+T_1T_2)\,g_{2,i,i}\,g_{3,j,i}}{T_1T_2}+\frac{(-1+2T_2)\,(-1+T_1T_2)\,g_{2,j,i}\,g_{3,j,i}}{T_1T_2^2}+\frac{(-1+T_1T_2)\,g_{2,j,j}\,g_{3,j,i}}{T_1T_2}-\frac{T_1\,g_{1,i,i}\,g_{3,j,j}}{-1+T_1}+g_{1,j,i}\,g_{3,j,j}+g_{2,j,i}\,g_{3,j,j}$$

In[∘]:= **Simplify**

$$\Big[R_1[-1,i,j]==-\frac{1}{2}-\frac{g_{1,j,i}\,g_{2,i,i}}{T_1}-\frac{g_{1,j,j}\,g_{2,i,i}}{-1+T_1}+\frac{T_1\,(-1+T_2)\,g_{1,i,i}\,g_{2,j,i}}{(-1+T_1)\,T_2}-\frac{(-T_1-T_2+T_1T_2)\,g_{1,j,i}\,g_{2,j,i}}{T_1T_2}-g_{1,j,j}\,g_{2,j,i}+\frac{T_1\,g_{1,i,i}\,g_{2,j,j}}{-1+T_1}-g_{1,j,i}\,g_{2,j,j}+g_{3,i,i}+\frac{g_{1,j,i}\,g_{3,i,i}}{T_1}+\frac{g_{1,j,j}\,g_{3,i,i}}{-1+T_1}-\frac{(-1+T_2)\,g_{2,j,i}\,g_{3,i,i}}{T_2}-g_{2,j,j}\,g_{3,i,i}-\frac{(-1+T_1T_2)\,g_{1,i,i}\,g_{3,j,i}}{(-1+T_1)\,T_2}+\frac{(-1+T_1T_2)\,g_{1,j,i}\,g_{3,j,i}}{T_1T_2}-\frac{(-1+T_1T_2)\,g_{2,i,i}\,g_{3,j,i}}{T_1T_2}+\frac{(-1+2T_2)\,(-1+T_1T_2)\,g_{2,j,i}\,g_{3,j,i}}{T_1T_2^2}+\frac{(-1+T_1T_2)\,g_{2,j,j}\,g_{3,j,i}}{T_1T_2}-\frac{T_1\,g_{1,i,i}\,g_{3,j,j}}{-1+T_1}+g_{1,j,i}\,g_{3,j,j}+g_{2,j,i}\,g_{3,j,j}\Big]$$

Out[∘]=

True

*pdf*

In[∘]:=

```
θ[{1, i0_, j0_}, {1, i1_, j1_}] =
  -T1 (T3 - 1) g1,j1,i0 g2,i1,i0 g3,j0,i1 + (T3 - 1) g1,j1,j0 g2,i1,i0 g3,j0,i1 +
   T1 (T3 - 1) g1,j1,i0 g2,j1,i0 g3,j0,i1 - (T3 - 1) g1,j1,j0 g2,j1,i0 g3,j0,i1;
```

$$\theta[\{1,i0\_,j0\_\},\{1,i1\_,j1\_\}]=-T_1(T_3-1)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}+(T_3-1)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}+T_1(T_3-1)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}-(T_3-1)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1};$$

In[∘]:= **Simplify**

$$[\theta[\{1,i0,j0\},\{1,i1,j1\}]==-T_1(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}+(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}+T_1(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+(1-T_1T_2)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}]$$

Out[∘]=

True

*pdf*

In[∘]:=

$$\theta[\{1,i0\_,j0\_\},\{-1,i1\_,j1\_\}]=(T_3-1)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}-T_1^{-1}(T_3-1)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}-(T_3-1)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+T_1^{-1}(T_3-1)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1};$$

In[∘]:=
```
Simplify[θ[{1, i0, j0}, {-1, i1, j1}] ==
  (-1 + T1 T2) g1,j1,i0 g2,i1,i0 g3,j0,i1 - ((-1 + T1 T2) g1,j1,j0 g2,i1,i0 g3,j0,i1)/T1 +
   (1 - T1 T2) g1,j1,i0 g2,j1,i0 g3,j0,i1 + ((-1 + T1 T2) g1,j1,j0 g2,j1,i0 g3,j0,i1)/T1]
```

$$(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}-\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_1}+(1-T_1T_2)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_1}$$

Out[∘]=
True

pdf
```
θ[{-1, i0_, j0_}, {1, i1_, j1_}] = CF[
  T1^-1 T2^-1 (T3 - 1) (g1,j1,i0 g2,i1,i0 g3,j0,i1 -
     T1 g1,j1,j0 g2,i1,i0 g3,j0,i1 - g1,j1,i0 g2,j1,i0 g3,j0,i1 + T1 g1,j1,j0 g2,j1,i0 g3,j0,i1)];
```

$$\theta[\{-1,i0\_,j0\_\},\{1,i1\_,j1\_\}]=\mathrm{CF}\left[T_1^{-1}T_2^{-1}(T_3-1)\left(g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}-T_1\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}-g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+T_1\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}\right)\right];$$

In[∘]:=
```
Simplify[θ[{-1, i0, j0}, {1, i1, j1}] ==
```

$$\frac{(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_1T_2}-\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_2}-\frac{(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_1T_2}+\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_2}]$$

Out[∘]=
True

pdf

$$\theta[\{-1,i0\_,j0\_\},\{-1,i1\_,j1\_\}]=\mathrm{CF}\left[\left(1-T_3^{-1}\right)\left(-T_1^{-1}\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}+g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}+T_1^{-1}\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}-g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}\right)\right];$$

In[∘]:=
```
Simplify[θ[{-1, i0, j0}, {-1, i1, j1}] ==
```

$$-\frac{(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_1^2T_2}+\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_1T_2}+\frac{(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_1^2T_2}-\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_1T_2}]$$

Out[∘]=
True

pdf

In[∘]:=

$$\Gamma_1[\varphi\_,k\_]=-\varphi/2+\varphi\,g_{3,k,k};$$

tex

We call the invariant computed $\theta$:

pdf

In[∘]:=
```
θ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ν, α, β, gEval, c, z},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} ⧴ (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                                              0    -1    ))];
  Δ = T^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A];
  G = Inverse[A]; gEval[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} ⧴ (G⟦α, β⟧ /. T → T_ν)];
  z = gEval[Sum_{k1=1}^n Sum_{k2=1}^n θ[Cs⟦k1⟧, Cs⟦k2⟧]];
  z += gEval[Sum_{k=1}^n R_1 @@ Cs⟦k⟧];
  z += gEval[Sum_{k=1}^{2 n} Γ_1[φ⟦k⟧, k]];
  {Δ, (Δ /. T → T_1) (Δ /. T → T_2) (Δ /. T → T_3) z} // Factor];
```

exec

```
nb2tex$PDFWidth /= 1.25;
```

## Some Knots

tex

\needspace{15mm}
{\bf\red Some Knots.}

pdf

In[∘]:=
```
Expand[θ[Knot[3, 1]]]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$

exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

In[ ]:=
```
PolyPlot[0] = Graphics[{}];
PolyPlot[p_] := Module[{crs, m1, m2, maxc, minc, s, hex},
  crs = CoefficientRules[T1^(m1=-Exponent[p,T1,Min]) T2^(m2=-Exponent[p,T2,Min]) p, {T1, T2}];
  maxc = N@Log@Max@Abs[Last /@ crs];
  minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
  If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
  hex = Table[{Cos[α], Sin[α]} / Cos[2 π / 12] / 2, {α, 2 π / 12, 2 π, 2 π / 6}];
  Graphics[crs /. ({x1_, x2_} → c_) :→ {
      If[c == 0, White, Lighter[If[c > 0, Red, Blue], 0.88 s[Abs@c]]],
      Polygon[({{1, -1/2}, {0, √3/2}}.{x1 + m1, x2 + m2} + #) & /@ hex] }] ]
```

exec

```
nb2tex$PDFWidth /= 1.25;
```

tex

```
\parpic[r]{$
 {\includegraphics[height=0.45in]{K11n34.png}
  \atop\text{\tiny K11n34}}
 {\includegraphics[height=0.45in]{K11n42.png}
  \atop\text{\tiny K11n42}}
$}
```

pdf

In[ ]:=
```
GraphicsRow[PolyPlot[θ[Knot[#]][[2]]] &
  /@ {"3_1", "K11n34", "K11n42"}]
```

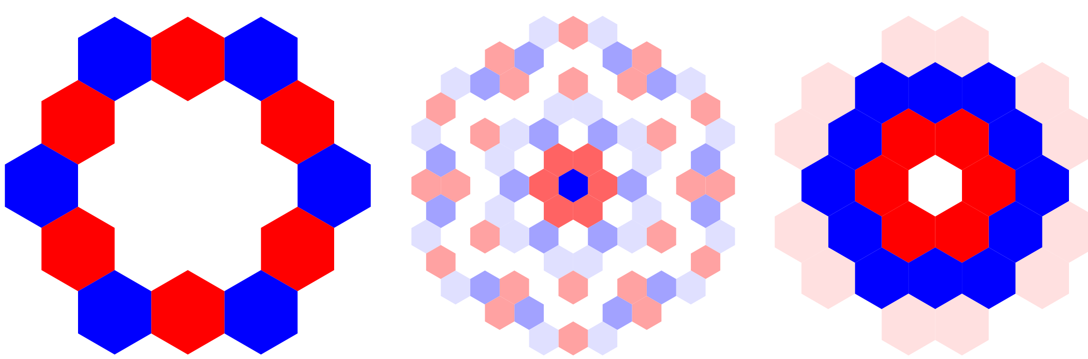
pdf

KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

pdf

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[ ]=
pdf



tex

```
\parpic[r]{$
```

{\includegraphics[height=0.6in]{../../Projects/Gallery/Conway.png}
\atop\text{\scriptsize Conway}}
{\includegraphics[height=0.6in]{../../Projects/Gallery/PhotoNotAvailable.png}
\atop\text{\scriptsize Kinoshita}}
{\includegraphics[height=0.6in]{../../Projects/Gallery/Terasaka.jpg}
\atop\text{\scriptsize Terasaka}}
$}
So $\theta$ detects knot mutation and separates the Conway knot K11n34 from the Kinoshita-Terasaka knot K11n42!

\vskip 8mm

%\needspace{50mm}
\parpic[r]{$
{\includegraphics[height=0.6in]{../../Projects/Gallery/Gompf.jpg}
\atop\text{\scriptsize Gompf}}
{\includegraphics[height=0.6in]{../../Projects/Gallery/Scharlemann.jpg}
\atop\text{\scriptsize Scharlemann}}
{\includegraphics[height=0.6in]{../../Projects/Gallery/Thompson.jpg}
\atop\text{\scriptsize Thompson}}
$}
The 48-crossing Gompf-Scharlemann-Thompson knot \cite{GompfScharlemannThompson:Counterexample} is significant because it may be a counterexample to the slice-ribbon conjecture:

\[ \resizebox{\linewidth}{!}{\import{../Waco-2203/}{GST48-Marked.pdf_t}} \]

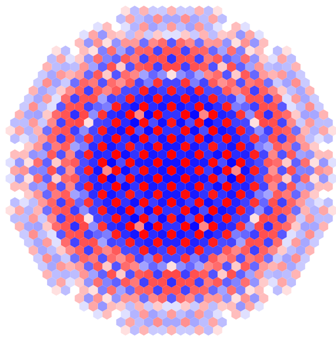
exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

In[∘]:= **AbsoluteTiming@**
**PolyPlot**$\left[\theta\left[\mathrm{EPD}\left[X_{14,1}, \overline{X}_{2,29}, X_{3,40}, X_{43,4}, \overline{X}_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, \overline{X}_{9,28}, X_{10,41}, X_{42,11}, \overline{X}_{27,12}, X_{30,15}, \overline{X}_{16,61}, \overline{X}_{17,72}, \overline{X}_{18,83}, X_{19,34}, \overline{X}_{89,20}, \overline{X}_{21,92}, \overline{X}_{79,22}, \overline{X}_{68,23}, \overline{X}_{57,24}, \overline{X}_{25,56}, X_{62,31}, X_{73,32}, X_{84,33}, \overline{X}_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, \overline{X}_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, \overline{X}_{63,74}, \overline{X}_{64,85}, \overline{X}_{76,65}, \overline{X}_{87,66}, \overline{X}_{67,94}, \overline{X}_{75,86}, \overline{X}_{88,77}, \overline{X}_{78,93}\right]\right][[2]]\right]$

Out[∘]=
pdf

{11.7333,  }

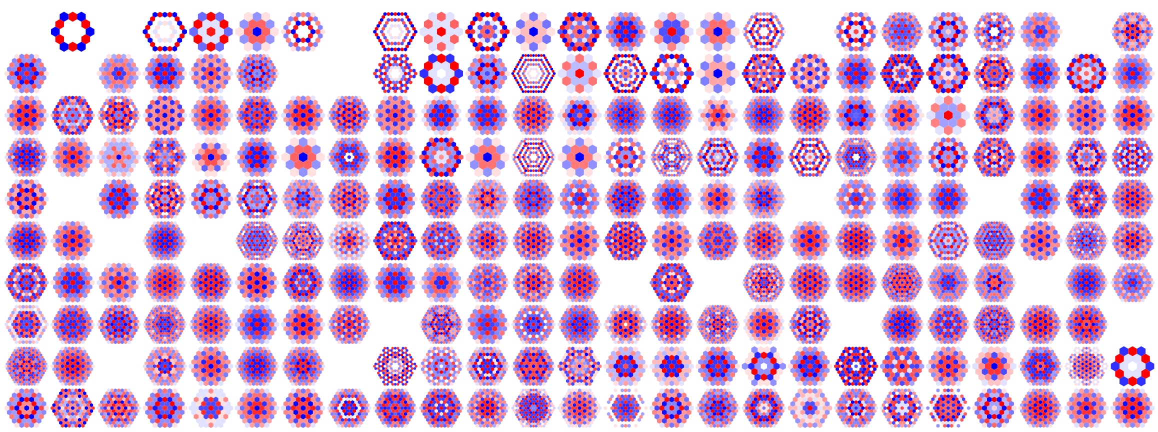
exec

In[∘]:= **nb2tex$PDFWidth /= 1.25;**

In[∘]:= **tab250 = {0}~Join~Table[θ[K][[2]], {K, AllKnots[{3, 10}]}];**

In[∘]:= **g250 = GraphicsGrid[Partition[PolyPlot /@ tab250, 25], Spacings → 0]**

Out[∘]=



In[∘]:= **Export["g250.png", g250, ImageSize → 2400]**

Out[∘]=

g250.png