Pensieve header: The rank 2 mod ϵ^2 invariant using integration techniques; continues UC4A2.nb and Theta.nb at pensieve://Projects/HigherRank/.

Initialization

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Beijing-2407"];
Once[<< IType.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/AP/Talks/Beijing-2407 to compute rotation numbers.
```

pdf

In[∘]:=
```
T3 = T1 T2;  i_+ := i + 1;
$π = (CF@Normal[# + O[ϵ]^2] /. {π_is__ :> B^-1 π_is, x_is__ :> B^-1 x_is, p_is__ :> B p_is} /.
      ϵ B^b_ /; b < 0 → 0 /. B → 1) &;
```

pdf

In[∘]:=
```
vs_i_ := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
ℱ[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K]〚2〛];
vs[K_] := Union @@ Table[{vs_i}, {i, Features[K]〚1〛}]
```

The Lagrangian

tex

\needspace{30mm}
{\bf\red The Lagrangian.}

exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

```
ℒ[X_{i_,j_}[1]] := T3 𝔼[Plus[
    Sum_{ν=1}^{3} (x_{νi} (p_{νi+} − p_{νi}) + x_{νj} (p_{νj+} − p_{νj}) + (T_ν − 1) x_{νi} (p_{νi+} − p_{νj+})),
    p_{3j} x_{2i} (T1 x_{1i} − x_{1j}),
    ϵ (T3 − 1) p_{1j} x_{3i} (p_{2j} − p_{2i}),
    ϵ (1 / 2 − p_{3i} x_{3i} − T3 p_{1j} p_{2j} x_{1i} x_{2i} + p_{2j} p_{3i} x_{2j} x_{3i} − T2 p_{2j} p_{3j} x_{2i} x_{3j} +
       (T3 − 1) p_{3j} x_{3i} (T1 p_{1j} x_{1i} + T2 p_{2j} x_{2i}) + (p_{1j} x_{1j} (p_{2i} x_{2i} − p_{3i} x_{3i}) +
          T1 p_{1i} x_{1i} (p_{3j} x_{3j} − p_{2j} x_{2j}) + (T3 − 1) p_{1j} x_{1j} (p_{2j} x_{2i} − T1 p_{3j} x_{3i})) / (T1 − 1)) ]]
```

pdf

$$\mathcal{L}[X_{i\_,j\_}[-1]] := T_3^{-1}\,\mathbb{E}\Big[\text{Plus}\Big[$$
$$\sum_{\nu=1}^{3}\left(x_{\nu i}\,(p_{\nu i^+}-p_{\nu i})+x_{\nu j}\,(p_{\nu j^+}-p_{\nu j})+\left(T_\nu^{-1}-1\right)x_{\nu i}\,(p_{\nu i^+}-p_{\nu j^+})\right),$$
$$T_2^{-1}\left(p_{3j}\,x_{1j}\,x_{2i}-T_1^{-1}\,p_{3j}\,x_{1i}\,x_{2i}\right),$$
$$\epsilon\,T_1^{-1}\,((T_3-1)\,p_{1j}\,p_{2i}\,x_{3i}-(T_3-1)\,p_{1j}\,p_{2j}\,x_{3i}),$$
$$\epsilon\left(-1/2+p_{3i}\,x_{3i}-T_1^{-1}\,p_{1j}\,p_{2i}\,x_{1i}\,x_{2i}-\left(1-T_1^{-1}-T_2^{-1}\right)p_{1j}\,p_{2j}\,x_{1i}\,x_{2i}-p_{1j}\,p_{2j}\,x_{1j}\,x_{2i}-\right.$$
$$p_{1j}\,p_{2j}\,x_{1i}\,x_{2j}+T_1^{-1}\,p_{1j}\,p_{3i}\,x_{1i}\,x_{3i}-\left(1-T_2^{-1}\right)p_{2j}\,p_{3i}\,x_{2i}\,x_{3i}-p_{2j}\,p_{3i}\,x_{2j}\,x_{3i}+p_{1j}\,p_{3j}\,x_{1i}\,x_{3j}+$$
$$p_{2j}\,p_{3j}\,x_{2i}\,x_{3j}+\left(1-T_3^{-1}\right)p_{3j}\,x_{3i}\left(p_{2j}\,x_{2j}+p_{1j}\,x_{1i}-p_{2i}\,x_{2i}+\left(2-T_2^{-1}\right)p_{2j}\,x_{2i}\right)+$$
$$\left(T_1\left(1-T_2^{-1}\right)p_{1i}\,p_{2j}\,x_{1i}\,x_{2i}-p_{1j}\,p_{2i}\,x_{1j}\,x_{2i}+T_1\,p_{1i}\,p_{2j}\,x_{1i}\,x_{2j}-\right.$$
$$\left.\left.T_2^{-1}\,(T_3-1)\,p_{1i}\,p_{3j}\,x_{1i}\,x_{3i}+p_{1j}\,p_{3i}\,x_{1j}\,x_{3i}-T_1\,p_{1i}\,p_{3j}\,x_{1i}\,x_{3j}\right)\Big/(T_1-1)\right)\Big]\Big]$$

pdf

In[∘]:=

$$\mathcal{L}[C_{i\_}[\varphi\_]] := T_3^{\varphi}\,\mathbb{E}\left[\sum_{\nu=1}^{3}x_{\nu i}\,(p_{\nu i^+}-p_{\nu i})+\epsilon\,\varphi\,(p_{3,i}\,x_{3,i}-1/2)\right]$$

exec

```
nb2tex$PDFWidth /= 1.25;
```

Reidemeister 3

tex

{\bf\red Reidemeister 3.}

pdf

In[∘]:=

$$\text{Short}\left[\text{lhs} = \int\mathcal{F}[i,j,k]\ \mathcal{L}\,/@\,(X_{i,j}[1]\,X_{i^+,k}[1]\,X_{j^+,k^+}[1])\ d\{vs_i,vs_j,vs_k,vs_{i^+},vs_{j^+},vs_{k^+}\}\right]$$

Out[∘]//Short=

pdf

$$T_1^3\,T_2^3\,\mathbb{E}\left[\frac{3\,\epsilon}{2}+\ll 138\gg\right]$$

pdf

In[∘]:=

$$\text{rhs} = \int\mathcal{F}[i,j,k]\ \mathcal{L}\,/@\,(X_{j,k}[1]\,X_{i,k^+}[1]\,X_{i^+,j^+}[1])\ d\{vs_i,vs_j,vs_k,vs_{i^+},vs_{j^+},vs_{k^+}\};\ \text{lhs} == \text{rhs}$$

Out[∘]=

pdf

True

The Trefoil

tex

\needspace{25mm}
\parpic[r]{\includegraphics[width=0.6in]{Trefoil.jpg}}
{\bf\red The Trefoil.}

pdf

In[∘]:=
```
K = Knot[3, 1]; ∫ℒ[K] ⅆ vs[K]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$$-\frac{\mathbb{i}\, T_1^2\, T_2^2\, \mathbb{E}\left[-\frac{\epsilon\,(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4)}{(1-T_1+T_1^2)\,(1-T_2+T_2^2)\,(1-T_1 T_2+T_1^2 T_2^2)}\right]}{(1-T_1+T_1^2)\,(1-T_2+T_2^2)\,(1-T_1 T_2+T_1^2 T_2^2)}$$

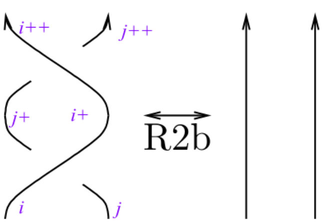
In[∘]:=

$$-\frac{\mathbb{i}\, T_1^2\, T_2^2\, \mathbb{E}\left[-\frac{\epsilon\,(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4)}{(1-T_1+T_1^2)\,(1-T_2+T_2^2)\,(1-T_1 T_2+T_1^2 T_2^2)}\right]}{(1-T_1+T_1^2)\,(1-T_2+T_2^2)\,(1-T_1 T_2+T_1^2 T_2^2)}$$

Out[∘]=

$$-\frac{\mathbb{i}\, T_1^2\, T_2^2\, \mathbb{E}\left[-\frac{\epsilon\,(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4)}{(1-T_1+T_1^2)\,(1-T_2+T_2^2)\,(1-T_1 T_2+T_1^2 T_2^2)}\right]}{(1-T_1+T_1^2)\,(1-T_2+T_2^2)\,(1-T_1 T_2+T_1^2 T_2^2)}$$

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=
```
lhs = ∫ℱ[i, j] ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}}
rhs = ∫ℱ[i, j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}};
lhs == rhs
```

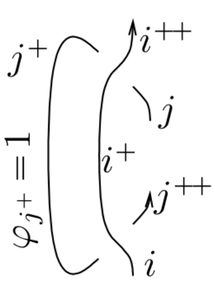
Out[∘]=

$$\mathbb{E}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}]$$

Out[∘]=

True

Invariance Under R2c



In[∘]:=
```
lhs = ∫𝓕[i, j] ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1]) d{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{j+2}}
rhs = ∫𝓕[i, j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0]) d{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{j+2}};
lhs == rhs
```

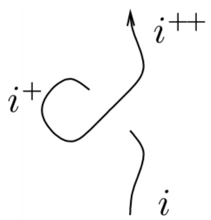
Out[∘]=

$$-i\, T_1 T_2\, \mathbb{E}\left[\frac{\epsilon}{2} + p_{1,2+i}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j} + \epsilon\, p_{3,3+j}\,\pi_{3,j}\right]$$

Out[∘]=

True

Invariance Under R1l



In[∘]:=
```
lhs = ∫𝓕[i] ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) d{vs_i, vs_{i⁺}, vs_{i+2}}
rhs = ∫𝓕[i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) d{vs_i, vs_{i⁺}, vs_{i+2}};
lhs == rhs
```

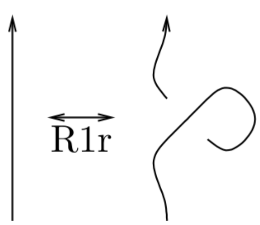
Out[∘]=

$$-i\, \mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$$

Out[∘]=

True

Invariance Under R1r



In[∘]:=

$$\texttt{lhs} = \int \mathcal{F}[i]\ \mathcal{L}\ /@\ (X_{i,i+2}[1]\ C_{i+1}[-1])\ d\{vs_i, vs_{i^+}, vs_{i+2}\}$$

$$\texttt{rhs} = \int \mathcal{F}[i]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{vs_i, vs_{i^+}, vs_{i+2}\};$$

$$\texttt{lhs} == \texttt{rhs}$$

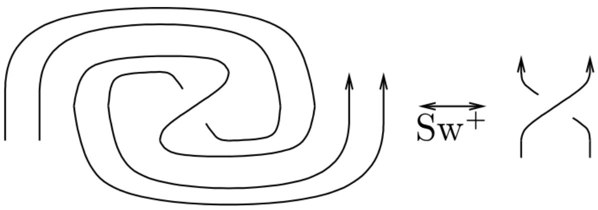
Out[∘]=

$$-i\,\mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$$

Out[∘]=

True

Invariance Under Sw



In[∘]:=

$$\texttt{lhs} = \int \mathcal{F}[i, j]\ \mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}\}$$

$$\texttt{rhs} = \int \mathcal{F}[i, j]\ \mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}\};$$

$$\texttt{lhs} == \texttt{rhs}$$

Out[∘]=

$$\begin{aligned}
T_1 T_2\,\mathbb{E}\Big[& \frac{\epsilon}{2} + T_1\,p_{1,3+i}\,\pi_{1,i} + (1 - T_1)\,p_{1,3+j}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + T_2\,p_{2,3+i}\,\pi_{2,i} + \\
& \frac{\epsilon\,T_2\,p_{2,3+i}\,\pi_{2,i}}{-1 + T_1} + (1 - T_2)\,p_{2,3+j}\,\pi_{2,i} - \epsilon\,T_1 T_2\,p_{1,3+j}\,p_{2,3+j}\,\pi_{1,i}\,\pi_{2,i} + T_1\,p_{3,3+j}\,\pi_{1,i}\,\pi_{2,i} + \\
& \frac{\epsilon\,T_2\,p_{1,3+j}\,p_{2,3+i}\,\pi_{1,j}\,\pi_{2,i}}{-1 + T_1} + \epsilon\,T_2\,p_{1,3+j}\,p_{2,3+j}\,\pi_{1,j}\,\pi_{2,i} - p_{3,3+j}\,\pi_{1,j}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} - \frac{\epsilon\,p_{2,3+j}\,\pi_{2,j}}{-1 + T_1} - \\
& \frac{\epsilon\,T_1^2\,p_{1,3+i}\,p_{2,3+j}\,\pi_{1,i}\,\pi_{2,j}}{-1 + T_1} + \epsilon\,T_1\,p_{1,3+j}\,p_{2,3+j}\,\pi_{1,i}\,\pi_{2,j} - \epsilon\,T_2\,(-1 + T_1 T_2)\,p_{1,3+j}\,p_{2,3+i}\,\pi_{3,i} + \\
& \epsilon\,T_2\,(-1 + T_1 T_2)\,p_{1,3+j}\,p_{2,3+j}\,\pi_{3,i} + T_1 T_2\,p_{3,3+i}\,\pi_{3,i} - \frac{\epsilon\,T_1 T_2\,p_{3,3+i}\,\pi_{3,i}}{-1 + T_1} + \\
& (1 - T_1 T_2)\,p_{3,3+j}\,\pi_{3,i} - \frac{\epsilon\,T_1 T_2\,p_{1,3+j}\,p_{3,3+i}\,\pi_{1,j}\,\pi_{3,i}}{-1 + T_1} + \epsilon\,T_2\,(-1 + T_1 T_2)\,p_{2,3+i}\,p_{3,3+j}\,\pi_{2,i}\,\pi_{3,i} + \\
& \epsilon\,T_1 T_2\,p_{2,3+j}\,p_{3,3+i}\,\pi_{2,j}\,\pi_{3,i} + \epsilon\,(1 - T_1 T_2)\,p_{2,3+j}\,p_{3,3+j}\,\pi_{2,j}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j} + \frac{\epsilon\,T_1\,p_{3,3+j}\,\pi_{3,j}}{-1 + T_1} + \\
& \frac{\epsilon\,T_1^2\,p_{1,3+i}\,p_{3,3+j}\,\pi_{1,i}\,\pi_{3,j}}{-1 + T_1} - \epsilon\,T_1\,p_{1,3+j}\,p_{3,3+j}\,\pi_{1,i}\,\pi_{3,j} - \epsilon\,T_2\,p_{2,3+j}\,p_{3,3+j}\,\pi_{2,i}\,\pi_{3,j}\Big]
\end{aligned}$$

Out[∘]=

True