Pensieve header: The rank 2 mod ϵ^2 invariant using integration techniques; continues UC4A2.nb and Theta.nb at pensieve://Projects/HigherRank/.

Initialization

```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Beijing-2407"];
        Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/AP/Talks/Beijing-2407 to compute rotation numbers.
```

pdf

```
In[•]:= T3 = T1 T2;
        i_+ := i + 1;
```

The Lagrangian

exec

```
In[•]:= nb2tex$PDFWidth *= 1.25;
```

pdf

```
In[•]:= ℒ[X_{i_,j_}[1]] := T3 𝔼[Plus[
            ∑_{ν=1}^{3} (x_{νi} (p_{νi+} - p_{νi}) + x_{νj} (p_{νj+} - p_{νj}) + (T_ν - 1) x_{νi} (p_{νi+} - p_{νj+})),
            p_{3j} x_{2i} (T1 x_{1i} - x_{1j}),
            ϵ (T3 - 1) p_{1j} x_{3i} (p_{2j} - p_{2i}),
            ϵ (1 / 2 - p_{3i} x_{3i} - T3 p_{1j} p_{2j} x_{1i} x_{2i} + p_{2j} p_{3i} x_{2j} x_{3i} - T2 p_{2j} p_{3j} x_{2i} x_{3j} +
                (T3 - 1) p_{3j} x_{3i} (T1 p_{1j} x_{1i} + T2 p_{2j} x_{2i}) + (p_{1j} x_{1j} (p_{2i} x_{2i} - p_{3i} x_{3i}) +
                    T1 p_{1i} x_{1i} (p_{3j} x_{3j} - p_{2j} x_{2j}) + (T3 - 1) p_{1j} x_{1j} (p_{2j} x_{2i} - T1 p_{3j} x_{3i})) / (T1 - 1)),
            O[ϵ]^2
        ]]
```

In[•]:=
```
Simplify[First[T3^-1 ℒ[Xi,j[1]]] ==
  (((-p1,i + p1,i⁺) x1,i + (-1 + T1) (p1,i⁺ - p1,j⁺) x1,i + (-p1,j + p1,j⁺) x1,j + (-p2,i + p2,i⁺) x2,i +
     (-1 + T2) (p2,i⁺ - p2,j⁺) x2,i + T1 p3,j x1,i x2,i - p3,j x1,j x2,i + (-p2,j + p2,j⁺) x2,j +
     (-p3,i + p3,i⁺) x3,i + (-1 + T1 T2) (p3,i⁺ - p3,j⁺) x3,i + (-p3,j + p3,j⁺) x3,j) +
    (1/2 - T1 T2 p1,j p2,j x1,i x2,i + p1,j p2,i x1,j x2,i/(-1 + T1) + (-1 + T1 T2) p1,j p2,j x1,j x2,i/(-1 + T1) -
     T1 p1,i p2,j x1,i x2,j/(-1 + T1) + (1 - T1 T2) p1,j p2,i x3,i + (-1 + T1 T2) p1,j p2,j x3,i -
     p3,i x3,i + T1 (-1 + T1 T2) p1,j p3,j x1,i x3,i - p1,j p3,i x1,j x3,i/(-1 + T1) -
     T1 (-1 + T1 T2) p1,j p3,j x1,j x3,i/(-1 + T1) + T2 (-1 + T1 T2) p2,j p3,j x2,i x3,i +
     p2,j p3,i x2,j x3,i + T1 p1,i p3,j x1,i x3,j/(-1 + T1) - T2 p2,j p3,j x2,i x3,j) ϵ + O[ϵ]^2)]
```

Out[•]=
```
O[ϵ]^2 == 0
```

pdf

In[•]:=
```
ℒ[Xi_,j_[-1]] := T3^-1 𝔼[Plus[
    Sum_{ν=1}^{3} (xνi (pνi⁺ - pνi) + xνj (pνj⁺ - pνj) + (Tν^-1 - 1) xνi (pνi⁺ - pνj⁺)),
    T2^-1 (p3j x1j x2i - T1^-1 p3j x1i x2i),
    ϵ T1^-1 ((T3 - 1) p1j p2i x3i - (T3 - 1) p1j p2j x3i),
    ϵ (-1/2 + p3i x3i - T1^-1 p1j p2i x1i x2i - (1 - T1^-1 - T2^-1) p1j p2j x1i x2i - p1j p2j x1j x2i -
       p1j p2j x1i x2j + T1^-1 p1j p3i x1i x3i - (1 - T2^-1) p2j p3i x2i x3i - p2j p3i x2j x3i + p1j p3j x1i x3j +
       p2j p3j x2i x3j + (1 - T3^-1) p3j x3i (p2j x2j + p1j x1i - p2i x2i + (2 - T2^-1) p2j x2i) +
       (T1 (1 - T2^-1) p1i p2j x1i x2i - p1j p2i x1j x2i + T1 p1i p2j x1i x2j -
          T2^-1 (T3 - 1) p1i p3j x1i x3i + p1j p3i x1j x3i - T1 p1i p3j x1i x3j) / (T1 - 1)),
    O[ϵ]^2
  ]]
```

In[•]:=
```
Simplify[First[T3 ℒ[Xi,j[-1]]] ==
  ((((-p1,i + p1,i⁺) x1,i + (-1 + 1/T1) (p1,i⁺ - p1,j⁺) x1,i + (-p1,j + p1,j⁺) x1,j + (-p2,i + p2,i⁺) x2,i +
      (-1 + 1/T2) (p2,i⁺ - p2,j⁺) x2,i - (p3,j x1,i x2,i)/(T1 T2) + (p3,j x1,j x2,i)/T2 + (-p2,j + p2,j⁺) x2,j +
      (-p3,i + p3,i⁺) x3,i + (-1 + 1/(T1 T2)) (p3,i⁺ - p3,j⁺) x3,i + (-p3,j + p3,j⁺) x3,j) +
    (-1/2 - (p1,j p2,i x1,i x2,i)/T1 + (T1 (-1 + T2) p1,i p2,j x1,i x2,i)/((-1 + T1) T2) -
      ((-T1 - T2 + T1 T2) p1,j p2,j x1,i x2,i)/(T1 T2) - (p1,j p2,i x1,j x2,i)/(-1 + T1) - p1,j p2,j x1,j x2,i +
      (T1 p1,i p2,j x1,i x2,j)/(-1 + T1) - p1,j p2,j x1,i x2,j + ((-1 + T1 T2) p1,j p2,i x3,i)/T1 -
      ((-1 + T1 T2) p1,j p2,j x3,i)/T1 + p3,i x3,i + (p1,j p3,i x1,i x3,i)/T1 -
      ((-1 + T1 T2) p1,i p3,j x1,i x3,i)/((-1 + T1) T2) + ((-1 + T1 T2) p1,j p3,j x1,i x3,i)/(T1 T2) +
      (p1,j p3,i x1,j x3,i)/(-1 + T1) - ((-1 + T2) p2,j p3,i x2,i x3,i)/T2 - ((-1 + T1 T2) p2,i p3,j x2,i x3,i)/(T1 T2) +
      ((-1 + 2 T2) (-1 + T1 T2) p2,j p3,j x2,i x3,i)/(T1 T2^2) - p2,j p3,i x2,j x3,i + ((-1 + T1 T2) p2,j p3,j x2,j x3,i)/(T1 T2) -
      (T1 p1,i p3,j x1,i x3,j)/(-1 + T1) + p1,j p3,j x1,i x3,j + p2,j p3,j x2,i x3,j) ϵ + O[ϵ]^2)]
```

Out[•]=
$$O[\epsilon]^2 == 0$$

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In[•]:=
```
ℒ[Ci_[φ_]] := T3^φ 𝔼[∑_{ν=1}^{3} xνi (pνi⁺ - pνi) + ϵ φ (p3,i x3,i - 1/2) + O[ϵ]^2]
```

In[•]:=
```
ℒ[Ci[φ]]
```

Out[•]=
$$(T_1 T_2)^{\varphi}\, \mathbb{E}\left[((-p_{1,i} + p_{1,1+i})\, x_{1,i} + (-p_{2,i} + p_{2,1+i})\, x_{2,i} + (-p_{3,i} + p_{3,1+i})\, x_{3,i}) + \varphi \left(-\frac{1}{2} + p_{3,i}\, x_{3,i}\right) \epsilon + O[\epsilon]^2 \right]$$

exec

In[•]:=
```
nb2tex$PDFWidth /= 1.25;
```

pdf

In[•]:=
```
<< FormalGaussianIntegration.m
```

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In[•]:=
```
ps_i_ := Sequence[p_{1,i}, p_{2,i}, p_{3,i}];
xs_i_ := Sequence[x_{1,i}, x_{2,i}, x_{3,i}];
vs_i_ := Sequence[ps_i, xs_i];
ℱ[is___] := 𝔼[Sum[π_{v,i} p_{v,i}, {i, {is}}, {v, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K][[2]]];
vs[K_] := Union @@ Table[{vs_i}, {i, Features[K][[1]]}]
```

pdf

Reidemeister 3

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In[•]:=
```
BalancingHack = ReplaceAll[ϵSeries[most___, last_] :>
    ϵSeries[most, CF[last /. {π_ss__ :> B^-1 π_ss, p_ss__ :> B p_ss}] /. B^b_ /; b < 0 → 0]];
```

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In[•]:=
```
Short[lhs = ∫ℱ[i, j, k] ℒ /@ (X_{i,j}[1] X_{i⁺,k}[1] X_{j⁺,k⁺}[1]) 𝕕{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}} //
    BalancingHack]
```

Out[•]//Short=
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$$T_1^3\,T_2^3\,\mathbb{E}\left[\epsilon\text{Series}\left[T_1^2\,p_{1,2+i}\,\pi_{1,i}-(-1+T_1)\,T_1\,p_{1,2+j}\,\pi_{1,i}+(1-T_1)\,p_{1,2+k}\,\pi_{1,i}+\right.\right.$$
$$\left.\left.\text{<<24>>}+T_1\,T_2\,p_{3,2+j}\,\pi_{3,j}+(1-T_1\,T_2)\,p_{3,2+k}\,\pi_{3,j}+p_{3,2+k}\,\pi_{3,k},\ \frac{3}{2}+\text{<<107>>}\right]\right]$$

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In[•]:=
```
rhs = ∫ℱ[i, j, k] ℒ /@ (X_{j,k}[1] X_{i,k⁺}[1] X_{i⁺,j⁺}[1]) 𝕕{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}} //
    BalancingHack;
lhs == rhs
```

Out[•]=
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True

pdf

The Trefoil

pdf

In[•]:=
```
K = Knot[3, 1]; ∫ℒ[K] 𝕕 vs[K]
```

Out[•]=
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$$-\frac{\mathbb{i}\,T_1^2\,T_2^2\,\mathbb{E}\left[\epsilon\text{Series}\left[0,\ -\frac{1-T_1+T_1^2-T_2-T_1^3\,T_2+T_2^2+T_1^4\,T_2^2-T_1\,T_2^3-T_1^4\,T_2^3+T_1^2\,T_2^4-T_1^3\,T_2^4+T_1^4\,T_2^4}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1\,T_2+T_1^2\,T_2^2\right)}\right]\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1\,T_2+T_1^2\,T_2^2\right)}$$