In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Beijing-2407"];
Once[<< IType.m];
T3 = T1 T2;
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Beijing-2407 to compute rotation numbers.
```

exec

In[∘]:=
```
nb2tex$PDFWidth *= 1.25;
```

The Programs

tex

{\red\bf A faster program,} in which the Feynman diagrams are ``pre-computed'' (see theta.nb at \web{ap}):

pdf

In[∘]:=

$$R_1[1, i\_, j\_] = 1/2 - T_3\, g_{1ji}\, g_{2ji} - g_{3ii} + g_{2jj}\, g_{3ii} + T_1\,(T_3 - 1)\, g_{1ji}\, g_{3ji} + T_2\,(T_3 - 1)\, g_{2ji}\, g_{3ji} - T_2\, g_{2ji}\, g_{3jj} + (g_{1jj}\, g_{2ii} + (T_3 - 1)\, g_{1jj}\, g_{2ji} - T_1\, g_{1ii}\, g_{2jj} - g_{1jj}\, g_{3ii} - T_1\,(T_3 - 1)\, g_{1jj}\, g_{3ji} + T_1\, g_{1ii}\, g_{3jj}) / (T_1 - 1);$$

In[∘]:=

$$\text{Simplify}\Big[R_1[1, i, j] == \frac{1}{2} + \frac{g_{1,j,j}\, g_{2,i,i}}{-1 + T_1} - T_1\, T_2\, g_{1,j,i}\, g_{2,j,i} + \frac{(-1 + T_1 T_2)\, g_{1,j,j}\, g_{2,j,i}}{-1 + T_1} - \frac{T_1\, g_{1,i,i}\, g_{2,j,j}}{-1 + T_1} - g_{3,i,i} - \frac{g_{1,j,j}\, g_{3,i,i}}{-1 + T_1} + g_{2,j,j}\, g_{3,i,i} + T_1\,(-1 + T_1 T_2)\, g_{1,j,i}\, g_{3,j,i} - \frac{T_1\,(-1 + T_1 T_2)\, g_{1,j,j}\, g_{3,j,i}}{-1 + T_1} + T_2\,(-1 + T_1 T_2)\, g_{2,j,i}\, g_{3,j,i} + \frac{T_1\, g_{1,i,i}\, g_{3,j,j}}{-1 + T_1} - T_2\, g_{2,j,i}\, g_{3,j,j}\Big]$$

Out[∘]=

True

pdf

In[∘]:=

$$R_1[-1, i\_, j\_] = -1/2 - T_1^{-1}\, g_{1ji}\, g_{2ii} - \left(1 - T_1^{-1} - T_2^{-1}\right) g_{1ji}\, g_{2ji} - g_{1jj}\, g_{2ji} - g_{1ji}\, g_{2jj} + g_{3ii} + T_1^{-1}\, g_{1ji}\, g_{3ii} - \left(1 - T_2^{-1}\right) g_{2ji}\, g_{3ii} - g_{2jj}\, g_{3ii} + \left(1 - T_3^{-1}\right) g_{1ji}\, g_{3ji} - \left(1 - T_3^{-1}\right) g_{2ii}\, g_{3ji} + \left(2 - T_2^{-1}\right)\left(1 - T_3^{-1}\right) g_{2ji}\, g_{3ji} + \left(1 - T_3^{-1}\right) g_{2jj}\, g_{3ji} + g_{1ji}\, g_{3jj} + g_{2ji}\, g_{3jj} + \left(T_1\left(1 - T_2^{-1}\right) g_{1ii}\, g_{2ji} - g_{1jj}\, g_{2ii} + T_1\, g_{1ii}\, g_{2jj} + g_{1jj}\, g_{3ii} - T_2^{-1}\,(T_3 - 1)\, g_{1ii}\, g_{3ji} - T_1\, g_{1ii}\, g_{3jj}\right) / (T_1 - 1);$$

In[∘]:=
$$\text{Simplify}\Big[R_1[-1, i, j] == -\frac{1}{2} - \frac{g_{1,j,i}\, g_{2,i,i}}{T_1} - \frac{g_{1,j,j}\, g_{2,i,i}}{-1+T_1} + \frac{T_1\,(-1+T_2)\, g_{1,i,i}\, g_{2,j,i}}{(-1+T_1)\, T_2} - \frac{(-T_1 - T_2 + T_1 T_2)\, g_{1,j,i}\, g_{2,j,i}}{T_1 T_2} - g_{1,j,j}\, g_{2,j,i} + \frac{T_1\, g_{1,i,i}\, g_{2,j,j}}{-1+T_1} - g_{1,j,i}\, g_{2,j,j} + g_{3,i,i} + \frac{g_{1,j,i}\, g_{3,i,i}}{T_1} + \frac{g_{1,j,j}\, g_{3,i,i}}{-1+T_1} - \frac{(-1+T_2)\, g_{2,j,i}\, g_{3,i,i}}{T_2} - g_{2,j,j}\, g_{3,i,i} - \frac{(-1+T_1 T_2)\, g_{1,i,i}\, g_{3,j,i}}{(-1+T_1)\, T_2} + \frac{(-1+T_1 T_2)\, g_{1,j,i}\, g_{3,j,i}}{T_1 T_2} - \frac{(-1+T_1 T_2)\, g_{2,i,i}\, g_{3,j,i}}{T_1 T_2} + \frac{(-1+2 T_2)\,(-1+T_1 T_2)\, g_{2,j,i}\, g_{3,j,i}}{T_1 T_2^2} + \frac{(-1+T_1 T_2)\, g_{2,j,j}\, g_{3,j,i}}{T_1 T_2} - \frac{T_1\, g_{1,i,i}\, g_{3,j,j}}{-1+T_1} + g_{1,j,i}\, g_{3,j,j} + g_{2,j,i}\, g_{3,j,j}\Big]$$

Out[∘]=

True

pdf

In[∘]:=
$$\theta[\{1, i0\_, j0\_\}, \{1, i1\_, j1\_\}] = -T_1\,(T_3-1)\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1} + (T_3-1)\, g_{1,j1,j0}\, g_{2,i1,i0}\, g_{3,j0,i1} + T_1\,(T_3-1)\, g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1} - (T_3-1)\, g_{1,j1,j0}\, g_{2,j1,i0}\, g_{3,j0,i1};$$

In[∘]:=
$$\text{Simplify}[\theta[\{1, i0, j0\}, \{1, i1, j1\}] == -T_1\,(-1+T_1 T_2)\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1} + (-1+T_1 T_2)\, g_{1,j1,j0}\, g_{2,i1,i0}\, g_{3,j0,i1} + T_1\,(-1+T_1 T_2)\, g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1} + (1-T_1 T_2)\, g_{1,j1,j0}\, g_{2,j1,i0}\, g_{3,j0,i1}]$$

Out[∘]=

True

pdf

In[∘]:=
$$\theta[\{1, i0\_, j0\_\}, \{-1, i1\_, j1\_\}] = (T_3-1)\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1} - T_1^{-1}\,(T_3-1)\, g_{1,j1,j0}\, g_{2,i1,i0}\, g_{3,j0,i1} - (T_3-1)\, g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1} + T_1^{-1}\,(T_3-1)\, g_{1,j1,j0}\, g_{2,j1,i0}\, g_{3,j0,i1};$$

In[∘]:=
$$\text{Simplify}\Big[\theta[\{1, i0, j0\}, \{-1, i1, j1\}] == (-1+T_1 T_2)\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1} - \frac{(-1+T_1 T_2)\, g_{1,j1,j0}\, g_{2,i1,i0}\, g_{3,j0,i1}}{T_1} + (1-T_1 T_2)\, g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1} + \frac{(-1+T_1 T_2)\, g_{1,j1,j0}\, g_{2,j1,i0}\, g_{3,j0,i1}}{T_1}\Big]$$

Out[∘]=

True

pdf

In[∘]:=
$$\theta[\{-1, i0\_, j0\_\}, \{1, i1\_, j1\_\}] = T_1^{-1} T_2^{-1}\,(T_3-1)\,(g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1} - T_1\, g_{1,j1,j0}\, g_{2,i1,i0}\, g_{3,j0,i1} - g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1} + T_1\, g_{1,j1,j0}\, g_{2,j1,i0}\, g_{3,j0,i1});$$

In[•]:=
```
Simplify[θ[{-1, i0, j0}, {1, i1, j1}] ==
  ((-1 + T1 T2) g1,j1,i0 g2,i1,i0 g3,j0,i1)/(T1 T2) - ((-1 + T1 T2) g1,j1,j0 g2,i1,i0 g3,j0,i1)/T2 -
  ((-1 + T1 T2) g1,j1,i0 g2,j1,i0 g3,j0,i1)/(T1 T2) + ((-1 + T1 T2) g1,j1,j0 g2,j1,i0 g3,j0,i1)/T2]
```

Out[•]=

True

pdf

In[•]:=
```
θ[{-1, i0_, j0_}, {-1, i1_, j1_}] =
  (1 - T3^-1) (-T1^-1 g1,j1,i0 g2,i1,i0 g3,j0,i1 + g1,j1,j0 g2,i1,i0 g3,j0,i1 +
     T1^-1 g1,j1,i0 g2,j1,i0 g3,j0,i1 - g1,j1,j0 g2,j1,i0 g3,j0,i1);
```

In[•]:=
```
Simplify[θ[{-1, i0, j0}, {-1, i1, j1}] ==
  -((-1 + T1 T2) g1,j1,i0 g2,i1,i0 g3,j0,i1)/(T1^2 T2) + ((-1 + T1 T2) g1,j1,j0 g2,i1,i0 g3,j0,i1)/(T1 T2) +
  ((-1 + T1 T2) g1,j1,i0 g2,j1,i0 g3,j0,i1)/(T1^2 T2) - ((-1 + T1 T2) g1,j1,j0 g2,j1,i0 g3,j0,i1)/(T1 T2)]
```

Out[•]=

True

pdf

In[•]:=
```
Γ1[φ_, k_] = -φ / 2 + φ g3,k,k;
```

tex

We call the invariant computed θ:

pdf

In[•]:=
```
θ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ν, α, β, gEval, c, z},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 
                                                                 0     -1   ))];
   Δ = T^((-Total[φ]-Total[Cs[[All,1]]])/2) Det[A];
   G = Inverse[A]; gEval[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} :> (G[[α, β]] /. T → T_ν)];
   z = gEval[Sum_{k1=1}^n Sum_{k2=1}^n θ[Cs[[k1]], Cs[[k2]]]];
   z += gEval[Sum_{k=1}^n R1 @@ Cs[[k]]];
   z += gEval[Sum_{k=1}^{2 n} Γ1[φ[[k]], k]];
   {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) z} // Factor];
```

exec

```
nb2tex$PDFWidth /= 1.25;
```

Some Knots

tex

```
\needspace{15mm}
{\bf\red Some Knots.}
```

pdf

```
Expand[θ[Knot[3, 1]]]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=
pdf

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$

exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

In[•]:=

```
PolyPlot[0] = Graphics[{}];
PolyPlot[p_] := Module[{crs, m1, m2, maxc, minc, s, hex},
  crs = CoefficientRules[T1^(m1=-Exponent[p,T1,Min]) T2^(m2=-Exponent[p,T2,Min]) p, {T1, T2}];
  maxc = N@Log@Max@Abs[Last /@ crs];
  minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
  If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
  hex = Table[{Cos[α], Sin[α]} / Cos[2 π / 12] / 2, {α, 2 π / 12, 2 π, 2 π / 6}];
  Graphics[crs /. ({x1_, x2_} → c_) :> {
      If[c == 0, White, Lighter[If[c > 0, Red, Blue], 0.88 s[Abs@c]]],
      Polygon[(({{1, -1/2}, {0, Sqrt[3]/2}}).{x1 + m1, x2 + m2} + #) & /@ hex] }] ]
```

exec

```
nb2tex$PDFWidth /= 1.25;
```

tex

```
\parpic[r]{$
 {\includegraphics[height=0.45in]{K11n34.png}
  \atop\text{\tiny K11n34}}
 {\includegraphics[height=0.45in]{K11n42.png}
  \atop\text{\tiny K11n42}}
$}
```

pdf

In[◦]:=
```
GraphicsRow[PolyPlot[θ[Knot[#]][[2]]] &
  /@ {"3_1", "K11n34", "K11n42"}]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

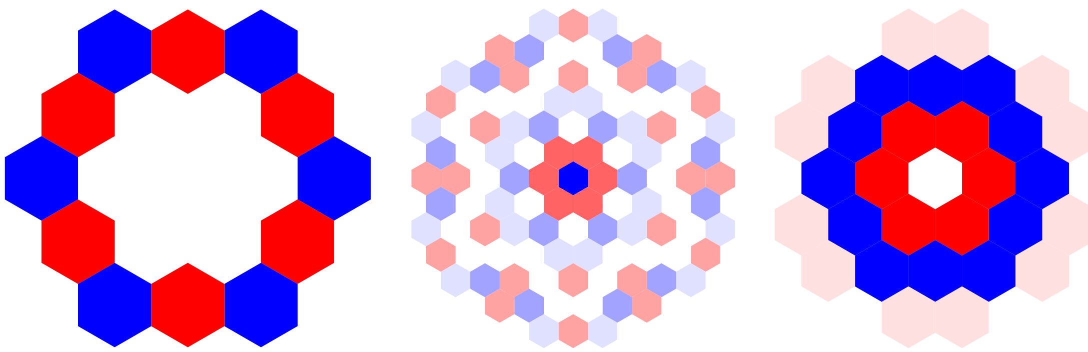
pdf

KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

pdf

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[◦]=
pdf



tex

\parpic[r]{$
 {\includegraphics[height=0.6in]{../../Projects/Gallery/Conway.png}
  \atop\text{\scriptsize Conway}}
 {\includegraphics[height=0.6in]{../../Projects/Gallery/PhotoNotAvailable.png}
  \atop\text{\scriptsize Kinoshita}}
 {\includegraphics[height=0.6in]{../../Projects/Gallery/Terasaka.jpg}
  \atop\text{\scriptsize Terasaka}}
$}
So θ detects knot mutation and separates the Conway knot K11n34 from the Kinoshita-Terasaka knot K11n42!

\vskip 8mm

%\needspace{50mm}
\parpic[r]{$
 {\includegraphics[height=0.6in]{../../Projects/Gallery/Gompf.jpg}
  \atop\text{\scriptsize Gompf}}
 {\includegraphics[height=0.6in]{../../Projects/Gallery/Scharlemann.jpg}
  \atop\text{\scriptsize Scharlemann}}
 {\includegraphics[height=0.6in]{../../Projects/Gallery/Thompson.jpg}

\atop\text{\scriptsize Thompson}}
$}
The 48-crossing Gompf-Scharlemann-Thompson knot \cite{GompfScharlemannThompson:Counterexample} is significant because it may be a counterexample to the slice-ribbon conjecture:

\[\resizebox{\linewidth}{!}{\import{../Waco-2203/}{GST48-Marked.pdf_t}} \]

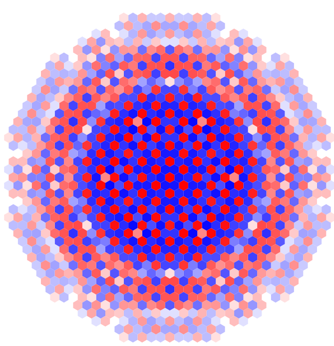
exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

In[ ]:= **AbsoluteTiming@**
PolyPlot[θ[**EPD**[$X_{14,1}$, $\overline{X}_{2,29}$, $X_{3,40}$, $X_{43,4}$, $\overline{X}_{26,5}$, $X_{6,95}$, $X_{96,7}$, $X_{13,8}$, $\overline{X}_{9,28}$, $X_{10,41}$, $X_{42,11}$, $\overline{X}_{27,12}$, $X_{30,15}$, $\overline{X}_{16,61}$, $\overline{X}_{17,72}$, $\overline{X}_{18,83}$, $X_{19,34}$, $\overline{X}_{89,20}$, $\overline{X}_{21,92}$, $\overline{X}_{79,22}$, $\overline{X}_{68,23}$, $\overline{X}_{57,24}$, $\overline{X}_{25,56}$, $X_{62,31}$, $X_{73,32}$, $X_{84,33}$, $\overline{X}_{50,35}$, $X_{36,81}$, $X_{37,70}$, $X_{38,59}$, $\overline{X}_{39,54}$, $X_{44,55}$, $X_{58,45}$, $X_{69,46}$, $X_{80,47}$, $X_{48,91}$, $X_{90,49}$, $X_{51,82}$, $X_{52,71}$, $X_{53,60}$, $\overline{X}_{63,74}$, $\overline{X}_{64,85}$, $\overline{X}_{76,65}$, $\overline{X}_{87,66}$, $\overline{X}_{67,94}$, $\overline{X}_{75,86}$, $\overline{X}_{88,77}$, $\overline{X}_{78,93}$]][[2]]]

Out[ ]=
pdf

{27.1284,  }

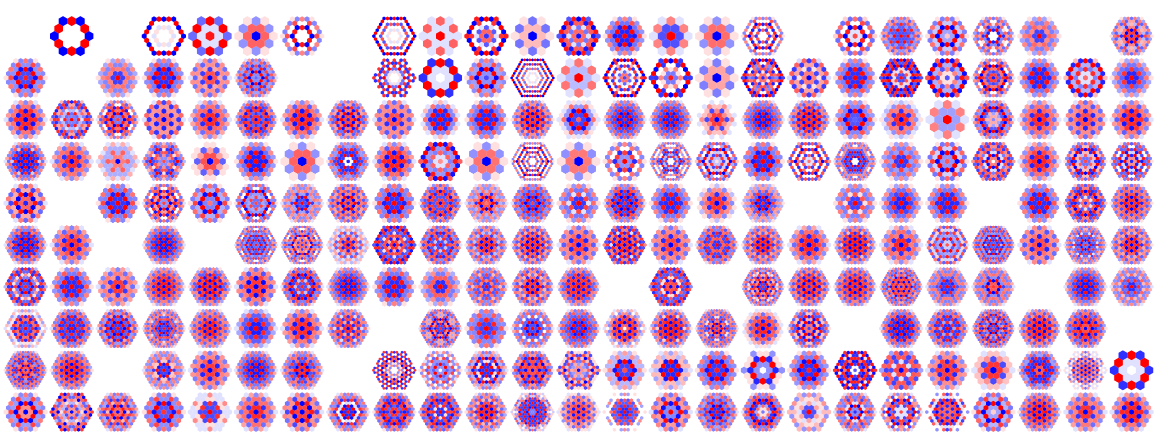
exec

In[ ]:=
```
nb2tex$PDFWidth /= 1.25;
```

In[ ]:=
```
tab250 = {0}~Join~Table[θ[K][[2]], {K, AllKnots[{3, 10}]}];
```

In[ ]:=
```
g250 = GraphicsGrid[Partition[PolyPlot /@ tab250, 25], Spacings → 0]
```

Out[ ]=



In[ ]:=
```
Export["g250.png", g250, ImageSize → 2400]
```
Out[ ]=
```
g250.png
```